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IDENTIFICATION PROBLEMS OF DAMPED SINE-GORDON EQUATIONS WITH CONSTANT PARAMETERS

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ABSTRACT. We study the problems of identification for the damped sine-Gordon equations with constant parameters. That is, we establish the existence and necessary conditions for the optimal constant parameters based on the fundamental optimal control theory and the transposition method studied in Lions and Magenes [5].

1. Introduction

The dynamics of a series of small-area Josephson junctions connected by superconducting strips is described by a partial differential equation

$$\frac{\Phi}{2\pi L}\frac{\partial^2\theta}{\partial x^2} - C\frac{\Phi}{2\pi}\frac{\partial^2\theta}{\partial t^2} - G\frac{\Phi}{2\pi}\frac{\partial\theta}{\partial t} = I_c \sin\theta - I_b, \tag{1.1}$$

where C, G, L and Φ are capacitance, conductance, inductance and flux quantum, respectively, and I_c is the maximum supercurrent in each Josephson junction, I_b is a current supplied by an external source and $\theta = \theta(t, x)$ is the phase difference in the Josephson junction at time t and position x. Multiplying $-2\pi/(C\Phi)$ on the both sides and putting $y = \theta$ in (1.1) we have

$$\frac{\partial^2 y}{\partial t^2} + \alpha\frac{\partial y}{\partial t} - \beta\Delta y + \gamma \sin y = f, \tag{1.2}$$

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where $\Delta = \partial^2/\partial x^2$ and $\alpha = G/C, \beta = 1/(CL), \gamma = 2\pi I_c/(C\Phi)$, $f = \gamma I_b/I_c$. We call the equation (1.2) the (one dimensional) damped sine-Gordon equation by a current source. In particular, if all physical parameters are measured in the unit length $\lambda = \Phi/(2\pi L I_c)$ and in the unit time $\tau = C\Phi/(2\pi I_c)$, then (1.2) is simplified by

$$\frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial y}{\partial t} - \Delta y + \sin y = f, \tag{1.3}$$

where $\alpha = G/C$ and $f = I_b/I_c$.

Many scientists have had great interests in α and f appeared in (1.3). For example, in Nakajima and Onodera [8], they studied the property of parameters by numerical simulations based on the finite difference method and in Bishop, *et al.* [2] and Levi [3], they verified numerically that the equation (1.3) leads by special choices of parameters and forcing functions to chaotic behaviours. In Temam [10], he has extensively studied the stability of (1.3). In Ha and Nakagiri [6], they studied the optimal control problems of the controlled system governed by (1.3), and in Park, *et al.* [9], they studied the problems of identification for (1.3).

In this paper we are going to study the problems of identification of a general equation described by

$$\frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial y}{\partial t} - \beta \Delta y + \gamma \sin y = \delta f \tag{1.4}$$

in R^n $(n \leq 3)$, where α, β, γ and δ are constants and f is a prescribed source function. In our identification problems all parameters α, β, γ and δ are assumed to be unknown. We remark that in [9] they didn't study the identification problems of (1.4) for the case where the parameter β is unknown. For solving the identification problems of (1.4) we will utilize the method which is used by Lions [4] for solving the optimal control problems. Whenever this method is utilized, we have to prove the Gâteaux differentiability of the solution map from the set of parameters to the solution space of (1.4). When β is unknown, we can not carry over the same analysis as we do for the case of β being known, because we have to solve the equations having forcing terms defined by the Laplacian operator Δ. The equations can not be solved by the standard variational method. In order to overcome this difficulty, we will utilize the method of transposition studied in Lions and Magenes [5]. We note that for the optimal control problems we often use the transposition method to describe the adjoint state equations in the weak sense, but for our identification problems we use the method to prove the Gâteaux differentiability of the solution map and to characterize the the Gâteaux

differential of the solution map. In Ahmed [1], he only used this method of proving the existence of the optimal parameters for linear second order equations with the same spatial differential order. For example, for fitting (1.4) with one equation studied in [1], it is enough to replace the parameter α and the nonlinear term $\gamma \sin y$ to the operator $-\alpha\Delta$ and 0, respectively, which is an easier case than our case.

In this paper we will establish new results on the existence and necessary conditions of the identification problems for the damped sin-Gordon equations as a class of nonlinear hyperbolic equations of physical importance.

This paper is composed of three sections. In studying the identification problems for (1.4) we are required to give the fundamental results on the weak solutions of (1.4). Hence in section 2, we explain the existence, uniqueness and regularity of solutions for damped sine-Gordon equations. In section 3 we state and solve the constant parameter identification problems for damped sine-Gordon equations.

2. Preliminaries

Let Ω be an open bounded set of the n ($n \leq 3$) dimensional Euclidean space R^n with a piecewise smooth boundary $\Gamma = \partial\Omega$. Let $Q = (0,T) \times \Omega, \Sigma = (0,T) \times \Gamma, R = (-\infty, \infty)$ and $R^+ = [0, \infty)$. We consider an initial boundary value problem for damped sine-Gordon equations described by

$$\begin{aligned} &\frac{\partial^2 y}{\partial t^2}(t,x) + \alpha \frac{\partial y}{\partial t}(t,x) - \beta \Delta y(t,x) + \gamma \sin y(t,x) = \delta f(t,x) \quad \text{in} \quad Q, \\ &\qquad\qquad y(t,x) = 0 \quad \text{on} \quad \Sigma, \\ &y(0,x) = y_0(x) \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in} \quad \Omega, \end{aligned} \tag{2.1}$$

where $\beta > 0, \alpha, \gamma, \delta \in R$, Δ is the Laplacian, f is a given function and y_0, y_1 are initial conditions.

To set (2.1) into the evolution equations or variational forms, we introduce two Hilbert spaces $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ by taking account of the Dirichlet boundary condition. We endow these spaces with inner products and norms as follows:

$$(\psi, \phi) = \int_\Omega \psi(x)\phi(x)dx, \quad |\psi| = (\psi,\psi)^{1/2} \quad \forall \phi, \psi \in L^2(\Omega);$$

$$((\psi, \phi)) = \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}\psi(x) \frac{\partial}{\partial x_i}\phi(x)dx, \|\psi\| = ((\psi,\psi))^{1/2} \quad \forall \phi, \psi \in H_0^1(\Omega).$$

Then a pair (V, H) is a Gelfand triple space with a notation, $V \hookrightarrow H \equiv H' \hookrightarrow V'$ and $V' = H^{-1}(\Omega)$, which means that embeddings $V \subset H$ and $H \subset V'$ are continuous, dense and compact. By c_1 and c_2 we denote the embedding constants of theirs, respectively. By $\langle \cdot, \cdot \rangle$ we denote the dual pairing between V' and V. Let us define a bilinear form on V by

$$a(\phi, \varphi) = \int_\Omega \nabla\phi(x) \cdot \nabla\varphi(x) dx = (\!(\phi, \varphi)\!) \quad \forall \phi, \varphi \in H_0^1(\Omega).$$

This bilinear form $a(\cdot, \cdot)$ is symmetric and bounded on $V \times V$. It is also coercive on $V \times V$, i.e.,

$$a(\phi, \phi) \geq \|\phi\|^2 \quad \forall \phi \in H_0^1(\Omega).$$

By the boundedness of $a(\cdot, \cdot)$ we can define the bounded linear operator $A \in \mathcal{L}(V, V')$, the space of bounded linear operators of V into V', by the relation $a(\phi, \psi) = \langle A\phi, \psi \rangle$ for $\phi, \psi \in V$. The operator A is an isomorphism from V onto V' and has a dense domain $D(A)$ in H, but it is not bounded in H. We define the functions $\sin y$ and $\cos y$ from H into H by

$$(\sin y)(x) = \sin y(x) \quad \text{and} \quad (\cos y)(x) = \cos y(x) \quad \text{for all} \quad x \in \Omega.$$

Using the operator A and nonlinear function $\sin y$, we convert (2.1) to a Cauchy problem in H:

$$\begin{aligned} &\frac{d^2y(t)}{dt^2} + \alpha \frac{dy(t)}{dt} + \beta A y(t) + \gamma \sin y(t) = \delta f(t), \quad t \in (0, T), \\ &y(0) = y_0 \in V, \quad \frac{dy}{dt}(0) = y_1 \in H. \end{aligned} \tag{2.2}$$

Let us define the solution Hilbert space $W(0, T)$ by

$$W(0, T) = \{g | g \in L^2(0, T; V), g' \in L^2(0, T; H), g'' \in L^2(0, T; V')\}$$

with inner product

$$(f, g)_{W(0,T)} = \int_0^T \left((\!(f(t), g(t))\!) + (f'(t), g'(t)) + (f''(t), g''(t))_{V'} \right) dt,$$

where $(\cdot, \cdot)_{V'}$ is the inner product of V'. We denote by $\mathcal{D}'(0, T)$ the space of distributions on $(0, T)$.

Let us define further the definition of weak solutions for (2.2).

DEFINTION 2.1. A function y is said to be a weak solution of (2.2) if $y \in W(0,T)$ and y satisfies

$$\langle y''(\cdot), \phi\rangle + (\alpha y'(\cdot), \phi) + (\!(\beta y(\cdot), \phi)\!) + (\gamma \sin y(\cdot), \phi) = (\delta f(\cdot), \phi)$$
$$\text{for all } \phi \in V \text{ in the sense of } \mathcal{D}'(0,T),$$
$$y(0) = y_0, \quad y'(0) = y_1.$$

For the existence and uniqueness of weak solutions for (2.2), we can prove the following theorem. For the proof, see Ha and Nakagiri [7].

THEOREM 2.2. *Let $\alpha, \gamma, \delta \in R$, $\beta > 0$ and f, y_0, y_1 be given satisfying*

$$f \in L^2(0,T;H), \quad y_0 \in V, \quad y_1 \in H. \tag{2.3}$$

Then the equation (2.2) has a unique weak solution y in $W(0,T)$ and y has the regularity

$$y \in C([0,T];V), \quad y' \in C([0,T];H). \tag{2.4}$$

Furthermore y satisfies

$$|y'(t)|^2 + \|y(t)\|^2 \leq c(\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,T;H)}), \ t \in [0,T], \tag{2.5}$$

where c is a constant depending on α, β, γ and δ.

REMARK 2.3. If we replace the nonlinear term $\sin y(t)$ with the term $B(t)y(t)$ in (2.2), then Theorem 2.2 is true, provided that the multiplier operator $B(\cdot)$ belongs to $L^\infty(0,T;\mathcal{L}(H,H))$.

3. Problems of identification

In this section we study the problems of identification for damped sine-Gordon equations described by

$$\begin{aligned} &y'' + \alpha y' + (\beta_0 + \beta^2)Ay + \gamma \sin y = \delta f \ \text{ in } \ (0,T), \\ &y(0) = y_0, \quad y'(0) = y_1, \end{aligned} \tag{3.1}$$

where $\beta_0 > 0$ is fixed. In (3.1) we replace the diffusion parameter β to $\beta_0 + \beta^2$ to obtain the linear space of parameters α, β, γ and δ. Hence the diffusion term in (3.1) never disappear and is uniformly coercive for all $\beta \in R$.

For setting the problems of identification we assume that the parameters α, β, γ and δ appeared in (3.1) are unknown and we take $\mathcal{P} = R^4$ as the set of parameters $(\alpha, \beta, \gamma, \delta)$. The Euclidean norm of $\mathcal{P}$ is denoted by $|\cdot|$. For simplicity we set $q = (\alpha, \beta, \gamma, \delta) \in \mathcal{P}$. Since for each $q \in \mathcal{P}$ there exists a unique weak solution $y = y(q) \in W(0,T)$ of (3.1), we can uniquely define the solution map $q \to y(q)$ of $\mathcal{P}$ into $W(0,T)$.

Let $\mathcal{K}$ be a Hilbert space of observations and let $\|\cdot\|_{\mathcal{K}}$ be its norm. The observation of $y(q)$ is assumed to be given by

$$z(q) = Cy(q) \in \mathcal{K}, \tag{3.2}$$

where C is a bounded linear observation operator of $W(0,T)$ into $\mathcal{K}$.

The cost functional attached to (3.1) with (3.2) is given by

$$J(q) = \|Cy(q) - z_d\|_{\mathcal{K}}^2 \quad \text{for} \quad q \in \mathcal{P}, \tag{3.3}$$

where $z_d \in \mathcal{K}$ is a desired value of $y(q)$.

Assume that an admissible subset $\mathcal{P}_{ad}$ of $\mathcal{P}$ is convex and closed. The identification problems subject to (3.3) and (3.1) are usually divided into the existence and characterization problems. That is,

(i) The problem of finding an element $q^* \in \mathcal{P}_{ad}$ such that

$$\inf_{q \in \mathcal{P}_{ad}} J(q) = J(q^*); \tag{3.4}$$

(ii) The problem of giving a characterization to such the q^*.

As usual we call q^* the optimal parameter and $y(q^*)$ the optimal state. It is well-known that there are no general method of solving (i), and in many cases some stronger conditions on the data in (3.1) are required to solve (i). Here we assume that $\mathcal{P}_{ad}$ is a compact subset of $\mathcal{P}$ and we solve (i). It is also well-known that we can solve (ii) by deriving necessary conditions on q^*. If $J(q)$ is Gâteaux differentiable at q^* in the direction $q - q^*$, then q^* has to satisfy

$$DJ(q^*)(q - q^*) \geq 0 \quad \text{for all} \quad q \in \mathcal{P}_{ad}, \tag{3.5}$$

where $DJ(q^*)$ denotes the Gâteaux derivative of $J(q)$ at $q = q^*$ in the direction $q - q^*$. We analyze the inequality (3.5) by introducing an adjoint state equation for (3.1) and deduce necessary conditions on q^*.

3.1. Existence of optimal parameters

In this subsection we assume that $\mathcal{P}_{ad}$ is a compact subset of $\mathcal{P}$ and we show the existence of q^*. The following theorem is essential to solve the problem (i) and (ii).

THEOREM 3.1. *The map $q \to y(q) : \mathcal{P} \to W(0,T)$ is weakly continuous. That is, $y(q_n) \to y(q)$ weakly in $W(0,T)$ as $q_n \to q$ in R^4.*

Proof. Let us assume $q_n = (\alpha_n, \beta_n, \gamma_n, \delta_n) \to q = (\alpha, \beta, \gamma, \delta)$ in R^4, i.e., $\alpha_n \to \alpha, \beta_n \to \beta, \gamma_n \to \gamma, \delta_n \to \delta$ in R. Let $y_n = y(q_n)$ be the weak solution of

$$\begin{aligned} &y'' + \alpha_n y' + (\beta_n^2 + \beta_0)Ay + \gamma_n \sin y = \delta_n f \ \text{ in } \ (0,T), \\ &y(q_n;0) = y_0, \ \ y'(q_n;0) = y_1. \end{aligned} \tag{3.6}$$

It follows from (2.5) that

$$|y_n'(t)|^2 + \|y_n(t)\|^2 \le c(q_n)(\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,T;H)}) \ \ \forall t \in [0,T],$$

where $c(q_n)$ depends on $\alpha_n, \beta_n, \gamma_n$ and δ_n. Since $\beta_n^2 + \beta_0 \ge \beta_0$ for all n, the sequence $\{c(q_n)\}$ is bounded in R^+. Hence $\{y_n\}$ is bounded in $L^\infty(0,T;V)$ and $\{y_n'\}$ is bounded in $L^\infty(0,T;H)$. Also we can easily verify that $\{y_n''\}$ is bounded in $L^2(0,T;V')$ by applying the boundednesses of $\{y_n\}$, $\{y_n'\}$, $\{Ay_n\}$ in $L^2(0,T;V')$, the boundedness of $\{q_n\}$ in R^4 and the inequality $|\sin y_n| \le |y_n|$ to the first equation in (3.6). Hence we can extract a subsequence of $\{y_n\}$, denoting it by $\{y_n\}$ again, and choose $z \in W(0,T)$ such that

$$\begin{aligned} &y_n \to z \ \text{ weakly in } \ L^2(0,T;V), \\ &y_n' \to z' \ \text{ weakly in } \ L^2(0,T;H), \\ &y_n'' \to z'' \ \text{ weakly in } \ L^2(0,T;V'), \\ &z(0) = y_0, \ z'(0) = y_1. \end{aligned} \tag{3.7}$$

Since the embedding $V \hookrightarrow H$ is compact, by the classical compactness theorem the embedding $L^2(0,T;V) \cap W^{1,2}(0,T;H) \hookrightarrow L^2(0,T;H)$ is compact. Since $\{y_n\} \subset L^2(0,T;V) \cap W^{1,2}(0,T;H)$, we see by the first one in (3.7) that

$$y_n \to z \ \text{ strongly in } \ L^2(0,T;H). \tag{3.8}$$

Since $\sin y$ is continuous on H, we have

$$\sin y_n \to \sin z \ \text{ strongly in } \ L^2(0,T;H). \tag{3.9}$$

Finally we take the limit $n \to \infty$ on the weak form of (3.6) by using (3.7) and (3.9). Then z is a weak solution of

$$\begin{aligned} &z'' + \alpha z' + (\beta^2 + \beta_0)Az + \gamma \sin z = \delta f \ \text{ in } \ (0,T), \\ &z(0) = y_0, \ \ z'(0) = y_1 \end{aligned} \tag{3.10}$$

in the sense of Definition 2.1. Hence by the uniqueness of weak solutions, we have $z = y(q)$. Therefore we show that $y(q_n) \to y(q)$ weakly in

$W(0,T)$ without extracting a subsequence $\{q_n\}$ again by the uniqueness of weak solutions. □

The following theorem follows immediately from Theorem 3.1 and the lower semi-continuity of norms.

THEOREM 3.2. *If $\mathcal{P}_{ad} \subset \mathcal{P} = R^4$ is compact, then there exists at least one optimal parameter $q^* \in \mathcal{P}_{ad}$ for the cost* (3.3).

3.2. Necessary conditions

For proving that $J(q)$ is Gâteaux differentiable at q^* in a space, we have to estimate the quotients $z_\lambda = (y(q_\lambda) - y(q^*))/\lambda$, where $q_\lambda = q^* + \lambda(q - q^*)$, $\lambda \in (0,1]$. We set $y_\lambda = y(q_\lambda)$ and $y^* = y(q^*)$ for simplicity. Generally it is desirable to estimate z_λ in the solution space $W(0,T)$. But since the second order evolution equations for z_λ have the forcing term containing the diffusion operator, it is not easy or impossible to solve the equations by the standard variational manner in [7]. Hence we will restrict ourselves to estimate $(z_\lambda(T), z_\lambda) \in H \times L^2(0,T;H)$ as $\lambda \to 0$ based on the method of transposition in [5].

Let us begin to prove the weak Gâteaux differentiability of the solution map $q \to (y(q;T), y(q))$ of $\mathcal{P}$ into $H \times L^2(0,T;H)$ through the method of transposition and characterize its Gâteaux derivatives. For $\lambda \in [0,1]$ we consider the terminal value problems described by linear damped evolution equations

$$\begin{aligned} &\phi'' - \alpha^*\phi' + (\beta^{*2} + \beta_0)A\phi + B(t,q,\lambda)\phi = \delta^* g \ \text{ in } \ (0,T), \\ &\phi(T) = 0, \ \phi'(T) = \phi_1, \end{aligned} \tag{3.11}$$

where $\phi_1 \in H, g \in L^2(0,T;H), B(t,q,0) = \gamma^* \cos y^*$ and

$$B(t,q,\lambda) = \gamma^* \int_0^1 \cos(\theta y_\lambda + (1-\theta)y^*)\, d\theta \ \text{ for } \ \lambda \in (0,1].$$

Here we note that the functions $B(t,q,\lambda)$, $\lambda \in [0,1]$ are only multipliers.

It can be proved that the problem (3.11) has an unique weak solution $\phi = \phi(\phi_1, g) \in W(0,T)$ if we take $B(t) := B(t,q,\lambda)$ and consider the reversed time flow $t \to T - t$ (cf. Remark 2.3). Further we have the estimate

$$|\phi'(t)|^2 + \|\phi(t)\|^2 \le c(|\phi_1|^2 + \|g\|^2_{L^2(0,T;H)}), \ t \in [0,T], \tag{3.12}$$

where c is a constant independent of λ and q.

We now explain the method of transposition. For fixed q and λ we define X_λ as a space of (ϕ_1, ϕ), where $\phi = \phi(\phi_1, g)$ is the solution of

(3.11) for given $(\phi_1, g) \in H \times L^2(0,T;H)$. We define the inner product on X_λ as

$$(\Phi, \Psi)_{X_\lambda} = (\phi_1, \psi_1) + (g,h)_{L^2(0,T;H)} \quad \text{for } \Phi = (\phi_1, \phi), \Psi = (\psi_1, \psi) \in X_\lambda,$$

where $\Phi = \phi(\phi_1, g)$ and $\psi = \phi(\psi_1, h)$. Then it is easily verified that $(X_\lambda, (\cdot,\cdot)_{X_\lambda})$ is a Hilbert space and the map $(\phi_1, \phi) \to (\phi_1, g)$ of X_λ onto $H \times L^2(0,T;H)$ is an isomorphism. For simplicity of notations, let us define the linear operator $\mathcal{L}_\lambda : X_\lambda \to L^2(0,T;H)$ by

$$\mathcal{L}_\lambda(\phi) = \phi'' - \alpha^*\phi' + (\beta^{*2} + \beta_0)A\phi + B(t,q,\lambda)\phi.$$

By the method of transposition due to Lions and Magenes [5], for a bounded linear function l on X_λ there is a unique solution $(\zeta_1, \zeta) \in H \times L^2(0,T;H)$ such that

$$(3.13)\quad (\zeta_1, \phi_1) + \int_0^T (\zeta(t), \mathcal{L}_\lambda(\phi)(t))dt = l(\phi_1, \phi) \quad \text{for all } (\phi_1, \phi) \in X_\lambda.$$

Note that the solution (ζ_1, ζ) depends on λ.

THEOREM 3.3. *The map $q \to (y(q;T), y(q))$ of $\mathcal{P}$ into $H \times L^2(0,T;H)$ is weakly Gâteaux differentiable. That is, for any fixed $q^* = (\alpha^*, \beta^*, \gamma^*, \delta^*)$ and $q = (\alpha, \beta, \gamma, \delta)$ the weak Gâteaux derivative $(z_1, z) = (Dy(q^*;T)(q-q^*), Dy(q^*)(q-q^*))$ of $(y(q;T), y(q))$ at $q = q^*$ in the direction $q - q^*$ exists in $H \times L^2(0,T;H)$ and it is a unique solution of the integral equation*

$$(3.14)\quad -(z_1, \phi_1) + \int_0^T (z(t), \mathcal{L}_0(\phi)(t))dt = \int_0^T \langle (\alpha^* - \alpha)y^{*'}(t) + 2\beta^*(\beta^* - \beta)Ay^*(t) + (\gamma^* - \gamma)\sin y^*(t) + (\delta - \delta^*)f(t), \phi(t)\rangle dt$$

for all $(\phi_1, \phi) \in X_0$, where $y^* = y(q^*)$ and

$$\mathcal{L}_0(\phi) = \phi'' - \alpha^*\phi' + (\beta^{*2} + \beta_0)A\phi + \gamma^*(\cos y^*)\phi.$$

Proof. For fixed q we set $q_\lambda = q^* + \lambda(q - q^*)$, $\lambda \in (0,1]$. We recall the simplified notations $y_\lambda = y(q_\lambda)$ and $y^* = y(q^*)$, which are weak solutions to (2.2) for given parameters q_λ and q^*, respectively. Then $q_\lambda \in \mathcal{P}$ and $|q_\lambda - q^*| = \lambda|q - q^*| \to 0$ as $\lambda \to 0$. Also by Theorem 3.1 we have

$$(3.15)\qquad y_\lambda \to y^* \quad \text{weakly in } W(0,T) \text{ as } \lambda \to 0,$$

which also yields

$$(3.16)\qquad y_\lambda \to y^* \quad \text{strongly in } L^2(0,T;H) \text{ as } \lambda \to 0.$$

Since y_λ is a weak solution and the boundedness of $\{q_\lambda\}$, by (2.5) we have

$$c_3 := \sup\{\|y_\lambda(t)\|^2 + |y'_\lambda(t)|^2 : \ (t,\lambda) \in [0,T] \times [0,1]\} < \infty. \tag{3.17}$$

For $\lambda \in (0,1]$ the quotient $z_\lambda = (y_\lambda - y^*)/\lambda$ satisfies

$$\begin{aligned} &z''_\lambda + \alpha^* z'_\lambda + (\beta^{*2} + \beta_0)Az_\lambda + \gamma^* \frac{\sin y_\lambda - \sin y^*}{\lambda} \\ &= (\alpha^* - \alpha)y'_\lambda + [2\beta^*(\beta^* - \beta) - \lambda(\beta - \beta^*)^2]Ay_\lambda \\ &\quad +(\gamma^* - \gamma)\sin y_\lambda + (\delta - \delta^*)f \ \text{ in } \ (0,T), \\ &z_\lambda(0) = z'_\lambda(0) = 0, \end{aligned}$$

or by the mean value theorem equivalently

$$\begin{aligned} &z''_\lambda + \alpha^* z'_\lambda + (\beta^{*2} + \beta_0)Az_\lambda + \gamma^* \Big(\int_0^1 \cos(\theta y_\lambda + (1-\theta)y^*)\, d\theta \Big) z_\lambda \\ &= (\alpha^* - \alpha)y'_\lambda + [2\beta^*(\beta^* - \beta) - \lambda(\beta - \beta^*)^2]Ay_\lambda \\ &\quad +(\gamma^* - \gamma)\sin y_\lambda + (\delta - \delta^*)f \ \text{ in } \ (0,T), \\ &z_\lambda(0) = z'_\lambda(0) = 0. \end{aligned} \tag{3.18}$$

Multiplying by $(\phi_1, \phi) \in X_\lambda$ to the both sides of (3.18) and integrating it over $[0,T]$ we have

$$-(z_\lambda(T), \phi_1) + \int_0^T (z_\lambda(t), \mathcal{L}_\lambda(\phi)(t))\, dt = \int_0^T \langle f_\lambda(t), \phi(t)\rangle\, dt, \tag{3.19}$$

where

$$\begin{aligned} f_\lambda(t) \ = \ &(\alpha^* - \alpha)y'_\lambda(t) + [2\beta^*(\beta^* - \beta) - \lambda(\beta - \beta^*)^2]Ay_\lambda(t) \\ &+(\gamma^* - \gamma)\sin y_\lambda(t) + (\delta - \delta^*)f(t), \end{aligned}$$

which is estimated by

$$\begin{aligned} \|f_\lambda\|_{V'} \ \le \ &|\alpha^* - \alpha| \ \|y'_\lambda(t)\|_{V'} + [2|\beta^*| \ |\beta^* - \beta| + \lambda(\beta - \beta^*)^2]\|y_\lambda(t)\| \\ &+|\gamma^* - \gamma| \ |y_\lambda(t)| + |\delta - \delta^*| \ \|f(t)\|_{V'}. \end{aligned}$$

Hence $\{f_\lambda\}_\lambda$ is uniformly bounded in $L^2(0,T;V')$ and let c_4 be the least upper bound of it. Let us take $l(\phi_1,\phi) = \int_0^T \langle f_\lambda(t), \phi(t)\rangle\, dt$ in (3.13). Then l is a bounded linear functional on X_λ. Indeed,

$$\begin{aligned} |l(\phi_1,\phi)| \ &\le \ \int_0^T |\langle f_\lambda(t), \phi(t)\rangle|\, dt \le \int_0^T \|f_\lambda(t)\|_{V'}\|\phi(t)\|\, dt \\ &\le \ \|f_\lambda\|_{L^2(0,T;V')}\sqrt{c}T\left(|\phi_1|^2 + \|g\|^2_{L^2(0,T;H)}\right)^{1/2} \\ &\le \ \sqrt{c}c_4 T\|(\phi_1,\phi)\|_{X_\lambda}. \end{aligned} \tag{3.20}$$

Hence (3.19) always admits the unique weak solution in the sense of (3.13) and $(z_\lambda(T), z_\lambda)$ becomes the solution of (3.19). Since $z_\lambda(T) \in H$ and $z_\lambda \in L^2(0,T;H)$, we can take $\phi_1 = -z_\lambda(T)$ and ϕ such that $\mathcal{L}_\lambda(\phi) = z_\lambda$ in (3.19). Then we have

$$|z_\lambda(T)|^2 + \int_0^T |z_\lambda(t)|^2 \, dt = \int_0^T \langle f_\lambda(t), \phi(t)\rangle \, dt. \tag{3.21}$$

From (3.20) the right hand side of (3.21) is estimated by

$$\int_0^T |\langle f_\lambda(t), \phi(t)\rangle| \, dt \le \sqrt{c} c_4 T \left(|z_\lambda(T)|^2 + \|z_\lambda\|^2_{L^2(0,T;H)}\right)^{1/2}. \tag{3.22}$$

By (3.21) and (3.22) we have

$$|z_\lambda(T)|^2 + \|z_\lambda\|^2_{L^2(0,T;H)} \le \sqrt{c} c_4 T \Big(|z_\lambda(T)|^2 + \|z_\lambda\|^2_{L^2(0,T;H)}\Big)^{1/2},$$

which implies

$$|z_\lambda(T)|^2 + \|z_\lambda\|^2_{L^2(0,T;H)} \le c c_4^2 T^2 < \infty.$$

This implies that $\{z_\lambda(T)\}$ is bounded in H and $\{z_\lambda\}$ is bounded in $L^2(0,T;H)$. Hence we can extract subsequences, denoting them by $\{z_\lambda(T)\}$ and $\{z_\lambda\}$ again, and find $z_1 \in H$ and $z \in L^2(0,T;H)$ such that

$$\begin{aligned} &z_\lambda(T) \to z_1 \ \text{ weakly in } \ H, \\ &z_\lambda \to z \ \text{ weakly in } \ L^2(0,T;H). \end{aligned} \tag{3.23}$$

It is easily proved by (3.15) and (3.16) that

$$\lim_{\lambda \to 0} f_\lambda = f_0 \ \text{ weakly in } \ L^2(0,T;V'), \tag{3.24}$$

where

$$f_0 = (\alpha^* - \alpha) y^{*'} + 2\beta^*(\beta^* - \beta) A y^* + (\gamma^* - \gamma) \sin y^* + (\delta - \delta^*) f.$$

Multiplying (3.18) by $(\phi_1, \phi) \in X_0$ and integrating it over $[0,T]$ we have

$$-(z_\lambda(T), \phi_1) + \int_0^T (z_\lambda(t), \mathcal{L}_0(\phi)(t)) \, dt + \epsilon(\lambda) = \int_0^T \langle f_\lambda(t), \phi(t)\rangle \, dt, \tag{3.25}$$

where

$$\epsilon(\lambda) = \int_0^T (z_\lambda(t), (B(t,q,\lambda) - B(t,q,0))\phi(t)) \, dt.$$

We shall prove that $\epsilon(\lambda) \to 0$ as $\lambda \to 0$ for each $(\phi_1, \phi) \in X_0$. Since $(z_\lambda(t), (B(t,q,\lambda) - B(t,q,0))\phi(t))$ is represented by

$$\gamma^* \int_\Omega \int_0^1 [\cos(\theta y_\lambda(t,x) + (1-\theta)y^*(t,x)) - \cos y^*(t,x)] z_\lambda(t,x) d\theta \phi(t,x) dx,$$

by using the Lipschitz continuity of the cosine function we can estimate the integrand of $\epsilon(\lambda)$ as follows:

$$\begin{aligned} & |(z_\lambda(t), (B(t,q,\lambda) - B(t,q,0))\phi(t))| \\ \leq\ & |\gamma^*|\ \|z_\lambda(t)\|_{L^4(\Omega)} \|\phi(t)\|_{L^4(\Omega)} |y_\lambda(t) - y^*(t)| \\ \leq\ & c_5^2 |\gamma^*|\ |z_\lambda(t)|\ |\phi(t)|\ |y_\lambda(t) - y^*(t)|, \end{aligned} \tag{3.26}$$

where c_5 is a constant such that $\|\psi\|_{L^4(\Omega)} \leq c_5 |\psi|$ for $\psi \in V$, which is possible when $n \leq 3$. Finally by integrating (3.26) over $[0,T]$ we have from (3.16)

$$\begin{aligned} |\epsilon_\lambda(t)| \ \leq\ & c_5^2 |\gamma^*| \left(\max_{t \in [0,T]} |\phi(t)| \right) \|z_\lambda\|_{L^2(0,T;H)} \|y_\lambda - y^*\|_{L^2(0,T;H)} \\ & \to 0 \ \text{ as } \ \lambda \to 0. \end{aligned}$$

Here we used the boundedness of $\{z_\lambda\}$ in $L^2(0,T;H)$ and $\phi \in C([0,T];H)$.

Hence by (3.23) and (3.24), we take $\lambda \to 0$ in the both side hands of (3.25) to have

$$(z_1, \phi_1) + \int_0^T (z(t), \mathcal{L}_0(\phi)(t))\ dt = \int_0^T \langle f_0(t), \phi(t) \rangle\ dt \quad \forall (\phi_1, \phi) \in X_0. \tag{3.27}$$

Now we set $l(\phi_1, \phi) = \int_0^T \langle f_0(t), \phi(t) \rangle\ dt$ in (3.27). Then it is clear that l is a bounded linear functional on X_0. Hence the equation (3.27) has an unique solution $(z_1, z) \in H \times L^2(0,T;H)$ in the sense of (3.13). Therefore the weak limit (z_1, z) becomes an unique solution of (3.27), and by the uniqueness of solutions (z_1, z) is shown to be the weak Gâteaux derivative $(Dy(q^*;T)(q-q^*), Dy(q^*)(q-q^*))$.

Since $(y(q;T), y(q))$ is weakly Gâteaux differentiable at any $q = q^*$ in $H \times L^2(0,T;H)$, we will deduce the necessary conditions on q^* for this restricted class of distributive and terminal value observations. That is, we consider the special cost functional $J(q)$ given by

$$J(q) = \kappa_1 |y(q;T) - z_d^1|^2 + \kappa_2 \|y(q) - z_d^2\|^2_{L^2(0,T;H)}, \tag{3.28}$$

where $z_d^1 \in H$, $z_d^2 \in L^2(0,T;H)$ and $\kappa_i \geq 0, i = 1,2$ such that $\kappa_1 + \kappa_2 > 0$. Let $\mathcal{P}_{ad}$ be a closed and convex subset of $\mathcal{P}$. In what follows we suppose

that q^* is the optimal parameter for the cost $J(q)$ in (3.28) on $\mathcal{P}_{ad}$, i.e., q^* satisfies (3.4). Then the necessary optimality condition (3.5) is rewritten as

(3.29)
$$(\kappa_1(y(q^*;T)-z_d^1),z_1)+\int_0^T(\kappa_2(y(q^*)-z_d^2)(t),z(t))dt\geq 0\quad\forall q\in\mathcal{P}_{ad},$$

where (z_1,z) is the solution of the integral equation (3.14).

Let us introduce the adjoint state p defined by the weak solution of the adjoint system

(3.30)
$$\begin{cases} p''-\alpha^*p'+(\beta^{*2}+\beta_0)Ap+\gamma^*(\cos y^*)p=\kappa_2(y(q^*)-z_d^2) \ \text{ in } \ (0,T),\\ p(T)=0,\quad p'(T)=-\kappa_1(y(q^*;T)-z_d^1).\end{cases}$$

or simply, by using the operator $\mathcal{L}_0$,

(3.31)
$$\begin{cases} \mathcal{L}_0(p)=\kappa_2(y(q^*)-z_d) \ \text{ in } \ (0,T),\\ p(T)=0,\quad p'(T)=-\kappa_1(y(q^*;T)-z_d^1).\end{cases}$$

Since $\kappa_2(y(q^*)-z_d^2)\in L^2(0,T;H)$ and $\kappa_1(y(q^*;T)-z_d^1)\in H$, there is an unique weak solution $p\in W(0,T)$ of (3.31) and $(p'(T),p)\in X_0$. Hence by Theorem 3.3, if we take $\phi_1=p'(T)$ and $\phi=p$ in (3.14), then we have

$$(z_1,\kappa_1(y(q^*;T)-z_d^1))+\int_0^T(z(t),\kappa_2(y(q^*;t)-z_d^2(t)))dt$$
$$=\int_0^T\langle(\alpha^*-\alpha)y^{*'}+2\beta^*(\beta^*-\beta)Ay^*+(\gamma^*-\gamma)\sin y^*+(\delta-\delta^*)f,p\rangle\,dt,$$

and by (3.29)

$$\int_0^T\langle(\alpha^*-\alpha)y^{*'}+2\beta^*(\beta^*-\beta)Ay^*+(\gamma^*-\gamma)\sin y^*+(\delta-\delta^*)f,p\rangle\,dt\geq 0$$

for all $q\in\mathcal{P}_{ad}$.

Summarizing these we have the following theorem.

THEOREM 3.4. *The optimal parameter q^* for the cost* (3.28) *is characterized by the states $y=y(q^*),p=p(q^*)$ of two systems*

(3.32)
$$\begin{aligned} &y''+\alpha^*y'+(\beta^{*2}+\beta_0)Ay+\gamma^*\sin y=\delta^*f \ \ \textit{in} \ \ (0,T),\\ &y(0)=y_0,\quad y'(0)=y_1,\end{aligned}$$

(3.33)
$$\begin{aligned} &p''-\alpha^*p'+(\beta^{*2}+\beta_0)Ap+\gamma^*(\cos y)p=\kappa_2(y-z_d^2) \ \text{ in } \ (0,T),\\ &p(T)=0,\quad p'(T)=-\kappa_1(y(T)-z_d^1)\end{aligned}$$

and one inequality

$$(3.34)\quad \int_0^T \langle(\alpha^*-\alpha)y'+2\beta^*(\beta^*-\beta)Ay+(\gamma^*-\gamma)\sin y+(\delta-\delta^*)f, p\rangle\ dt \geq 0$$

for all $q \in \mathcal{P}_{ad}$.

Let us deduce the bang-bang principle from (3.34) for the case where $\mathcal{P}_{ad}$ is given by $\mathcal{P}_{ad} = [\alpha_1, \alpha_2] \times [0, \beta_1] \times [\gamma_1, \gamma_2] \times [\delta_1, \delta_2]$. In this case the necessary condition (3.34) is equivalent to

$$(3.35)\quad \int_0^T ((\alpha^* - \alpha)y'(t), p(t))\ dt \geq 0 \quad \forall \alpha \in [\alpha_1, \alpha_2],$$

$$(3.36)\quad \int_0^T (\beta^*(\beta^* - \beta)Ay(t), p(t))\ dt \geq 0 \quad \forall \beta \in [0, \beta_1],$$

$$(3.37)\quad \int_0^T ((\gamma^* - \gamma)\sin y(t), p(t))\ dt \geq 0 \quad \forall \gamma \in [\gamma_1, \gamma_2],$$

$$(3.38)\quad \int_0^T ((\delta - \delta^*)f(t), p(t))\ dt \geq 0 \quad \forall \delta \in [\delta_1, \delta_2].$$

First let us analyze (3.35). Put $a = \int_Q \frac{\partial y}{\partial t}(x,t)p(x,t)\ dxdt$ and assume that $a \neq 0$. Then (3.35) is rewritten simply by

$$(\alpha^* - \alpha)a \geq 0 \quad \forall \alpha \in [\alpha_1, \alpha_2].$$

Consequently it is easily verified that α^* is given by

$$\alpha^* = \frac{1}{2}\{\mathrm{sign}(a) + 1\}\alpha_2 - \frac{1}{2}\{\mathrm{sign}(a) - 1\}\alpha_1.$$

Secondly let us analyze (3.36). Also put $b = \int_Q \nabla y(x,t) \cdot \nabla p(x,t)\ dxdt$ and assume $b \neq 0$. Then (3.36) is written by

$$\beta^*(\beta^* - \beta)b \geq 0 \quad \forall \beta \in [0, \beta_1].$$

Consequently it is easily verified that β^* is given by

$$\beta^* = \frac{1}{2}\{\mathrm{sign}(b) + 1\}\beta_1 \quad \text{or} \quad \beta^* = 0.$$

Similarly, form (3.37)-(3.38) we can show that

$$\begin{aligned} \gamma^* &= \frac{1}{2}\{\mathrm{sign}(c) + 1\}\gamma_2 - \frac{1}{2}\{\mathrm{sign}(c) - 1\}\gamma_1, \\ \delta^* &= \frac{1}{2}\{\mathrm{sign}(d) + 1\}\delta_2 - \frac{1}{2}\{\mathrm{sign}(d) - 1\}\delta_1, \end{aligned}$$

provided that

$$c = \int_Q \sin y(t,x)p(x,t)\,dxdt \neq 0,$$
$$d = \int_Q f(x,t)p(x,t)\,dxdt \neq 0.$$

These are the so called bang-bang principle for the optimal parameter $q^* = (\alpha^*, \beta^*, \gamma^*, \delta^*)$.

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