NEW RESULTS ON STABILITY PROPERTIES FOR THE FEYNMAN INTEGRAL VIA ADDITIVE FUNCTIONALS

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ABSTRACT. It is known that the analytic operator-valued Feynman integral exists for some "potentials" which are so singular that they must be given by measures rather than by functions. Corresponding stability results involving monotonicity assumptions have been established by the author and others. Here in our main theorem we prove further stability theorem without monotonicity requirements.

1. Introduction

Consider the function

$$(1) \qquad F_t^\mu(w) = e^{-A_t^\mu(w)},$$

where $A_t^\mu(\cdot)$ is the additive functional corresponding to an appropriate measure μ. This function can be considered as a generalization of the function

$$(2) \qquad F_t^V(w) = e^{-\int_0^t V(w(s))\,ds},$$

where V is a potential.

In [1], the definition of the analytic operator-valued Feynman integral was extended to the function (1) (Actually, μ can be a potential which is too singular to be given by a function) and existence theorems for the analytic operator-valued Feynman integral of the function (1) were proved under appropriate conditions on the measures involved by making use of Dirichlet forms and Markov processes. These results enlarged the existence theory for the analytic operator-valued Feynman integral.

Extensions of the existence theory raise questions as to the possibility of extending the stability theory from [12] which was inspired by [18]. The study of stability theorems for the Feynman integral where potentials are given by a class of signed smooth measures rather than ordinary potentials was initiated in [6] and extended in [19]. Because of the difficulties of dealing with convergence of forms corresponding to measures instead of potential functions, monotonicity assumptions played an important role in [6] and [19]. It has been a long standing desire for the author to get rid of the monotonicity assumptions in [6] and [19]. Pursuing this goal, we prove Theorem 25 below, which is related in spirit to Theorem 3.6 in [19] but the nature of the monotonicity is reversed. Our main result is Theorem 30 which involves *no monotonicity assumption* and has Theorem 3.6 in [19] as its corollary. The results of the present paper depend on arguments which are rather different than those in [6] and [19] and at the end of this paper we analyze the relationship between the dominated convergence theorem from [18] (See also [15]) and Theorem 30.

2. Additive functionals, generalized Kato class measures, smooth measures and the Feynman-Kac formula

Our primary goal in this paper is to develop stability properties for the Feynman integral determined by signed smooth measures. In order to obtain these results in Section 5, we need to review definitions and key results related to positive continuous additive functionals, measures in the generalized Kato class, smooth measures, closed forms and their associated operators.

Generalized Kato class measures were considered in connection with Schrödinger semigroups [24] and Feynman-Kac formulae and the concept of smooth measures was introduced by M. Fukushima in the description of the class of Revuz measures associated with positive continuous additive functionals in the Dirichlet space setting [9].

Recently, the theory of additive functionals in the framework of Dirichlet forms has been shown to be powerful in the study of Schrödinger operators $H^\mu = -\frac{1}{2}\Delta + \mu$ (μ is an appropriate measure) and the related semigroups [2-4]. The correspondence between generalized Kato class measures and PCAF's (that is, piecewise continuous additive functionals) was studied in [4] and the correspondence between smooth measures and PCAF's was studied in [9]. As a result of Theorem 5.1.3 in [9] we have the special function (1) associated with a smooth measure

μ and this relationship is the first bridge which enables us to deal with the Feynman integral via additive functionals determined by smooth measures. Many valuable results concerned with generalized Kato class measures and smooth measures can be found in [2-4].

Let $H^1(R^d)$ be the standard Sobolev space, i.e.,

$$H^1(R^d) \equiv \{u \in L_2(R^d, m) | \frac{\partial u}{\partial x_i} \in L_2(R^d, m), 1 \le i \le d\},$$

where $L_2(R^d, m)$ denotes the space of R-valued functions on R^d which are square integrable with respect to Lebesgue measure m and the derivatives are taken in the distributional sense. Throughout this paper, we write $L_2(R^d)$ instead of $L_2(R^d, m)$. For a form q and an operator H, $D(q)$ and $D(H)$ stand for the domains of q and H, respectively. We let $\mathcal{E}$ denote the classical Dirichlet form, that is, the bilinear form acting on $D(\mathcal{E}) \equiv H^1(R^d)$:

$$\mathcal{E}(u, v) = \frac{1}{2} \int_{R^d} \nabla u \cdot \nabla v \, dm$$

and for $u, v \in D(\mathcal{E})$, we define

$$\mathcal{E}_1(u, v) = \frac{1}{2} \int_{R^d} \nabla u \cdot \nabla v \, dm + \int_{R^d} uv \, dm.$$

We now give the definitions of capacity, smooth measure and generalized Kato class measure; see [1] for more detailed discussions about these subjects.

DEFINITION 1. Given an open set $G \subset R^d$, let

$$Cap\,(G) = \inf\{\mathcal{E}_1(u, u) \mid u \in H^1(R^d) \text{ and } u \ge 1 \, a.e. \text{ on } G\}.$$

For an arbitrary set $A \subset R^d$, let

$$Cap\,(A) = \inf\{Cap\,(G) \mid A \subset G \subset R^d, G \text{ is open}\}.$$

DEFINITION 2. A positive Borel measure μ on R^d is called smooth if μ charges no set of zero capacity and if there exists an increasing sequence $\{F_n\}$ of compact sets such that

$$\mu(F_n) < \infty \text{ for } n \ge 1$$

and

$$\lim_{n \to \infty} Cap\,(K - F_n) = 0 \text{ for any compact set } K \subset R^d.$$

We shall denote by S the family of all smooth measures and by S_σ the family of all σ-finite smooth measures.

We show in the following lemma that every smooth measure defined on R^d as in Definition 2 above is a σ-finite measure. We are concerned with this issue because the existence theorems in [1] were developed under the assumption of the σ-finiteness of the positive and negative variations of a generalized signed smooth measure; that is, they assumed that $\mu = \mu^+ - \mu^-$ is a generalized signed measure and that both μ^+ and μ^- are in S_σ. (See p.271 and p.289 in [1]). It was observed but not proved in [1, Remark, p.284] that $\mu(R^d - \cup_n F_n) = 0$ where μ is a smooth measure again as in Definition 2 above and $\{F_n\}$ is the corresponding sequence of compact sets with respect to μ. We prove this fact here from which it easily follows that any smooth measure is σ-finite. The proof is not difficult but we have not seen this fact before and so it may be unknown to other readers.

LEMMA 3. *Every smooth measure defined on R^d is a σ-finite measure.*

Proof. Let μ be a smooth measure. Then there exists an increasing sequence $\{F_n\}$ of compact sets such that $\mu(F_n) < \infty$ for each $n \in N$. The crucial part of this proof is to show that $Cap(R^d - \cup_n F_n) = 0$. To prove this fact, let $R^d = \cup_m A_m$ where $A_m = \prod_d [-m, m]$ for each $m \in N$. Then $Cap(R^d - \cup_n F_n) = Cap[\cup_m (A_m - (\cup_n F_n))] = \sup_m Cap[A_m - (\cup_n F_n)]$ using a property of capacity (See Theorem 3.3.1 in [1]). And $\sup_m Cap[A_m - (\cup_n F_n)] = \sup_m Cap[\cap_n (A_m - F_n)] \leq \sup_m \inf_n Cap(A_m - F_n) = \sup_m \lim_n Cap(A_m - F_n)$. By the definition of smooth measure, $\lim_n Cap(A_m - F_n) = 0$, for each $m \in N$ and this implies $Cap(R^d - \cup_n F_n) = 0$. On the other hand, we have $\mu(R^d - \cup_n F_n) = 0$ since μ charges no set of zero capacity. Now let $G_m = F_m \cup (R^d - \cup_n F_n)$. Then we conclude that $\cup_m G_m = R^d$ and $\mu(G_m) < \infty$ for each $m \in N$, i.e., μ is a σ-finite measure. $\square$

One implication of the preceding lemma is that the assumption of σ-finiteness in the paper [19] is redundant and so can be eliminated.

DEFINITION 4. A positive Borel measure μ on R^d is said to be in the generalized Kato class if

$$\lim_{\alpha \to 0^+} \sup_{x \in R^d} \int_{|x-y| \leq \alpha} \frac{\mu(dy)}{|x-y|^{d-2}} = 0, \quad d \geq 3,$$

$$\lim_{\alpha \to 0^+} \sup_{x \in R^d} \int_{|x-y| \leq \alpha} (\log|x-y|^{-1})\mu(dy) = 0, \quad d = 2,$$

$$\sup_{x \in R^d} \int_{|x-y| \leq 1} \mu(dy) < \infty, \quad d = 1.$$

We denote by GK_d the generalized Kato class.

Using Lemma 3, Theorem 2.1 in [3] and the fact that every measure μ in the generalized Kato class is a Radon measure; i.e. $\mu(K) < \infty$ for all compact sets K, we have the following proposition.

PROPOSITION 5. $GK_d \subset S_\sigma = S$.

Next, we turn to the definition of positive continuous additive functional which will be abbreviated by PCAF for convenience. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P_x)$ be the canonical Brownian motion on R^d [5]. Let t be a nonnegative real number. For each ω in $\Omega = C([0, \infty), R^d)$, the collection of all continuous functions from $[0, \infty)$ to R^d, we define a function $\theta_t \omega : [0, \infty) \to R^d$ by $(\theta_t \omega)(s) = \omega(t+s)$ for all $s \in [0, \infty)$.

DEFINITION 6. A function $A : [0, \infty) \times \Omega \to R$ is called a PCAF in the classical sense if $A(t, \cdot) = A_t$ is $\mathcal{F}_t$-measurable for each t and there exists $\Lambda \in \mathcal{F}$ (called a defining set of A) satisfying the following properties:

1. $P_x(\Lambda) = 1$ for all $x \in R^d$.
2. $\theta_t \omega \in \Lambda$ for all $\omega \in \Lambda$.
3. For each $\omega \in \Lambda$, the function $A.(\omega) : [0, \infty) \to R$ is continuous, increasing and vanishes at 0 and is additive in the sense that

$$A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$$

for all $t, s \geq 0$.

For a nonnegative bounded Borel measurable function V on R^d, we consider a function A^V defined on $[0, \infty) \times \Omega$ by

$$A^V(t, \omega) = A_t^V(\omega) = \int_0^t V(\omega(s)) \, ds$$

for all (t, ω) in $[0, \infty) \times \Omega$. This is a typical example of a positive continuous additive functional and we are familiar with the following function appearing in the classical Feynman-Kac formula:

$$F_t^V(w) = e^{-\int_0^t V(w(s))\,ds}.$$

In the present paper, we are interested in the following function

$$F_t^\mu(w) = e^{-A_t^\mu(w)},$$

where the additive functional $A_t^\mu(\cdot)$ replaces $\int_0^t V(\omega(s))\,ds$ and an appropriate measure μ replaces the potential V.

Now we have a generalized definition of PCAF made by M. Fukushima which extends the notion of a PCAF in the classical sense.

DEFINITION 7. A function $A : [0, \infty) \times \Omega \to R$ is called a PCAF in Fukushima's sense if $A(t, \cdot) = A_t$ is $\mathcal{F}_t$-measurable for each $t \geq 0$ and there exist $\Lambda \in \mathcal{F}$ and $N \in \mathcal{B}(R^d)$ satisfying the following properties:

1. $P_x(\Lambda) = 1$ for all $x \in R^d - N$, where $Cap\,(N) = 0$.
2. $\theta_t\omega \in \Lambda$ for all $\omega \in \Lambda$.
3. For each $\omega \in \Lambda$, the function $A.(\omega) : [0, \infty) \to R$ is continuous, increasing and vanishes at 0 and is additive in the sense that

$$A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$$

for all $t, s \geq 0$.

Λ is called a defining set of A, and N is called an exceptional set of A.

In the rest of this section we investigate various properties of closed forms corresponding to signed smooth measures. For a signed Borel measure $\mu = \mu^+ - \mu^-$ on R^d (where μ^+ and μ^- are the usual positive and negative variations of μ, respectively), we write $\mu \in S - GK_d$ if $\mu^+ \in S$ and $\mu^- \in GK_d$ and μ is called a signed smooth measure. For $\mu \in S - GK_d$, we define $\mathcal{Q}_\mu$ and $\mathcal{E}_\mu$ as follows:

$$\mathcal{Q}_\mu(u, v) \equiv \int_{R^d} uv\,d\mu = \int_{R^d} uv\,d\mu^+ - \int_{R^d} uv\,d\mu^-$$

for all $u, v \in D(\mathcal{Q}_\mu) \equiv L_2(R^d, |\mu|) \cap L_2(R^d)$ and

$$\mathcal{E}_\mu(u, v) \equiv \mathcal{E}(u, v) + \mathcal{Q}_\mu(u, v)$$

for all $u, v \in D(\mathcal{E}_\mu) \equiv D(\mathcal{E}) \cap D(\mathcal{Q}_\mu)$.

For $\mu \in S - GK_d$, let A^{μ^+} be the PCAF in Fukushima's sense corresponding to μ^+ and A^{μ^-} be the PCAF in the classical sense corresponding to μ^-. (The existence of A^{μ^+} and A^{μ^-} are guaranteed by [1, Theorem 3.3.10] and [1, Theorem 3.2.3], respectively). We let $A_t^\mu = A_t^{\mu^+} - A_t^{\mu^-}$. Then $(A_t^\mu)_{t \geq 0}$ is a continuous additive functional in Fukushima's sense which has finite variation on every bounded interval [9]. Let us introduce the notation

$$(3) \qquad p_t^\mu f(x) = E_x[e^{-A_t^\mu(w)} f(\omega(t))]$$

provided that the right-hand side in (3) makes sense for $f \in L_2(R^d)$ where E_x stands for the expectation with respect to P_x and P_x is the probability measure associated with the Brownian paths in R^d which start at x at time 0.

Let $\mathcal{H}$ be a real or complex Hilbert space equipped with the inner product $\langle , \rangle$ and the norm $\|\cdot\|$. From [16], we have the following theorem.

THEOREM 8. *Let q be a densely defined, symmetric closed form in $\mathcal{H}$ which is bounded below by γ. Then there exists a unique bounded below self-adjoint operator H satisfying that for any $\xi \leq \gamma$, $D(q) = D((H - \xi)^{\frac{1}{2}})$ and $q(u,v) = \langle (H - \xi)^{\frac{1}{2}} u, (H - \xi)^{\frac{1}{2}} v \rangle + \xi \langle u, v \rangle$, for all $u, v \in D(q)$. Furthermore, $q(u,v) = \langle Hu, v \rangle$ for all $u \in D(H), v \in D(q)$.*

From [1, Proposition 3.4.7 and Theorem 3.4.8] and [2, Theorem 4.3], we have the following proposition. We have abridged and rephrased the existing results for the convenience of the reader.

PROPOSITION 9. *Let $\mu = \mu^+ - \mu^- \in S - GK_d$. Then*

1. *$\mathcal{E}_\mu$ is a densely defined symmetric bilinear form.*
2. *$\mathcal{E}_\mu$ is closed and bounded below.*
3. *$(p_t^\mu)_{t \geq 0}$ is a strongly continuous symmetric semigroup on $L_2(R^d)$.*

Moreover, let H^μ be the bounded below self-adjoint operator corresponding to $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ whose existence is guaranteed by Theorem 8, i.e. $H^\mu = -\frac{1}{2}\Delta + \mu$, the form sum of $-\frac{1}{2}\Delta$ and μ with domain dense in $L_2(R^d)$, and let $\tilde{H}^\mu$ be the infinitesimal generator of $(p_t^\mu)_{t \geq 0}$. Then

$$H^\mu = -\tilde{H}^\mu$$

and hence we have

$$(4) \qquad p_t^\mu f(x) = e^{-tH^\mu} f(x)$$

for all $f \in L_2(R^d)$.

REMARK 10. By (3) and (4), we obtain the Feynman-Kac formula

$$e^{-tH^\mu} f(x) = E_x[e^{-A_t^\mu(\omega)} f(\omega(t))]$$

for every $f \in L_2(R^d)$, $m - a.e.$ $x \in R^d$, and for all $t \geq 0$.

Now we extend $\mathcal{E}_\mu$ to the subspace $D(\mathcal{E}_\mu^C) \equiv D(\mathcal{E}_\mu) + iD(\mathcal{E}_\mu)$ of $L_2(R^d, C) \equiv L_2(R^d) + iL_2(R^d)$ where $i = \sqrt{-1}$.
Define $\mathcal{E}_\mu^C : D(\mathcal{E}_\mu^C) \longrightarrow C$ by

$$\mathcal{E}_\mu^C(u, v) \equiv \int_{R^d} \nabla u \cdot \overline{\nabla v} \, dm + \int_{R^d} u\bar{v} \, d\mu$$

for all $u, v \in D(\mathcal{E}_\mu^C)$.

The following propositions come from [1]. Proposition 11 describes the relationship between $\mathcal{E}_\mu^C$ and $\mathcal{E}_\mu$ and Proposition 12 shows how a natural extention $\mathcal{E}_\mu^C$ of $\mathcal{E}_\mu$ preserves the key properties of $\mathcal{E}_\mu$.

PROPOSITION 11. Let $\mu \in S - GK_d$. Then for $u = u_1 + iu_2, v = v_1 + iv_2 \in D(\mathcal{E}_\mu^C)$, $\mathcal{E}_\mu^C$ is represented as follows:

(5) $\mathcal{E}_\mu^C(u, v) = \mathcal{E}_\mu(u_1, v_1) + \mathcal{E}_\mu(u_2, v_2) + i[\mathcal{E}_\mu(u_2, v_1) - \mathcal{E}_\mu(u_1, v_2)]$.

PROPOSITION 12. Let $\mu = \mu^+ - \mu^- \in S - GK_d$. Then

1. $\mathcal{E}_\mu^C$ is a densely defined symmetric sesquilinear form.
2. $\mathcal{E}_\mu^C$ is bounded below and closed.

Moreover, let H_C^μ be the bounded below self-adjoint operator corresponding to $(\mathcal{E}_\mu^C, D(\mathcal{E}_\mu^C))$ whose existence is guaranteed by Theorem 8, i.e. $H_C^\mu = -\frac{1}{2}\Delta + \mu$, the form sum of $-\frac{1}{2}\Delta$ and μ with domain dense in $L_2(R^d, C)$. Then we obtain the following Feynman-Kac formula in the complex setting

$$(e^{-tH_C^\mu} u)(x) = E_x[e^{-A_t^\mu(\omega)} u(\omega(t))]$$

for every $u \in L_2(R^d, C)$, $m - a.e.$ $x \in R^d$, and for all $t \geq 0$.

3. The existence of the analytic in time operator-valued Feynman integral

We are now ready to introduce the definition and the existence theorem for the analytic (in time) operator-valued Feynman integral of functions in which we are especially interested. Given $\omega \in \Omega = C([0, \infty), R^d)$, let

$$F_t^\mu(\omega) = F^\mu(\omega) = e^{-A_t^\mu(\omega)},$$

where μ is a signed smooth measure and A_t^μ is given in Section 2. Let C, C_+ and $\overline{C}_+$ be the set of all complex numbers, all complex numbers with positive real part and all nonzero complex numbers with nonnegative real part, respectively.

DEFINITION 13. Given $t > 0$, $u \in L_2(R^d, C)$ and $x \in R^d$, consider the expression

$$(J^t(F^\mu)u)(x) = E_x\{e^{-A_t^\mu(\omega)}u(\omega(t))\}$$

(6)
$$= \int_{\Omega_x} e^{-A_t^\mu(\omega)}u(\omega(t))\, dP_x(\omega),$$

where Ω_x is the set of $\omega \in C([0, \infty), R^d)$ such that $\omega(0) = x$ and P_x is the probability measure associated with the Brownian paths in R^d which start at x at time 0. We say that the operator-valued function space integral $J^t(F^\mu)$ exists for $t > 0$ if (6) defines $J^t(F^\mu)$ as an element of $\mathcal{L}(L_2(R^d, C))$, the space of bounded linear operators on $L_2(R^d, C)$. If $J^t(F^\mu)$ exists for every $t > 0$ and, in addition, has an extension as a function of t to an analytic operator-valued function on C_+, and a strongly continuous function on $\overline{C}_+$, we say that $J^t(F^\mu)$ exists for all $t \in \overline{C}_+$. When t is purely imaginary, $J^t(F^\mu)$ is called the analytic (in time) operator-valued Feynman integral of F^μ.

The following theorem comes from [1]. In their elaborate paper [1], they obtained existence theorems for the analytic (in time) operator-valued Feynman integral more general than what follows. We just state the following one which is sufficient to pursue our purpose. We refer to [19] for a sketch of its proof.

THEOREM 14. Let $\mu = \mu^+ - \mu^- \in S - GK_d$ and let $\mathcal{E}_\mu^C$ be given in Section 2. Let H_C^μ be the self-adjoint operator corresponding to

$(\mathcal{E}_\mu^C, D(\mathcal{E}_\mu^C))$. Then $J^t(F^\mu)$ exists for all $t \in \overline{C}_+$ and has the representation

$$J^t(F^\mu) = e^{-tH_C^\mu}$$

for all $t \in \overline{C}_+$, where $e^{-tH_C^\mu}$ is given meaning via the Spectral Theorem applied to the self-adjoint operator H_C^μ. In particular, for $t \in R$, the analytic in time operator-valued Feynman integral $J^{it}(F^\mu)$ exists and we have

$$J^{it}(F^\mu) = e^{-itH_C^\mu},$$

where $\{e^{-itH_C^\mu}, t \in R\}$ is the unitary group corresponding to the self-adjoint operator H_C^μ.

4. Perturbation of closed forms

This section will be mainly devoted to an introduction of two important perturbation theorems for closed forms important perturbation theorems for closed forms (Theorem 17 and (Theorem17 and Theorem 18) which will play crucial roles in Section 5. From now on, let $\mathcal{H}$ denote a complex Hilbert space with the inner product $\langle , \rangle$ and the norm $\| \cdot \|$. For $x_n, x \in \mathcal{H}$, let $x_n \to x$ denote that x_n is strongly convergent to x, and for operators A_n, A on $\mathcal{H}$, let $A_n \to A$ indicate that A_n converges to A in the strong operator topology.

DEFINITION 15. Let $A, A_m, m = 1, 2, \cdots$ be self-adjoint operators on $\mathcal{H}$. We say that $\{A_m\}_{m=1}^\infty$ converges to A in the strong resolvent sense if

$$[I + iA_m]^{-1} \to [I + iA]^{-1},$$

where I denotes the identity operator and $i = \sqrt{-1}$.

From [17], we have the following theorem.

THEOREM 16. (Trotter, Kato, Rellich, Neveu) Let $H, H_m, m = 1, 2, \cdots$ be self-adjoint operators on $\mathcal{H}$. Then the following statements are equivalent:

1. $\{H_m\}_{m=1}^\infty$ converges to H in the strong resolvent sense.
2. $e^{-itH_m} \to e^{-itH}$ for all t in R.
3. $[I + i\lambda H_m]^{-1} \to [I + i\lambda H]^{-1}$ for all λ in R, $\lambda \neq 0$.
4. $e^{-itH_m} \to e^{-itH}$ uniformly in t on any compact subset of R.

If, in addition, the operators H_m and H are uniformly bounded below, then number 1 in the above implies:

$$e^{-tH_m} \to e^{-tH} \text{ uniformly in } t \text{ on any compact subset of } [0, +\infty).$$

The following monotone convergence theorem for closed forms is due to B. Simon [23] who refined a previously existing result of Kato using a canonical decomposition for any quadratic form that is bounded below.

THEOREM 17. *Let* $h_0, h_1, h_2, \cdots$ *, be closed, densely defined, and bounded below forms on* $\mathcal{H}$*. Suppose that*

$$h_1 \le h_2 \le \cdots \le h_0 .$$

Let $D(h_\infty) = \{ u \in \cap_n D(h_n) \mid \sup_n h_n(u, u) < \infty \}$ *with*

$$h_\infty(u, v) = \lim_{n \to \infty} h_n(u, v) .$$

Then h_∞ *is a densely defined closed form and* H_n *converges to* H *in the strong resolvent sense, where* H_n *and* H *are self-adjoint operators associated with* h_n *and* h_∞*, respectively.*

Now we will provide a perturbation theorem for closed forms that are bounded below, which was originally proved under the more general condition that the forms were closed and sectorial in $\mathcal{H}$ [16, Theorem 3.6, p.455].

THEOREM 18. *Let* t*,* t_n*,* $n = 1, 2, \cdots$ *be densely defined, bounded below, and closed forms in* $\mathcal{H}$ *satisfying the following properties:*

1. $D(t_n) \subset D(t), \quad n = 1, 2, \cdots .$
2. $t_n(u, u) \ge t(u, u)$ *for all* $u \in D(t_n)$*,* $n = 1, 2, \cdots .$
3. *There is a form core* D *of* t *such that* $D \subset \liminf D(t_n)$ *and*

$$\lim_{n \to \infty} t_n(u, u) = t(u, u) \quad \text{for all } u \in D.$$

Then H_n *converges to* H *in the strong resolvent sense, where* H_n *and* H *are self-adjoint operators associated with* t_n *and* t*, respectively.*

We finish this section by stating some basic results related to semi-bounded (i.e. bounded below) forms to help the reader get more understanding about our proof in Section 5. When a form t in $\mathcal{H}$, which is

bounded below by α, is given, we define a pre-Hilbert space under the inner product defined by

$$\langle u, v \rangle_t = t(u, v) + (1 - \alpha)\langle u, v \rangle, \quad u, v \in D(t).$$

We denote by $\| \cdot \|_t$ the norm induced by $\langle \, , \, \rangle_t$.

DEFINITION 19. Let t be a symmetric form in $\mathcal{H}$. A sequence $\{u_n\}$ of vectors will be said to be t-convergent to $u \in \mathcal{H}$, in symbol $u_n \xrightarrow{t} u$ as $n \to \infty$, if $u_n \in D(t)$, $u_n \to u$ and $t(u_n - u_m) \to 0$ as $m, n \to \infty$.

REMARK 20. Let t be a symmetric form in $\mathcal{H}$. It is easy to prove the following facts.

1. t is closed if and only if $u_n \xrightarrow{t} u$ implies that $u \in D(t)$ and $t(u_n - u) \to 0$ as $n \to \infty$.
2. $u_n \in D(t)$ and $\|u_n - u\|_t \to 0$ as $n \to \infty$ implies that $u_n \xrightarrow{t} u$.

THEOREM 21. Let t be a semibounded form in $\mathcal{H}$. If $u_n \xrightarrow{t} u$ and $v_n \xrightarrow{t} v$, then $\lim\limits_{n \to \infty} t(u_n, v_n)$ exists. In particular, this limit is equal to $t(u, v)$ when t is closed.

5. Stability properties for the Feynman integral

In order to develop our main results in this section, the only thing left is to scrutinize possible form cores. The importance of having form cores should be clear, because forms in many applications are given only on some dense subset rather than the whole domain.

In general, a smooth measure may not be a Radon measure. Even worse, if μ is a nowhere Radon smooth measure, then it may happen that $D(\mathcal{E}_\mu)$ contains no non-trivial continuous functions. Thus when we deal with a smooth measure μ it is necessary to find a relatively nice form core for a closed form $\mathcal{E}_\mu$. To this end we start from the notion of quasi everywhere, which will be abbreviated by q.e. from now on.

DEFINITION 22. Let A be a subset of R^d. A statement depending on $x \in A$ is said to hold q.e. on A if there exists a set $N \subset A$ of zero capacity such that the statement is true for every $x \in A - N$.

DEFINITION 23. Let u be a function defined q.e. on R^d. We call u quasi-continuous if there exists for any $\epsilon > 0$ an open set $G \subset R^d$ such that $Cap(G) < \epsilon$ and $u|_{R^d-G}$ is continuous. Here $u|_{R^d-G}$ denotes the restriction of u to $R^d - G$.

Let us define a class of functions $C_q(R^d)$ as follows:

$$C_q(R^d) = \{f \mid f \text{ is bounded, Borel measurable,}$$
$$\text{quasi-continuous and has compact support}\}.$$

PROPOSITION 24. Let $\mu \in S - GK_d$. Then $D(\mathcal{E}_\mu) \cap C_q(R^d)$ is a form core of $\mathcal{E}_\mu$. Moreover, $H^1(R^d) \cap C_q(R^d)$ is a form core of $\mathcal{E}_\mu$ when μ^+ is a Radon measure.

Proof. We already knew that $D(\mathcal{E}_\mu) \cap C_q(R^d) = H^1(R^d) \cap L_2(R^d, |\mu|) \cap C_q(R^d)$ is a form core of $\mathcal{E}_\mu$. (See [19, Proposition 3.5]). It is easy to prove that $C_q(R^d)$ is a subset of $L_2(R^d, |\mu|)$ when μ^+ is a Radon measure. □

We are now in a position to prove our main theorems. Theorem 25 is concerned with convergence from below and this is the counterpart of Theorem 3.6 in [19] which dealt with convergence from above. In Theorem 3.6 in [19], the limiting measure μ was given and the assumption was made that μ should be a signed measure. In case of Theorem 25 in this paper, only the sequence $\{\mu_n\}$ is given and the limiting measure μ, on which *no assumption* is made, is constructed. For the proof of Theorem 25, we use an operator theoretic method instead of the more concrete real analytic method which was used for the proof of Theorem 3.6 in [19].

THEOREM 25. Let $\mu_n, n = 1, 2, \cdots$ be signed measures on $(R^d, \mathcal{B}(R^d))$ satisfying the following properties:

1. For each $E \in \mathcal{B}(R^d), \{\mu_n(E)\}_{n=1}^\infty$ and $\{\mu_n^-(E)\}_{n=1}^\infty$ are nondecreasing sequences.
2. There exist finite measures ν and η such that

$$\mu_n^+ \leq \nu \in S \quad , \quad \mu_n^- \leq \eta \in GK_d$$

for all $n \in N$.

Let $\mu = \sup_n \mu_n^+ - \sup_n \mu_n^-$. For convenience, let $t_n = \mathcal{E}_{\mu_n}^C$ and $t = \mathcal{E}_\mu^C$ where $\mathcal{E}_{\mu_n}^C$ and $\mathcal{E}_\mu^C$ are given in Section 2. Then $\{H_n\}_{n=1}^\infty$ converges to H in the strong resolvent sense where H_n and H are self-adjoint operators associated with t_n and t, respectively.

REMARK 26. Using hypothesis 1 in the above theorem, we get $\{\mu_n^+(E)\}_{n=1}^\infty$ is a nondecreasing sequence for each $E \in \mathcal{B}(R^d)$. Then the definitions of GK_d and S and the hypothesis 2 together imply that for all $n \in N$, μ_n is a finite signed measure which belongs to $S - GK_d$.

REMARK 27. By virtue of the Vitali-Hahn-Sak's theorem, $\sup_n \mu_n^+$ and $\sup_n \mu_n^-$ are finite measures. A simple proof shows that the set function defined by $\mu = \sup_n \mu_n^+ - \sup_n \mu_n^-$ is a finite signed measure which belongs to $S - GK_d$. Moreover, μ is the limiting measure, i.e. $\mu_n(E) \to \mu(E)$ for each $E \in \mathcal{B}(R^d)$.

Proof of Theorem 25. For each $n \in N$, t_n is a densely defined closed form which is bounded below by Remark 26 and Proposition 12. A direct calculation shows that $\{t_n\}_{n \in N}$ is a nondecreasing sequence of forms which is bounded by $t_0 = \mathcal{E}_\nu^C$. Define a form

$$(7) \qquad q(f, f) = \lim_{n \to \infty} t_n(f, f) = \sup_{n \to \infty} t_n(f, f)$$

for all $f \in D(q) = \{f \in \cap_n D(t_n) | \sup_n t_n(f, f) < \infty\}$. By virtue of Theorem 17, q is a densely defined closed form and H_n converges to H in the strong resolvent sense where H_n and H are self-adjoint operators associated with t_n and q, respectively. So if we can show $q = t$, our proof will be done.

We claim that $q = t$. Note that we have a form core $D' = D + iD$ of t where $D = H^1(R^d) \cap C_q(R^d)$ by Remark 27 and Proposition 24. Moreover, we can show that $D' \subset D(q) \subset D(t)$ using

$$(8) \qquad \int |g|^2 \, d(\sup_n \mu_n^+) = \sup_n \int |g|^2 \, d\mu_n^+$$

and

$$(9) \qquad \int |g|^2 \, d(\sup_n \mu_n^-) = \sup_n \int |g|^2 \, d\mu_n^-$$

for all $g \in \bigcap_n D(\mathcal{E}_{\mu_n})$. Actually (8) and (9) are true because of the iterated limit theorem for a double sequence. Now let $f \in D(t)$. Noting that D' is a form core of t, we get a sequence $\{f_m\}$ in D' such that $||f_m - f||_t \to 0$ as $m \to \infty$. This implies that $f_m \underset{t}{\to} f$ by Remark 20. It is not difficult to show that $q(h,h) = t(h,h)$ for all $h \in D'$. Using this fact we can prove that $f_m \underset{q}{\to} f$. In fact, $f_m \to f$, $f_m \in D' \subset D(q)$ and $q(f_m - f_k) = t(f_m - f_k) \to 0$ as $m,k \to \infty$. Since q is closed, we see $f \in D(q)$ by Remark 20. Eventually we conclude that $D(t) = D(q)$. Furthermore, $t(f_m, f_m) = q(f_m, f_m)$ implies that $t(f,f) = q(f,f)$ in the light of Theorem 21. $\qquad\square$

We have the following two corollaries which are stability theorems for the analytic (in time) operator-valued Feynman integral.

COROLLARY 28. *Under the same conditions as in Theorem 25,*

$$(10) \qquad\qquad J^{it}(F^{\mu_n}) \to J^{it}(F^{\mu})$$

for all $t \in R$ in the strong operator topology.

Proof. In the light of Theorem 14, we get

$$J^{it}(F^{\mu_n}) = e^{-itH_C^{\mu_n}} \quad and \quad J^{it}(F^{\mu}) = e^{-itH_C^{\mu}},$$

where $H_C^{\mu_n}$ and H_C^{μ} are self-adjoint operators associated with $\mathcal{E}_{\mu_n}^C$ and $\mathcal{E}_{\mu}^C$, respectively. By Theorem 25 and Theorem 16, we get (10). $\qquad\square$

COROLLARY 29. *Assume the same hypotheses as in Theorem 25 and assume*

$$f_n \to f \quad in \quad L_2(R^d, C).$$

Then for all $t \in R$,

$$J^{it}(F^{\mu_n})f_n \to J^{it}(F^{\mu})f \quad in \quad L_2(R^d, C).$$

Stability theorems for the Feynman integral where ordinary potentials are considered rather than measures can be found in [18 and 11-15]. Monotonicity is not assumed in any of these results. On the other hand, monotonicity is an important assumption in [6 and 19]. It is desirable to obtain a stability theorem for the Feynman integral without the monotone condition. In this sense, the following theorem is a substantial improvement of Theorem 3.6 in [19]. In fact, $\{\mu_n(E)\}$ and

$\{\mu_n^-(E)\}$ were nonincreasing sequences and the forms corresponding to μ_n were uniformly bounded in Theorem 3.6 in [19]. These conditions are replaced in our next theorem by the simple assumption that the sequences $\{\mu_n(E)\}$ and $\{\mu_n^-(E)\}$ are bounded below.

THEOREM 30. *Let* μ, μ_n, $n = 1, 2, \cdots$ *be signed measures on* $(R^d,$ $\mathcal{B}(R^d))$ *satisfying the following properties:*

1. *For each* $E \in \mathcal{B}(R^d)$, $\mu_n(E) \to \mu(E)$ *as* $n \to \infty$, *where*

$$\mu_n(E) \geq \mu(E) \quad , \quad \mu_n^-(E) \geq \mu^-(E).$$

2. *There exist Radon measure* ν *and a measure* η *such that*

$$\mu_n^+ \leq \nu \in S \quad , \quad \mu_n^- \leq \eta \in GK_d$$

for all $n \in N$.

For convenience, let $t_n = \mathcal{E}_{\mu_n}^C$ *and* $t = \mathcal{E}_\mu^C$ *where* $\mathcal{E}_{\mu_n}^C$ *and* $\mathcal{E}_\mu^C$ *are given in Section 2. Then* $\{H_n\}_{n=1}^\infty$ *converges to* H *in the strong resolvent sense where* H_n *and* H *are self-adjoint operators associated with* t_n *and* t, *respectively.*

Proof. Since $\mu_n \in S - GK_d$, for each $n \in N, t_n$ is a densely defined closed form which is bounded below by Proposition 12. It is not difficult to show that $\mu = \mu^+ - \mu^- \in S - GK_d$ using hypotheses 1 and 2. Hence t is a densely defined closed form which is bounded below. Moreover, we can prove that $D(t_n) \subset D(t)$ for each $n \in N$ and $t_n(u, u) \geq t(u, u)$ for all $u \in D(t_n), n = 1, 2, \cdots$. Now let $D = H^1(R^d) \cap C_q(R^d)$. Noting that $\mu = \mu^+ - \mu^- \in S - GK_d$ and μ^+ is a Radon measure, we see that $D' = D + iD$ is a form core of t by Proposition 24. On the other hand, D' is a form core of t_n for each $n \in N$. Let $f = g + ih \in D'$. We claim that $\lim_{n\to\infty} t_n(f, f) = t(f, f)$. For this, it is sufficient to show that

$$(11) \qquad \lim_{n\to\infty} \int |g|^2 \, d\mu_n = \int |g|^2 \, d\mu$$

and

$$(12) \qquad \lim_{n\to\infty} \int |h|^2 \, d\mu_n = \int |h|^2 \, d\mu.$$

It is not difficult to show (11) because g is bounded with compact support. By essentially the same method as in the proof of (11) we can prove (12). Hence by virtue of Theorem 18 we conclude that $\{H_n\}$ converges to H in the strong resolvent sense where H_n and H are self-adjoint operators associated with t_n and t, respectively. $\qquad \square$

Here, as in Corollary 28 and Corollary 29, the strong resolvent convergence in the conclusion of Theorem 30 immediately yields the stability theorems for the analytic (in time) operator-valued Feynman integral. Finally we can show that Theorem 3.6 in [19] is a corollary of Theorem 30.

COROLLARY 31. *Let* μ, μ_n, $n = 1, 2, \cdots$ *be signed measures on* $(R^d, \mathcal{B}(R^d))$ *satisfying the following properties:*

1. *For each* $E \in \mathcal{B}(R^d)$, $\mu_n(E) \to \mu(E)$ *as* $n \to \infty$ *where* $\{\mu_n(E)\}_{n=1}^{\infty}$, $\{\mu_n^-(E)\}_{n=1}^{\infty}$ *are nonincreasing sequences.*
2. *There exist Radon measure* ν *and a measure* η *such that*

$$\mu_n^+ \leq \nu \in S \quad , \quad \mu_n^- \leq \eta \in GK_d$$

   for all $n \in N$.

For simplicity, let $t_n = \mathcal{E}_{\mu_n}^C$ *and* $t = \mathcal{E}_\mu^C$ *where* $\mathcal{E}_{\mu_n}^C$ *and* $\mathcal{E}_\mu^C$ *are given in Section 2. Assume that* t_n *is uniformly bounded by* $\alpha < 0$. *Then* $\{H_n\}_{n=1}^{\infty}$ *converges to* H *in the strong resolvent sense where* H_n *and* H *are self-adjoint operators associated with* t_n *and* t, *respectively.*

Proof. Let $E \in \mathcal{B}(R^d)$. By hypothesis 1, we get $\mu_n(E) \to \inf \mu_n(E) = \mu(E)$ and $\mu = \mu^+ - \mu^- = \inf \mu_n^+ - \inf \mu_n^-$. On the other hand, we can prove $\inf \mu_n^+$ and $\inf \mu_n^-$ are positive measures using the Vitali-Hahn-Sak's theorem. Now we conclude that $\mu_n(E) \geq \inf\{\mu_n(E)\} = \mu(E)$ and $\mu_n^-(E) \geq \inf\{\mu_n^-(E)\} \geq \mu^-(E)$. Then the conclusion follows from Theorem 30. □

REMARK 32. In [18], Lapidus proved a dominated convergence theorem for the modified Feynman integral involving ordinary potentials, that is, potentials given by functions. A slight improvement (also due to Lapidus) of this result is presented in [15]. When restricted to potentials given by functions, the conditions in Lapidus' result in [15] are more general than our result in the following sense: If $\nu = |f| \cdot m \in S$ (where m is Lebesgue measure) is a Radon measure then $f \in L_{loc}^1(R^d)$ and $\eta = |g| \cdot m \in GK_d$ if and only if $g \in K_d$ (where K_d is the class of all Kato functions), where ν and η are the dominating measures in Theorem 30. Also Theorem 30 has a bounded below condition for converging measures. However, the result here (i.e., Theorem 30) applies to many potentials which are not given by functions. (See examples in [1, pp. 285-286]).

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