CERTAIN CONFORMALLY INVARIANT CONNECTIONS OF RIZZA MANIFOLDS

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ABSTRACT. We introduce certain conformally invariant $h$-Finsler connection in a Rizza manifold. Using this connection, we find some conformally invariant Finsler tensors. The conformal flatness and the Kählerian Finsler manifold with respect to the above connection are investigated.

1. Introduction

Let $M$ be a $2n$-dimensional differentiable manifold admitting an almost complex structure $f^i_j(x)$ and Finsler metric $g_{ij}(x, y)$. If the couple $(f^i_j(x), g_{ij}(x, y))$ satisfies the Rizza condition:

$$\{g_{ij}(x, y) - g_{pq}(x, y)f^p_i(x)f^q_j(x)\}y^i = 0,$$

then the manifold $M$ is called a Rizza manifold ([4], [12], and [13]). In [7], Ichijyo has introduced the notions of a $(G_{ij}(x, y), N^i_j(x, y))$-structure and its conformal changes, where $G_{ij}(x, y)$ is a generalized Finsler metric and $N^i_j(x, y)$ is the generalized Chern's non-linear connection in a Rizza manifold. In [11], Park has introduced a conformally invariant $h$-Finsler connection $(M^i_j(x, y), M^i_j(x, y))$, where $M^i_j(x, y)$ is a non-linear connection constructed from the generalized Chern's non-linear connection $N^i_j(x, y)$ and a generalized Finsler metric $G_{ij}(x, y)$ and an almost complex structure $f^i_j(x)$. In the same paper, he found some conformally invariant Finsler tensors and investigated the conformal flatness of the $(G_{ij}, M^i_j)$-structure.

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In the present paper the author introduces a special conformally invariant $h$-Finsler connection $(H^i_j(x, y), H^i_j(x, y))$ in a Rizza manifold, where $H^i_j(x, y)$ is constructed from the generalized Chern’s non-linear connection $N^i_j(x, y)$ and $h_0$-torsion $P^i_jk(x, y)$. We find some conformally invariant Finsler tensors and conformal flatness of a $(G_{ij}, H^i_j)$-structure. We also investigate the Kaehlerian Finsler manifold with respect to the $(H^i_j(x, y), H^i_j(x, y))$-connection in a Rizza manifold.

2. Preliminaries

In a Rizza manifold, if we put

\begin{equation}
G_{ij}(x, y) = \frac{1}{2}(g_{ij}(x, y) + g_{pq}(x, y)f^p_if^q_j),
\end{equation}

then it is seen that $G_{ij}(x, y) = G_{ji}(x, y)$, $G_{ij}(x, y)$ is a positively homogeneous tensor of degree zero for $y^i$ and $G_{ij}(x, y)\xi^i\xi^j$ is positive definite. That is, $G_{ij}(x, y)$ is a generalized Finsler metric ([2], [8]). This is not a Finsler metric. With respect to this generalized Finsler metric $G_{ij}(x, y)$, we have

\begin{equation}
G_{pq}(x, y)f^p_i(x)f^q_j(x) = G_{ij}(x, y), \quad y^r\partial^r G_{ij}(x, y) = 0.
\end{equation}

Moreover, due to the Rizza condition (1.1) we can see easily

\begin{equation}
y^m f^r_m(x)\partial^r G_{ij}(x, y) = 0, \quad \partial^r G_{pq}(x, y)y^p y^q = 0,
G_{ij}(x, y) = G_{pq}(x, y)f^p_i(x)f^q_j(x).
\end{equation}

In a Rizza manifold $M$, the generalized Chern’s non-linear connection is given by ([8]):

\begin{equation}
N^i_j = \frac{1}{2}(G^{ih}\partial^h G_{hs} - f^i_h G^{hr} f^t_j \partial^r G_{rs} + s^i_{sj} - G^{i} G_{m} G_{s} S^m_{sj})
- G^{i} G_{hm} y^m S^r_{sj} + G^{i} G_{hr} f^r_s S^t_{msj})y^s,
\end{equation}

where $S^i_{kj} = (\partial_k f^i_r) f^r_j$.

It is known that if the given almost complex structure $f^i_j(x)$ in the Rizza manifold is integrable, then $N^i_j(x, y)$ defined by (2.4) coincides with the Chern’s non-linear connection ([7]). With respect to a generalized Finsler metric $G_{ij}(x, y)$ and the generalized Chern’s non-linear
connection \( N^i_j(x, y) \) respectively defined by (2.1) and (2.4), we introduce a symmetric Finsler connection \( FT = (\Gamma^i_j, N^i_j, C^i_j) \) as follows (\cite{[1]}, \cite{[9]}):

\[
\begin{align*}
\Gamma^i_j(x, y) &= \frac{1}{2} G^{im}(X_j G_{mk} + X_k G_{mj} - X_m G_{jk}), \\
C^i_j(x, y) &= \frac{1}{2} G^{im}(\partial_j G_{mk} + \partial_k G_{mj} - \partial_m G_{jk}),
\end{align*}
\]

(2.5)

where \( X_j = \partial_j - N^m_j \partial_m \), \( \partial_j = \partial / \partial x^j \) and \( \partial_j = \partial / \partial y^j \). Denoting the \( h \)-covariant, \( v \)-covariant derivative with respect to \( FT \) by \( \nabla \) and \( \hat{\nabla} \) respectively, we have directly

\[
\nabla_k G_{ij} = 0, \quad \hat{\nabla}_k G_{ij} = 0.
\]

The above Finsler connection \( FT = (\Gamma^i_j, N^i_j, C^i_j) \) is said to be the Finsler connection associated with the \( (G, N) \)-structure.

According to Matsumoto \cite{[9]}, we write the \( h \)-torsion and \( hv \)-torsion of \( FT \) as

\[
R^i_{jk} = X_k N^i_j - j/k, \quad P^i_{jk} = \partial_k N^i_j - \Gamma^i_j,
\]

(2.6)

and the \( h \)-curvature and \( hv \)-curvatures of \( FT \) as

\[
R^i_{jk} = K^i_{jk} + C^i_m R^m_{jk}, \quad P^i_{jk} = \partial_k \Gamma^i_j - Q^i_{jk},
\]

(2.7)

where \( j/k \) denotes the interchange of indices \( j \) and \( k \) of the preceding terms, and we put

\[
\begin{align*}
K^i_{jk} &= X_k \Gamma^i_j + \Gamma^m_k \Gamma^i_m - j/k, \\
Q^i_{jk} &= \nabla_j C^i_k - C^i_m P^m_{jk}.
\end{align*}
\]

(2.8)

3. Conformally invariant \( h \)-connections

In a Rizza manifold \( M \), let us consider the conformal changes:

\[
G_{ij}(x, y) \rightarrow \overline{G}_{ij}(x, y) = e^{2\sigma(x)} G_{ij}(x, y),
\]

(3.1)

where \( \sigma(x) \) is any scalar. Then we can see easily

\[
\begin{align*}
\partial_k \overline{G}_{ij}(x, y) &= e^{2\sigma(x)} \partial_k G_{ij}(x, y) + 2\sigma_k e^{2\sigma} G_{ij}(x, y), \\
\overline{G}^{ij}(x, y) &= e^{-2\sigma(x)} G^{ij}(x, y), \quad \partial_k \overline{G}_{ij}(x, y) = e^{2\sigma(x)} \partial_k G_{ij}(x, y).
\end{align*}
\]

(3.2)
The conformal change of the generalized Chern's non-linear connection $N^i_j$ defined (2.4) is given as follows ([7]):

\[ N^i_j = N^i_j + y^i \sigma_j - f^i_h y^h f^r_j \sigma_r, \]

where $\sigma_k = \partial_k \sigma(x)$, and we have

\[ \bar{N}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \sigma_k + \delta^i_k \sigma_j - G_{jk} G^{ir} \sigma_r, \]

(3.5)

If we put $P_k = P^r_{rk}$, then we have $\bar{P}_k = P_k - 2n \sigma_k$, from which

\[ \sigma_k = \frac{(P_k - \bar{P}_k)}{2n}. \]

Substituting (3.6) in (3.3) and (3.4), we have

\[ \bar{N}^i_{jk} + (y^i \bar{\bar{P}}_j - f^i_h y^h f^r_j \bar{\bar{P}}_r)/(2n) = N^i_j + (y^i P_j - f^i_h y^h f^r_j P_r)/(2n), \]

(3.7)

\[ \bar{\bar{N}}^i_{jk} + (\delta^i_j \bar{\bar{P}}_k + \delta^i_k \bar{\bar{P}}_j - G_{jk} G^{ir} \bar{\bar{P}}_r)/(2n) = \Gamma^i_{jk} + (\delta^i_j P_k + \delta^i_k P_j - G_{jk} G^{ir} P_r)/(2n), \]

respectively. If we put

\[ H^i_j = N^i_j + (y^i P_j - f^i_h y^h f^r_j P_r)/(2n), \]

(3.8)

\[ H^i_{jk} = \Gamma^i_{jk} + (\delta^i_j P_k + \delta^i_k P_j - G_{jk} G^{ir} P_r)/(2n), \]

then $\bar{H}^i_j = H^i_j$ and $\bar{H}^i_{jk} = H^i_{jk}$, that is, $(H^i_j, H^i_{jk})$ is a conformally invariant $h$-connection. Thus we have

**Theorem 3.1.** In a Rizza manifold $M$, we have a conformally invariant $h$-Finsler connection $(H^i_{jk}, H^i_j)$ given by (3.7) and (3.8).

In a Rizza manifold $M$ with a conformally invariant $h$-Finsler connection $FH = (H^i_{jk}, H^i_j, C^i_{jk})$ associated $(G, H)$-structure, the $h$- and
\( u \)-covariant derivatives of a Finsler tensor field, for example \( K^i_{\ j}(x, y) \), are given by

\[
\begin{align*}
\overset{\bullet}{\nabla}_k K^i_{\ j} &= \overset{\bullet}{X}_kK^i_{\ j} + K^r_{\ j}H^i_{\ rl}K^l_{\ r}k, \\
\tilde{\nabla}_k K^i_{\ j} &= \tilde{\partial}_k K^i_{\ j} + K^r_{\ j}C^i_{\ rl}K^l_{\ r}k,
\end{align*}
\]  

(3.9)

where \( \overset{\bullet}{X}_k = \partial_k - H^r_{\ kl}\partial_r \).

For the supporting element \( y^i \), we have

\[
\begin{align*}
\overset{\bullet}{\nabla}_k y^i &= H^i_{\ kl}y^k, \\
\tilde{\nabla}_k y^i &= \delta^i_k + C^i_{\ kl}y^l,
\end{align*}
\]

where the index 0 denotes the contraction with the supporting element \( y^i \). The Finsler tensor field \( \overset{\bullet}{D}^i_k \) given by

\[
\overset{\bullet}{D}^i_k = H^i_{\ kl}y^k
\]

is called the deflection tensor field with respect to connection \( FH = (H^i_{\ kl}, H^i_{\ j}, C^i_{\ j,l}) \). Thus the deflection tensor vanishes if and only if \( H^i_{\ j} = H^i_{\ kl} \).

The Ricci identities applying to \( G_{ij} \) are

\[
\begin{align*}
\overset{\bullet}{\nabla}_l \overset{\bullet}{\nabla}_k G_{ij} &= \overset{\bullet}{\nabla}_k \overset{\bullet}{\nabla}_l G_{ij} \\
&= -G_{rj} \overset{\bullet}{R}^r_{\ kl} - G_{ir} \overset{\bullet}{R}^r_{\ kl} - (\overset{\bullet}{\nabla}_r G_{ij}) T^r_{\ kl} = -G_{rj} \overset{\bullet}{R}^r_{\ kl} - (\overset{\bullet}{\nabla}_r G_{ij}) T^r_{\ kl},
\end{align*}
\]  

(3.10)

where torsion tensors \( T^i_{\ jk}, \overset{\bullet}{R}^i_{\ kl}, \overset{\bullet}{P}^i_{\ kl} \) and curvature tensors \( \overset{\bullet}{R}^i_{\ kl} \) are given by

\[
\begin{align*}
T^i_{\ jk} &= H^i_{\ jk} - j/k, \\
\overset{\bullet}{R}^i_{\ kl} &= \overset{\bullet}{X}_l H^i_{\ kl} - k/l, \\
\overset{\bullet}{P}^i_{\ kl} &= \tilde{\partial}_l H^i_{\ kl} - H^i_{\ kl}, \\
\overset{\bullet}{R}^i_{\ kl} &= \overset{\bullet}{K}^i_{\ kl} + C^i_{\ j} \overset{\bullet}{R}^j_{\ kl}, \\
\overset{\bullet}{P}^i_{\ kl} &= \tilde{\partial}_l H^i_{\ kl} - Q^i_{\ kl},
\end{align*}
\]

(3.11)
and

\begin{align}
\hbar K^i_{j\,kl} &= (\hbar X_l H^i_{j\,k} + H^r_{j\,r\,k} H^i_{r\,l}) - \frac{k}{l}, \\
\hbar Q^i_{j\,kl} &= \hbar \nabla_k C^i_{j\,l} + C^i_{j\,r\,l} P^r_{k\,l}.
\end{align}

(3.12)

Since \( \hbar T^i_{j\,k} = 0 \) and \( \hat{\nabla}_k G_{ij} = \nabla_k G_{ij} = 0 \), (3.10) is written as

\begin{align}
\hbar \nabla_l \nabla_k G_{ij} - \hbar \nabla_k \nabla_l G_{ij} &= -\hbar R^i_{jkl} - \hbar \tilde{R}^i_{jkl}, \\
\hbar \nabla_l \nabla_k G_{ij} &= -\hbar P^i_{jkl} - \hbar P^i_{jkl} - (\nabla_r G_{ij}) C^r_{k\,l},
\end{align}

(3.13)

where \( R^i_{jkl} = G_{rj} R^r_{i\,kl} \), \( P^i_{jkl} = G_{ir} P^r_{i\,kl} \).

On the other hand, substituting (3.7) and (3.8) in

\[ \hbar \nabla_k G_{ij} = \hbar X_k G_{ij} - \hbar G_{rj} H^r_{i\,k} - \hbar G_{ir} H^r_{j\,k} \]

and using (2.3) and \( \nabla_k G_{ij} = 0 \), we have

\begin{align}
\hbar \nabla_k G_{ij} &= \frac{\hbar}{n} P_k G_{ij}, \\
\hbar \nabla^i_{k} G_{ij} &= -\frac{\hbar}{n} P_k G_{ij}.
\end{align}

(3.14)

Therefore we get

\begin{align}
\hbar \nabla_l \nabla_k G_{ij} - \hbar \nabla_k \nabla_l G_{ij} &= (\nabla_l P_k - \nabla_k P_l) G_{ij}/n.
\end{align}

(3.15)

From (3.13), (3.14), (3.15) and \( H^i_{j\,k} = H^i_{k\,j} \) we have

\begin{align}
n(\hbar R^i_{jkl} + \hbar \tilde{R}^i_{jkl}) &= \hbar (X_k P_l - X_l P_k) G_{ij}, \\
n(\hbar P^i_{jkl} + \hbar P^i_{jkl}) &= -\hbar (\hat{\partial}_l P_k) G_{ij}.
\end{align}

(3.16)

Transvecting (3.16) by \( G^{ij} \) respectively, we have

\begin{align}
\hbar R^i_{jkl} G^{ij} &= X_k P_l - X_l P_k, \\
\hbar P^i_{jkl} G^{ij} &= -\hbar \hat{\partial}_l P_k.
\end{align}

(3.17)

This shows that if \( \hbar R^i_{jkl} = \hbar P^i_{jkl} = 0 \), then the \( (G, N) \)-structure in a Rizza manifold is conformally flat. In fact, from \( \hbar P^i_{jkl} = 0 \) and
\[ P_{ijkl}G^{ij} = \partial_i P_k, \text{ we have } P_k = P_k(x), \text{ and thus, from } R_{ijkl}G^{ij} = X_kP_l - X_lP_k, \text{ we have } \]
\[ \frac{\partial P_k}{\partial x^l} = \frac{\partial P_l}{\partial x^k}. \]

Therefore there exists a local function \( \sigma(x) \) on \( U(x) \) such that
\[ \frac{\partial \sigma(x)}{\partial x^k} = \frac{1}{2n} P_k. \]

By this local function \( \sigma(x) \), the local metric \( \overline{G}_{ij} = e^{-2\sigma(x)}G_{ij} \) is flat metric on \( U(x) \). In fact, from the equations (3.3) and (3.7) imply the equation \( \overline{N}_j^i = 0 \), and moreover, from the equations (3.4) and (3.8) imply the equation \( \overline{\Gamma}_j^i_k = 0 \). Hence the metrical condition \( \nabla_kG_{ij} = 0 \) implies \( G_{ij} = G_{ij}(y) \) on \( U(x) \).

The following definition is well-known: If for a any point \( p \) of \( M \) with \( (G, N) \)-structure, there exists a neighborhood \( (U, x') \) containing \( p \) such that \( G_{ij} \) is a locally Minkowski metric on \( U(x) \), the \( (G, N) \)-structure on \( M \) is said to be a conformally flat. Thus we have

**Theorem 3.2.** The \( (G, N) \)-structure on a Rizza manifold \( (M, G, f) \) is conformally flat if and only if
\[ R_{ijkl} = P_{ijkl} = 0 \]
are satisfied.

Next, we put
\[ \Gamma^i_j^k = \frac{1}{2} G^{ir}(X_jG_{rk} + X_kG_{jr} - X_rG_{jk}). \]

From (2.2) and (2.3), we get \( X_kG_{ij} = X_kG_{ij} \) and hence \( \Gamma^i_j^k = \Gamma^i_j^k \).

The triple \( \pi \Gamma = (\pi \Gamma^i_j^k, H^i_j, C^i_j) \) is, of course, a sort of Finsler connection ([9]). With respect to connection \( \pi \Gamma \), (2.6), (2.7) and (2.8) are expressed by
\[ R^i_j^k = X_k H^i_j - j/k, \quad P^i_j^k = \partial_k H^i_j - \Gamma^i_j^k, \]
\[ R^i_j^k = K^i_j^k + C^i_j^k, \quad R^i_j^k = \partial_k \Gamma^i_j^k - \Gamma^i_j^k, \]
(3.19)
4. The flatness of $(G,H)$-structure

In a Rizza manifold with $\Gamma^*$, we shall define the notion of flatness similar to a Finsler space with a generalized Finsler metric $G_{ij}$ and a non-linear connection $N^i_\cdot$ ([8]).

**Definition 4.1** Let $M$ be a Rizza manifold with $\Gamma^*$-connection admitting a $(G,H)$-structure. If, for any point $p$ of $M$, there exists a coordinate neighborhood $(U,x^i)$ containing $p$ such that $\dddot{X}_k G_{ij} = 0$ holds on $U$, then the $(G,H)$-structure is said to be weakly flat.

The definition of the flatness of $(G,H)$-structure is equivalent to the one of the $(G,N)$-structure. However, here we shall find some quantities of the flatness in a Rizza manifold with $\Gamma^*$-connection.

In a Rizza manifold $M$, if a $(G,H)$-structure is weakly flat, then from Definition 4.1, $M$ is covered by a system of local coordinate neighborhoods $\{(U,x^i)\}$ such that $\dddot{X}_k G_{ij} = 0$ holds good in each $U$. Therefore we have $\dddot{\Gamma}^i_{jk} = 0$ in each $U$ from (3.18). By virtue of (3.19), we get $\dddot{K}^i_{jk} = 0.$ And from (3.19)4, we have

\[(4.1)\quad \dddot{P}_{hijk} = -\dddot{Q}_{hijk},\]

where $\dddot{P}_{hijk} = G_{ir} \dddot{P}_{h^r_{jk}}, \quad \dddot{Q}_{hijk} = G_{ir} \dddot{Q}_{h^r_{jk}}$. Applying the Ricci identities to $\dddot{\nabla}_k G_{ij} = 0$ and $\dddot{\nabla}_k G_{ij} = 0$, we have

\[(4.2)\quad \dddot{R}_{hijk} = -\dddot{R}_{ihjk}, \quad \dddot{P}_{hijk} = -\dddot{P}_{ihjk},\]

where $\dddot{R}_{hijk} = G_{ir} \dddot{R}_{h^r_{jk}}$. Hence, from (4.1) and (4.2), we have

\[(4.3)\quad \dddot{Q}_{ihjk} + \dddot{Q}_{hijk} = 0.\]

Conversely, we suppose that $\dddot{K}^i_{jk} = 0$ and (4.3) holds good. By the second Bianchi identity for the connection $\Gamma^*$, we have

\[\dddot{\nabla}_j C_{khi} - \dddot{\nabla}_k C_{hji} + C_{jhr} P^r_{\cdot \cdot ki} - C_{kh^r} \dddot{P}^r_{\cdot \cdot jk} - \dddot{P}_{j^h_{ki}} + \dddot{P}_{kh^r_{ji}} = 0,\]
that is,

\[ Q_{khji} - Q_{jhki} + P_{jhki} + P_{khji} = 0. \]

Applying the Christoffel process with respect to \( k, h \) and \( j \) to (4.4) and using (4.3), we get

\[ 2P_{jhki} = Q_{khji} + Q_{hkji} + Q_{jkhk} - Q_{jkhk} - Q_{kjhi} - Q_{jkhk}. \]

From (4.3), the above equation is reduced to \( P_{jhki} = -Q_{jkhk} \). Therefore, from (3.19) and (4.3), we have \( \dot{\Gamma} \Gamma^i_k = 0 \). Thus \( \Gamma^i_j \) is a function of the position only on \( M \).

On the other hand, \( \dot{K}_{hjk} = 0 \) tells us that \( M \) is covered by a system of local coordinate neighborhoods such that \( \dot{\Gamma} \Gamma^i_k = 0 \) hold on each \( U \). Hence \( \nabla_k G_{ij} = 0 \) leads us to \( X_k G_{ij} = 0 \) on \( U \). Therefore the given \((G, H)\)-structure is weakly flat. Thus we have

**Theorem 4.1.** A \((G, H)\)-structure in a Rizza manifold with \( \dot{\Gamma} \)-connection is weakly flat if and only if

\[ K_{hjk} = 0, \quad Q_{ihjk} + Q_{hijk} = 0 \]

are satisfied.

**Definition 4.2.** In a Rizza manifold \( M \), with \( \dot{\Gamma} \)-connection if for every point \( p \) of \( M \), there exists a coordinate neighborhood \((U, x^i)\) containing \( p \) such that \( \partial_k G_{ij} = 0 \) and \( H^r_k \partial_r G_{ij} = 0 \) hold on \( U \), then the \((G, H)\)-structure is said to be flat.

**Theorem 4.2.** A \((G, H)\)-structure in Rizza manifold is flat if and only if

\[ K_{hjk} = 0, \quad Q_{ihjk} + Q_{hijk} = 0, \quad P_{k0}^r (C_{ijr} + C_{jir}) = 0 \]

are satisfied, where \( P_{k0}^r = P_{kji} x^i, \quad C_{ijr} = G_{jim} C_{r}^{m} \).

**Proof.** Let a \((G, H)\)-structure be flat. By Definition 4.2, the \((G, H)\)-structure is weakly flat. By virtue of Theorem 4.1, the former two equation of (4.6) is satisfied. The Rizza manifold \( M \) is covered by a system
of local coordinate neighborhoods \(\{(U, x^i)\}\) such that \(\partial_k G_{ij} = 0\) and \(H^r \hat{\partial}_r G_{ij} = 0\) in each \(U\). On the other hand, from \(\nabla_k G_{ij} = 0\) we have that

\[
(4.7) \quad \hat{\partial}_m G_{jk} = C_{mjk} + C_{mkj}.
\]

From (3.19)_2, we have \(\hat{P}^i_{jr} y^r = H^i_j - \hat{\Gamma}^i_{jr} y^r\). Since \(\hat{\Gamma}^i_{jr} = 0\) in \(U\), we have \(\hat{P}^i_{jr} y^r = H^i_j\). Therefore, from the definition of the flatness and (4.7), we get \((C_{mjk} + C_{mkj})\hat{P}^m_{ir} y^r = 0\).

Conversely, we suppose that (4.6) holds good. By virtue of Theorem 4.1, we see that the \((G, H)\)-structure is weakly flat. Hence, with respect to the assigned coordinate neighborhood \(U\) of the weak flatness, \(X_k G_{ij} = 0\), from which \(\hat{\Gamma}^i_{jk} = 0\). Thus, from (3.19)_2 \(\hat{P}^i_{jr} y^r = H^i_j\) holds in each \(U\). Therefore, from this equation and (4.7) and third equation of (4.6), we have \(H^m_k \hat{\partial}_m G_{ij} = 0\) in each \(U\). Since \(X_k G_{ij} = 0\) is shown, \(\hat{\partial}_k G_{ij} = 0\) is also true in each \(U\). Hence the given \((G, H)\)-structure is flat. \(\Box\)

5. A Kaehlerian Finsler \((G, H)\)-structure

In a Rizza manifold, as a non-linear connection, we shall adapt a non-linear connection \(H^i_j\) which is given by (3.7). In this case, a \((G, H)\)-structure satisfying \(\nabla_k f^i_j = 0\) is said a Kaehlerian Finsler \((G, H)\)-structure, where \(\nabla_k\) denotes the \(h\)-covariant derivative with respect to the \(\hat{F}\)-connection.

**THEOREM 5.1.** If the Kaehlerian Finsler \((G, H)\)-structure is flat, the relations

\[
\hat{R}^i_{jk} = \hat{P}^i_{jk} = 0, \quad \hat{P}^i_{h,jk} = \hat{Q}^i_{h,jk} = 0, \quad \hat{R}^i_{h,jk} = \hat{K}^i_{h,jk} = 0
\]

hold true.

**PROOF.** Since the given \((G, H)\)-structure is flat, from Definition 4.2, we see that the manifold is covered by a system of local coordinate neighborhoods \(\{(U, x^i)\}\) such that, in each \((U, x^i)\), \(G_{ij} = G_{ij}(y)\) and \(H^r \partial_r G_{ij} = 0\). Therefore \(X_k G_{ij} = 0\). Thus we get \(\hat{\Gamma}^i_{jk} = 0\). By
assumption, in the above coordinate neighborhood $U$, $\hat{\nabla}_k f^i_j = 0$ is also true. Hence it follows $\partial_k f^i_j = 0$, from which $S^i_j = (\partial_k f^i_{\ell}) f^\ell_j = 0$. Therefore the generalized Chern's non-linear connection $N^i_j$ given by (2.4) vanishes in $U$. From $N^i_j = 0$ and the latter of (2.6), we have $P_i = 0$. Therefore $\bar{H}^i_j = 0$ from (3.7). Hence, from $\bar{H}^i_j = 0$, $\bar{\Gamma}^i_jk = 0$ and $\bar{\nabla}_h C^i_jk = 0$, the proof is evidently completed. 

**Theorem 5.2.** Let $M$ be a $2n$-dimensional $(n > 2)$ Rizza manifold with the Kaehlerian $(G, H)$-structure. If $M$ is locally conformal to a flat $(G, H)$-structure, then $\hat{K}^i_jk=0$ holds true.

**Proof.** Let the $(\bar{G}, \bar{H})$ be flat and locally conformal to a Rizza manifold $M$ with the Kaehlerian Finsler $(G, H)$-structure. Then it follows that $\partial_k \bar{G}^i_j = 0$ and $\bar{H}^m_k \hat{\partial}_m \bar{G}^i_j = 0$ locally, and naturally $\bar{\nabla}_k \bar{G}^i_j = 0$ from which $\bar{\Gamma}^i_jk = 0$. Also $\partial_k (e^{2\sigma} G^i_j) = 0$ leads to

$$\partial_k \bar{G}^i_j = 2\sigma_k G^i_j$$

and

$$\partial_k \bar{G}^{ij} = -2\sigma_k G^{ij}.$$  

From (3.7) and $\bar{H}^m_k = H^m_k$, we have

$$H^m_k \hat{\partial}_m G^i_j = 0.$$  

From this equation and

$$0 = H^m_k \hat{\partial}_m (G^{ir} G_{rj}) = H^m_k \{ \hat{\partial}_m (G^{ir}) G_{rj} + G^{ir} \hat{\partial}_m G_{rj} \},$$

we have

$$H^m_k \hat{\partial}_m G^{ij} = 0.$$  

On the other hand, by the Rizzi identity for $\hat{\nabla}_k f^i_j = 0$, we have

$$0 = \hat{\nabla}_k \hat{\nabla}_h f^i_j - \hat{\nabla}_h \hat{\nabla}_k f^i_j = f^m_j R^i_{mhk} + f^m_i R^r_{jhm} - C^m_{i\ell} f^m_j R^r_{h\ell} + C^m_{ir} f^m_j R^r_{hk}$$

and

$$= f^m_j (\hat{R}^i_{mhk} - C^m_{i\ell} \hat{R}^r_{h\ell}) - f^m_i (\hat{R}^r_{mjh} - C^m_{ir} \hat{R}^r_{hk})$$

$$= f^m_j \hat{\kappa}^i_{mjh} - f^m_i \hat{\kappa}^r_{mjh}.$$
from (3.19), that is

\begin{equation}
(5.5) \quad f^i_m K^m_{jk} = f^m_j K^i_{hk}.
\end{equation}

Since \( \Gamma^i_{jk} = 0 \), (3.4) shows us that

\begin{equation}
(5.6) \quad \dot{\Gamma}^i_{jk} = -\delta^i_j \sigma_k - \delta^i_k \sigma_j + G_{jk} G^{im} \sigma_m.
\end{equation}

Using (5.1), (5.2), (5.3), (5.4) and (5.6), we get

\begin{equation}
\dot{K}^i_{jk} = \delta^i_k ( \partial_j \sigma_h + \sigma_j \sigma_h - G_{hj} \sigma^m \sigma_m ) - \delta^i_j ( \partial_k \sigma_h + \sigma_k \sigma_h - G_{hk} \sigma^m \sigma_m ) \\
+ G_{hj} G^{im} ( \partial_k \sigma_m + \sigma_k \sigma_m ) - G_{hk} G^{im} ( \partial_j \sigma_m + \sigma_j \sigma_m ).
\end{equation}

Putting \( \sigma_{jh} = \partial_j \sigma_h + \sigma_j \sigma_h - G_{hj} \sigma^m \sigma_m / 2 \), we have \( \sigma_{jh} = \sigma_{hj} \). Therefore we find

\begin{equation}
(5.7) \quad \dot{K}^i_{jk} = \delta^i_k \sigma_{jh} - \delta^i_j \sigma_{kh} + G_{hj} G^{im} \sigma_{km} - G_{hk} G^{im} \sigma_{jm}.
\end{equation}

Substituting (5.7) in (5.5), we have

\begin{equation}
(5.8) \quad f^i_m ( \delta^m_k \sigma_{jh} - \delta^m_j \sigma_{kh} + G_{hj} \sigma^m_k - G_{jk} \sigma^m_h ) \\
= f^m_j ( \delta^i_k \sigma_{hm} - \delta^i_h \sigma_{km} + G_{hm} \sigma^i_k - G_{km} \sigma^i_h ),
\end{equation}

where \( \sigma^m_r = G^{im} \sigma_{ir} \). Contracting with \( i \) and \( k \) in (5.8), we have

\begin{equation}
(5.9) \quad -f^m_h \sigma_{mj} + G_{jh} f^r_m \sigma^m_r - f^r_m G_{jr} \sigma^m_h \\
= 2n f^m_j \sigma_{hm} - f^m_j \sigma_{hm} + f_{mj} \sigma^m_m - f^m_j \sigma_{mh}.
\end{equation}

Transvecting (5.9) by \( G^{jh} \), we get \( f^r_m \sigma^m_r = 0 \) and \( f^r_m \sigma_{mr} = 0 \), where \( f^r_m = G^{im} f^i_r \). From \( f_{ij} = -f_{ji} \) and \( f^r_m G_{jr} \sigma^m_h = f_{jh} \sigma^m_h \), (5.9) leads us to

\begin{equation}
- f^m_h \sigma_{mj} = (2n - 3) \sigma_{hm} f^m_j + \sigma^m_m f_{mj}.
\end{equation}

Transvecting this equation by \( f^h_k \), we obtain

\begin{equation}
(5.10) \quad \sigma_{kj} = (2n - 3) \sigma_{pq} f^p_q f^q_j + \sigma^m_m G_{jk}.
\end{equation}

Moreover, transvecting (5.10) by \( G^{kj} \), we get \( 4(n - 1) \sigma^m_m = 0 \). Since we assume \( n > 2 \), we have \( \sigma^m_m = 0 \). Therefore (5.10) is written as \( \sigma_{kj} = (2n - 3)^2 \sigma_{kj} \). Thus we have \( \sigma_{kj} = 0 \). Consequently from (5.7) we obtain \( \dot{K}^i_{jk} = 0 \).

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References


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