POLYNOMIALS THAT GENERATE
A ROW OF PASCAL’S TRIANGLE

SEON-HONG KIM

Abstract. Let $p$ be an odd prime, and let $f(x)$ be the interpolating
polynomial associated with a table of data points $(j + 1, (\, ? ))$
for $0 \leq j \leq p$. In this article, we find congruence identities modulo
$p$ of $(p - 1)! f(x), (p - 2)! f(x)$, and $(p - 3)! f(x)$. Moreover we
present some conjectures of these types.

1. Introduction

Pascal’s triangle was originally developed by the ancient Chinese, but
Blaise Pascal was the first person who discovered all of the patterns it
contained. Assume that Pascal’s triangle starts from the zeroth row and
the zeroth element of a row. In the row numbered $n$ there appear the
coefficients of the expansion of the polynomial $(x + 1)^n$.

Pascal’s triangle has many patterns. Some of them are as follows.

(1) The sum of the numbers in any row is equal to $2^n$, where $n$ is
the number of the row.

(2) If the first element in a row is a prime number, all numbers in
that row (excluding the 1’s) are divisible by it.

(3) If a diagonal of numbers of any of the 1’s bordering the sides
of the triangle and ending on any number inside the selection is
equal to the number below the end of the selection that is not
on the same diagonal itself.

Besides above, there are many interesting patterns relating to so
called magic 11’s, Fibonacci’s sequence, polygonal numbers. Also it
is connected to Sierpinski’s triangle. For the details, see [1].

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In this article, we consider some congruence identities modulo an odd prime \( p \) of the interpolation formula from the \( p \text{th} \) row of Pascal's triangle, i.e., the coefficients of \((x + 1)^p\). In fact, let \( p \) be an odd prime and \( f(x) \) the interpolating polynomial associated with a table of data points \((j + 1, \binom{p}{j})\) for \( 0 \leq j \leq p \). Then we show in Section 2 that

\[(p - 1)! f(x) \equiv x^{p-1} + x^{p-2} + \cdots + x \pmod{p}.
\]

It easily follows that

\[
(p - 2)! f(x) \equiv (p - 1) \left(x^{p-1} + x^{p-2} + \cdots + x\right) \pmod{p},
\]

\[
(p - 3)! f(x) \equiv \frac{p + 1}{2} \left(x^{p-1} + x^{p-2} + \cdots + x\right) \pmod{p}
\]

provided all coefficients of the left sides are rational integers. Moreover we present some conjectures of these types.

2. Main result

In this section, we show the following proposition.

**Proposition 1.** Let \( p \) be an odd prime and \( f(x) \) the interpolating polynomial associated with a table of data points \((j + 1, \binom{p}{j})\) for \( 0 \leq j \leq p \). Then we have

\[(p - 1)! f(x) \equiv x^{p-1} + x^{p-2} + \cdots + x \pmod{p}.
\]

**Proof.** Let \( p \) be an odd prime and \( f(x) \) the interpolating polynomial associated with a table of data points \((j + 1, \binom{p}{j})\) for \( 0 \leq j \leq p \). We use Lagrange interpolation. Then

\[
f(x) = \sum_{j=0}^{p} \left( \prod_{i=0 \atop i \neq j}^{p} \frac{x - (i + 1)}{j - i} \right) \binom{p}{j}
\]

\[
= \sum_{j=0}^{p} \left( \frac{x - 1}{j} \frac{x - 2}{j - 1} \cdots \frac{x - j}{1} \frac{x - (j + 2)}{-1} \frac{x - (j + 3)}{-2} \right. \\
\left. \cdots \frac{x - (p + 1)}{-2} \right) {p! \over j!(p-j)!}
\]
\[
\begin{align*}
\sum_{j=0}^{p} (-1)^{p-j} \frac{p!}{(j!(p-j)!)^2} \left( \frac{(x-1)(x-2)\cdots(x-(p+1))}{x-(j+1)} \right) \\
= \frac{1}{p!} \sum_{j=0}^{p} (-1)^{j+1} \binom{p}{j}^2 \frac{(x-1)(x-2)\cdots(x-(p+1))}{x-(j+1)} \\
= \frac{1}{p!} \left( (x-2)\cdots(x-p)(x-1-(x-(p+1))) \\
+ \sum_{j=1}^{p-1} (-1)^j (p-2j) \binom{p}{j} R_j(x) \right) \\
= \frac{1}{(p-1)!} \left( (x-2)\cdots(x-p) \\
+ \sum_{j=1}^{p-1} (-1)^j \binom{p}{j}^2 R_j(x) - \frac{2}{p} \sum_{j=1}^{p-1} (-1)^j j \binom{p}{j}^2 R_j(x) \right),
\end{align*}
\]

where \( R_j(x) \) is an integral polynomial depending on \( j \). Hence

\((p-1)!f(x) \equiv (x-2)(x-3)\cdots(x-p) \pmod{p} \).

Since \( x^p - x \equiv 0 \pmod{p} \) for all \( 0 \leq x \leq p-1 \), we have

\( x(x-1)\cdots(x-(p-1)) \equiv x^p - x \pmod{p} \)

and

\( x(x-2)(x-3)\cdots(x-(p-1)) \equiv x^{p-1} + \cdots + x \pmod{p} \).

Hence

\( (x-p)(x-2)(x-3)\cdots(x-(p-1)) \equiv x^{p-1} + \cdots + x \pmod{p} \),

which proves the result. \( \square \)

The next easily follows from Proposition 1.
COROLLARY 2. With the same assumptions of Proposition 1, we have

\[(p-2)! f(x) \equiv (p-1) \left( x^{p-1} + x^{p-2} + \cdots + x \right) \pmod{p}, \]

\[(p-3)! f(x) \equiv \frac{p+1}{2} \left( x^{p-1} + x^{p-2} + \cdots + x \right) \pmod{p} \]

provided all coefficients of the left sides are rational integers.

PROOF. By Proposition 1,

\[(p-1)! f(x) \equiv x^{p-1} + x^{p-2} + \cdots + x \pmod{p} \]

\[\equiv (p-1)^2 \left( x^{p-1} + x^{p-2} + \cdots + x \right) \pmod{p}. \]

Hence we have

\[(p-2)! f(x) \equiv (p-1) \left( x^{p-1} + x^{p-2} + \cdots + x \right) \pmod{p} \]

\[\equiv (p-2) \frac{p+1}{2} \left( x^{p-1} + x^{p-2} + \cdots + x \right) \pmod{p} \]

and

\[(p-3)! f(x) \equiv \frac{p+1}{2} \left( x^{p-1} + x^{p-2} + \cdots + x \right) \pmod{p}. \]

\[\square \]

REMARK 3. With the same assumptions of Proposition 1, for \( k \geq 4 \), it seems that \( (p-k)! f(x) \equiv h \left( x^{p-1} + x^{p-2} + \cdots + x \right) \pmod{p} \) for some integer \( h = h(p) \) in terms of \( p \). Finding suitable \( h \)'s remains a question.

Now we consider congruence identities in the case of modulo \( n-1 \). A number \( k \) is called hexagonal if \( k = h(2h-1) \) for some \( h \geq 1 \). The author has investigated extensive interpolation formulas modulo \( n-1 \) from the coefficients of \( (x+1)^n \) and formulated the conjecture below. But it remains an open problem.

CONJECTURE. Let \( n \) be an integer \( \geq 3 \) and \( f(x) \) the interpolating polynomial associated with a table of data points \( (j+1, \binom{n}{j}) \) for \( 0 \leq j \leq n \). Then on modulo \( n-1 \) we have that \( (n-1)! f(x) \) is congruent to 0 if \( n = 2k+1 \) and \( k \) is hexagonal, and \( x^2(2(k-1)+4k^k+2(k-1)x^{2(k-1)}) \) if \( n = 2k+1 \) and \( k \) is odd prime, respectively.
Remark 4. One might think that there are "better" and "simpler" forms of congruence identities modulo an integer of the type of congruence identities in this article. However, due to author's search, not many interpolation formulas from the coefficients of \((x + 1)^n\) by multiplying a constant have good and simple forms of congruence identities modulo an integer. We could only find congruence identities modulo an integer in this article having simple forms and regular patterns.

References


School of Mathematical Sciences
Seoul National University
Seoul 151-742, Korea
E-mail: s-kim17@orgio.net