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FIXED POINT THEORY FOR MULTIMAPS IN EXTENSION TYPE SPACES

Ravi P. Agarwal, Donal O'Regan and Sehie Park

Abstract. New fixed point results for the $\mathfrak{A}_c^\kappa$ selfmaps are given. The analysis relies on a factorization idea. The notion of an essential map is also introduced for a wide class of maps. Finally, from a new fixed point theorem of ours, we deduce some equilibrium theorems.

1. Introduction

This paper presents new fixed point results for multivalued selfmaps, in particular the $\mathfrak{A}_c^\kappa(X, X)$ maps. The most general result in the literature [12] assumes X is convex and admissible (in the sense of Klee), but here we will show that it is enough to assume X is an extension space (so it could be an absolute retract), or an approximate extension space, or indeed a neighborhood extension space under some restrictions. In Section 3 we present the notion of an essential map and discuss some of its properties. Section 4 presents some quasi–equilibrium theorems.

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. We will follow mainly [1, 2, 3, 12].

Let Y be a convex subset of a Hausdorff topological vector space E. Recall a *polytope* P in Y is any convex hull of a nonempty finite subset of Y. A nonempty subset X of E is said to be *admissible* (in the sense of Klee) if for every compact subset K of X and every neighborhood V of 0, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E. For example, every convex subset of a Hausdorff locally convex topological vector space is admissible. For other examples, see [12] and references therein.

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Of particular importance in this paper will be the class $\mathfrak{A}_c^\kappa$ due to Park. Suppose X and Y are topological spaces. Given a class $\mathbb{X}$ of maps, $\mathbb{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (the set of nonempty subsets of Y) belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of finite compositions of maps in $\mathbb{X}$. We let

$$\mathcal{F}(\mathbb{X}) = \{X : \ Fix\, F \neq \emptyset \ \text{ for all } \ F \in \mathbb{X}(X,X)\},$$

where $Fix\, F$ denotes the set of fixed points of $F : X \to 2^X$.

A class $\mathfrak{A}$ of maps is defined by the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of single-valued continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is upper semicontinuous (u.s.c.) and compact-valued; and
(iii) for any polytope P, $F \in \mathfrak{A}_c(P,P)$ has a fixed point, where the intermediate spaces of compositions are suitably chosen for each $\mathfrak{A}$.

An *admissible* class $\mathfrak{A}_c^\kappa(X,Y)$ of maps $F : X \to 2^Y$ is one such that, for each F and each nonempty compact subset K of X there exists a map $G \in \mathfrak{A}_c(X,Y)$ satisfying $G(x) \subset F(x)$ for all $x \in K$.

Examples of $\mathfrak{A}_c^\kappa$ are classes of continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (u.s.c. with nonempty compact convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (u.s.c. with R_δ values), the acyclic maps $\mathbb{V}$ (u.s.c. with compact acyclic values), the Powers maps $\mathbb{V}_c$ (finite compositions of acyclic maps), the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are uniform spaces), admissible maps of Górniewicz, σ-selectional maps of Haddad and Lasry, permissible maps of Dzedzej, the class $\mathbb{K}_c^+$ of Lassonde, the class $\mathbb{V}_c^+$ of Park *et al.*, and approximable maps of Ben-El-Mechaiekh and Idzik, and others. For details on the admissible classes, see [12].

In [12] Park gave an elementary proof of the following result.

THEOREM 1.1. *Let E be a Hausdorff topological vector space and X an admissible, convex, compact subset of E. Then any map $F \in \mathfrak{A}_c^\kappa(X,X)$ has a fixed point.*

A class of maps $\mathcal{R}(X,Y)$ is said to be *admissible* (in the sense of Ben-El-Mechaiekh and Deguire [3]) if

(i) $\mathcal{R}$ contains the class $\mathbb{C}$; and

(ii) each $F \in \mathcal{R}_c$ is upper semicontinuous and closed-valued.

The following result is given in [3, Proposition 2.2].

THEOREM 1.2. *Let $\mathcal{R}$ be an admissible class of maps. Then the Hilbert cube I^∞ (subset of l^2 consisting of points $(x_1, x_2, ...)$ with $|x_i| \leq 1/i$ for all i) and the Tychonoff cube T (cartesian product of copies of the unit interval imbedded in a normed space) are in $\mathcal{F}(\mathcal{R}_c)$ provided the closed unit ball $\mathbf{B}^n = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$ is in $\mathcal{F}(\mathcal{R}_c)$ for all $n \geq 1$.*

From Theorem 1.1 or 1.2, we immediately have the following.

THEOREM 1.3. *I^∞ and T are in $\mathcal{F}(\mathfrak{A}_c^\kappa)$.*

REMARK 1.1. It is worth remarking that we do not need to introduce the class $\mathcal{R}$ (we did so to give credit to the authors in [1, 3]) since if we assume $\mathbf{B}^n \in \mathcal{F}(\mathcal{R}_c)$, then since $\mathbf{B}^n$ is a homeomorphic image of a polytope, we have for any polytope P, that $F \in \mathcal{R}_c(P, P)$ has a fixed point, where the intermediate spaces of compositions are suitably chosen for each $\mathcal{R}$. Thus Theorem 1.3 follows immediately from [3] and the fact that $\mathbf{B}^n \in \mathcal{F}(\mathfrak{A}_c^\kappa)$.

REMARK 1.2. Since I^∞ and T are compact, Theorem 1.3 is actually equivalent to

(1) I^∞ and T are in $\mathcal{F}(\mathfrak{A}_c)$.

However, considering a Browder type map F (having nonempty convex values and open fibers), we notice that $F \notin \mathfrak{A}_c$ but $F \in \mathfrak{A}_c^\kappa$.

For a subset K of a topological space X, we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets in X (usually we write $Cov(K) = Cov_X(K)$). Given a map $F : X \to 2^X$ and $\alpha \in Cov(X)$, a point $x \in X$ is said to be an *α–fixed point* of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$.

Given two maps $F, G : X \to 2^Y$ and $\alpha \in Cov(Y)$, F and G are said to be *α–close*, if for any $x \in X$ there exist $U_x \in \alpha$, $y \in F(x) \cap U_x$, and $w \in G(x) \cap U_x$.

2. Extension type spaces and fixed points

In this section, we show that various extension type spaces have the fixed point property with respect to the $\mathfrak{A}_c^\kappa$ selfmaps. For details and examples of such extension type spaces, see [1, 3] and references therein.

In the definitions in this section by a space we mean a Hausdorff topological space.

Let Q be a class of topological spaces. A space Y is an *extension space* for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subset X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$.

We now present a new fixed point result for the $\mathfrak{A}_c^\kappa$ maps.

THEOREM 2.1. *Let $X \in ES(compact)$ and $F \in \mathfrak{A}_c^\kappa(X, X)$ a compact map. Then F has a fixed point.*

Proof. It is known [8] that every compact space is homeomorphic to a closed subset of the Tychonoff cube T, so as a result $K = \overline{F(X)}$ can be embedded as a closed subset $K^\star$ of T; let $s : K \to K^\star$ be a homeomorphism. Also let $i : K \hookrightarrow X$ and $j : K^\star \hookrightarrow T$ be inclusions. Now since $X \in ES(\text{compact})$ and $i\,s^{-1} : K^\star \to X$, then $i\,s^{-1}$ extends to a continuous function $h : T \to X$. Let $G = j\,s\,F\,h$ and notice $G \in \mathfrak{A}_c^\kappa(T, T)$. Hence, Theorem 1.3 guarantees that there exists $x \in T$ with $x \in G(x)$. Let $y = h(x)$, so

$$y \in h\,j\,s\,F\,(y) \quad \text{i.e.} \quad y = h\,j\,s\,(q) \quad \text{for some} \quad q \in F\,(y).$$

Since $h\,j\,(z) = i\,s^{-1}\,(z)$ for $z \in K^\star$, we have $h\,j\,s\,(q) = (h\,j)\,s\,(q) = i\,(q) = q$, and so $y \in F(y)$. □

REMARK 2.1. If $X \in AR$ (an absolute retract as defined in [5]) then of course $X \in ES(\text{compact})$ [We know from the Arens–Eells theorem that X is r–dominated by a normed space E so there exist maps $r : E \to X$ and $s : X \to E$ with $r\,s = 1$. Now since any normed space is $ES(\text{compact})$, it follows immediately that $X \in ES(\text{compact})$]. So a special case of Theorem 2.1 occurs if $X \in AR$.

A space Y is an *approximate extension space* for Q (written $Y \in AES(Q)$) if for any $\alpha \in Cov\,(Y)$ and any pair (X, K) in Q with $K \subset X$ closed, and any continuous function $f_0 : K \to Y$, there exists a continuous function $f : X \to Y$ such that $f|_K$ is α–close to f_0.

We now extend Theorem 2.1 to approximate extension spaces. To prove this we need the following elementary result for α–fixed points (see [1, Lemma 1.2]).

LEMMA 2.2. *Let X be a regular topological space and $F : X \to 2^X$ an upper semicontinuous map with closed values. Suppose there exists a cofinal family of coverings $\theta \subset Cov\,(X)$ such that F has an α–fixed point for every $\alpha \in \theta$. Then F has a fixed point.*

THEOREM 2.3. *Let $X \in AES(compact)$ and $F \in \mathfrak{A}_c^\kappa(X, X)$ a compact map. Then F has a fixed point.*

Proof. Let K, $K^\star$, s, i and j be as in the proof of Theorem 2.1. Let $\alpha \in Cov\,(X)$ and let $h : T \to X$ be such that h and $i\,s^{-1}$ are α–close on $K^\star$ (guaranteed since $X \in AES(\text{compact})$). Let $G = j\,s\,F\,h$ and notice $G \in \mathfrak{A}_c^\kappa(T,T)$. Now Theorem 1.3 guarantees that there exists $x \in T$ with $x \in G(x)$. Let $y = h(x)$, so

$$y \in h\,j\,s\,F\,(y) \quad \text{i.e.} \quad y = h\,j\,s\,(q) \quad \text{for some} \quad q \in F\,(y).$$

Since $i\,s^{-1}$ and h are α–close on $K^\star$ there exists $U \in \alpha$ with $i\,s^{-1}(s(q)) \in U$ and $h\,j\,(s\,(q)) \in U$ i.e. $q \in U$ and $y \in U$. Thus

$$y \in U \quad \text{and} \quad F(y) \cap U \neq \emptyset \quad \text{since} \quad q \in F\,(y).$$

As a result F has an α–fixed point. Since α is arbitrary, Lemma 2.2 guarantees that F has a fixed point. □

DEFINITION 2.1. Let V be a subset of a Hausdorff topological vector space E. Then we say V is *Schauder admissible* if for every compact subset K of V and every covering $\alpha \in Cov_V(K)$, there exists a continuous function (called the Schauder projection) $\pi_\alpha : K \to V$ such that

(i) π_α and $i : K \hookrightarrow V$ are α–close;

(ii) $\pi_\alpha(K)$ is contained in a subset $C \subset V$ with $C \in AES(\text{compact})$.

If $V \in AES(\text{compact})$ then V is trivially Schauder admissible. If V is an open convex subset of a Hausdorff locally convex topological vector space E, then it is well known [1, Lemma 4.8] that V is Schauder admissible.

We next present a result of Himmelberg type [9].

THEOREM 2.4. *Let V be a Schauder admissible subset of a Hausdorff topological vector space E and $F \in \mathfrak{A}_c^\kappa(V,V)$ a compact map. Then F has a fixed point.*

Proof. Since $F(V) \subset K$, K compact, for each $\alpha \in Cov_V\,(K)$ there exist $\pi_\alpha : K \to V$ (as described in Definition 2.1) and a subset $C \subset V$ with $C \in AES(\text{compact})$ such that, by putting $F_\alpha \equiv \pi_\alpha F$,

$$F_\alpha\,(V) = \pi_\alpha\,F\,(V) \subset C.$$

Notice $F_\alpha \in \mathfrak{A}_c^\kappa(C,C)$ so Theorem 2.3 guarantees that there exists $x \in C$ with $x \in \pi_\alpha\,F\,(x)$ i.e. $x = \pi_\alpha\,(q)$ for some $q \in F\,(x)$. Now Definition 2.1 (i) guarantees that there exists $U \in \alpha$ with $\pi_\alpha\,(q) \in U$ and $i\,(q) \in U$ i.e. $x \in U$ and $q \in U$. Thus

$$x \in U \quad \text{and} \quad F(x) \cap U \neq \emptyset \quad \text{since} \quad q \in F\,(x).$$

As a result F has an α–fixed point. Since α is arbitrary, Lemma 2.2 guarantees that F has a fixed point. □

A space Y is a *neighborhood extension space* for Q (written $Y \in NES(Q)$) if for any pair (X, K) in Q with $K \subset X$ closed and any continuous function $f_0 : K \to Y$ there is a continuous extension $f : U \to Y$ of f_0 over a neighborhood U of K in X.

We would like to extend Theorem 2.3 to neighborhood extension spaces. However even in the case when F is admissible in the sense of Górniewicz [6] extra conditions need to be added (recall that maps admissible in the sense of Górniewicz are in the class $\mathfrak{A}_c^\kappa$).

Recall the following well known result [1, Lemma 4.7].

LEMMA 2.5. *Let T be a Tychonoff cube contained in a Hausdorff topological vector space. Then T is a retract of span (T).*

Let $X \in NES(\text{compact})$ and $F \in \mathfrak{A}_c^\kappa(X, X)$ a compact map.

Let K, $K^\star$, s and i be as in the proof of Theorem 2.1. Let U be an open neighborhood of $K^\star$ in T and $h : U \to X$ be a continuous extension of $i\,s^{-1} : K^\star \to X$ on U (guaranteed since $X \in NES(\text{compact})$). Let $j : K^\star \hookrightarrow U$ be the natural embedding so $h\,j = i\,s^{-1}$. Now consider *span* (T) in a Hausdorff locally convex topological vector space containing T. Now Lemma 2.5 guarantees that there exists a retraction $r : \mathit{span}\,(T) \to T$. Let $i^\star : U \hookrightarrow r^{-1}(U)$ be an inclusion and consider $G = i^\star\, j\, s\, F\, h\, r$. Notice $G \in \mathfrak{A}_c^\kappa(r^{-1}(U), r^{-1}(U))$. Assume

$$G \in \mathfrak{A}_c^\kappa(r^{-1}(U), r^{-1}(U)) \text{ has a fixed point.} \tag{2.1}$$

If (2.1) is true then there exists $x \in r^{-1}(U)$ with $x \in G\,x$. Let $y = h\,r\,(x)$, so

$$y \in h\,r\,i^\star\, j\, s\, F\,(y) \quad \text{i.e.} \quad y = h\,r\,i^\star\, j\, s\,(q) \quad \text{for some} \quad q \in F\,(y).$$

Since $h(z) = i\,s^{-1}(z)$ for $z \in K^\star$, we have $h\,r\,i^\star\, j\, s\,(q) = (h\,r\,i^\star\, j)\, s\,(q) = i\,(q)$, and so $y \in F(y)$.

Thus existence of a fixed point of F is guaranteed if (2.1) is satisfied; recall $G = i^\star\, j\, s\, F\, h\, r$ and $r^{-1}(U)$ is an open subset of a Hausdorff locally convex topological vector space.

For specific classes of maps (2.1) is known to be true. For example, if F is admissible in the sense of Górniewicz [6] and the Lefschetz set $\Lambda\,(F) \neq \{0\}$ then we know [6] that (2.1) holds. More generally, if F is permissible in the sense of Dzedzej [7] and $\Lambda\,(F) \neq \{0\}$ then (2.1) holds. It would be of interest to know other examples.

3. Essential maps

Throughout this section Y will be a completely regular topological space with $Y \in AES(\text{compact})$, so in particular the results in this section will hold if $Y \in ES(\text{compact})$ or $Y \in AR$. [Of course $Y \in AES(\text{compact})$ could be replaced by Y Schauder admissible in this section]. Also U will be an open subset of Y. In this section we consider a subclass $\mathcal{A}$ of $\mathfrak{A}_c^{\kappa}$. The subclass must have the following property: for subsets X_1, X_2 and X_3 of Hausdorff topological vector spaces

$$\text{if } F \in \mathcal{A}(X_2, X_3) \text{ and } f \in \mathbb{C}(X_1, X_2), \text{ then } Ff \in \mathcal{A}(X_1, X_3).$$

The theory in this section will work for any class of maps $\mathcal{A}$ which satisfy a normalization property. In particular one can view the class $\mathcal{A}$ as any class where we can get a Leray–Schauder type result. For example we could take $\mathcal{A}$ to be $\mathbb{V}$ since clearly (3.3) (and (3.4), (3.5)) hold.

DEFINITION 3.1. $F \in \mathcal{A}_{\partial U}(\overline{U}, Y)$ if $F \in \mathcal{A}(\overline{U}, Y)$ with F compact and $x \notin F(x)$ for $x \in \partial U$.

DEFINITION 3.2. A map $F \in \mathcal{A}_{\partial U}(\overline{U}, Y)$ is *essential* if for every $G \in \mathcal{A}_{\partial U}(\overline{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G(x)$.

THEOREM 3.1. (Homotopy Invariance) *Let Y and U be as above. Suppose $F \in \mathcal{A}_{\partial U}(\overline{U}, Y)$ is an essential map and $H \in \mathcal{A}(\overline{U} \times [0,1], Y)$ is a compact map. Also assume the following two properties hold:*

$$H(x,0) = F(x) \quad \text{for } x \in \overline{U} \tag{3.1}$$

and

$$x \notin H_t(x) \text{ for any } x \in \partial U \text{ and } t \in (0,1] \text{ (here } H_t(x) = H(x,t)). \tag{3.2}$$

Then H_1 has a fixed point in U.

Proof. Let

$$B = \left\{x \in \overline{U} : \; x \in H_t(x) \; \text{ for some } \; t \in [0,1]\right\}.$$

When $t = 0$, $H_t = F$ and since $F \in \mathcal{A}_{\partial U}(\overline{U}, Y)$ is essential there exists an $x \in U$ with $x \in F(x)$. Thus $B \neq \emptyset$. Since H is upper semicontinuous and compact, it is immediate that B is closed and compact. In addition (3.2) (together with $F \in \mathcal{A}_{\partial U}(\overline{U}, Y)$) implies $B \cap \partial U = \emptyset$. Thus (since Y is completely regular) there exists a continuous function $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Define a map R by $R(x) = H(x, \mu(x))$ for $x \in \overline{U}$. Let $j : \overline{U} \to \overline{U} \times [0,1]$ be given by $j(x) = (x, \mu(x))$. Note j is continuous so $R = H\,j \in \mathcal{A}(\overline{U}, Y)$. In addition, R is compact since H is. Also notice for $x \in \partial U$ that $R(x) = H_0(x) = F(x)$, and so $R \in \mathcal{A}_{\partial U}(\overline{U}, Y)$. Now $R|_{\partial U} = F|_{\partial U}$ and $F \in \mathcal{A}_{\partial U}(\overline{U}, Y)$ essential implies that there exists $x \in U$ with $x \in R(x)$ (i.e. $x \in H_{\mu(x)}(x)$). Thus $x \in B$ and so $\mu(x) = 1$. Consequently $x \in H_1(x)$. □

Next we give an example of an essential map.

THEOREM 3.2. (Normalization) *Let Y and U be as above with $0 \in U$. Suppose the following condition is satisfied:*

$$\text{(3.3)} \qquad \begin{cases} \text{for any map } \theta \in \mathcal{A}_{\partial U}(\overline{U}, Y) \text{ with } \theta|_{\partial U} = \{0\}, \\ \text{the map } J \text{ is in } \mathfrak{A}_c^{\kappa}(Y, Y); \text{ here} \\ J(x) = \begin{cases} \theta(x), & x \in \overline{U} \\ \{0\}, & x \in Y \setminus \overline{U}. \end{cases} \end{cases}$$

Then the zero map is essential in $\mathcal{A}_{\partial U}(\overline{U}, Y)$.

Proof. Let $\theta \in \mathcal{A}_{\partial U}(\overline{U}, Y)$ with $\theta|_{\partial U} = \{0\}$. We must show that there exists $x \in U$ with $x \in \theta(x)$. Define a map J by

$$J(x) = \begin{cases} \theta(x), & x \in \overline{U} \\ \{0\}, & x \in Y \setminus \overline{U}. \end{cases}$$

From (3.3) we know $J \in \mathfrak{A}_c^{\kappa}(Y, Y)$. Clearly J is compact since θ is. Hence, Theorem 2.3 implies that there exists $x \in Y$ with $x \in J(x)$. Now if $x \notin U$ we have $x \in J(x) = \{0\}$, which is a contradiction since $0 \in U$. Thus $x \in U$ so $x \in J(x) = \theta(x)$. □

Next we present another version of the normalization property when we are in the topological vector space setting. Let $Y \in AES(\text{compact})$ be a convex subset of a topological vector space E and let U be an open subset of Y with $0 \in U$. In addition assume there exists a continuous retraction $r : Y \to \overline{U}$.

THEOREM 3.3. (Normalization) *Let E, Y, U and r be as above and suppose the following condition is satisfied:*

$$\text{(3.4)} \qquad \begin{cases} \text{for any continuous function } \mu : Y \to [0,1] \text{ and} \\ \text{any map } G \in \mathcal{A}(Y, Y) \text{ we have } \mu\, G \in \mathfrak{A}_c^{\kappa}(Y, Y). \end{cases}$$

Then the zero map is essential in $\mathcal{A}_{\partial U}(\overline{U}, Y)$.

Proof. Let $\theta \in \mathcal{A}_{\partial U}(\overline{U}, Y)$ with $\theta|_{\partial U} = \{0\}$. Let

$$A = \{x \in \overline{U} : \; x \in \lambda\, \theta(x) \text{ for some } \lambda \in [0,1]\}.$$

Now $A \neq \emptyset$ is compact and $A \subset U$ (this is clear since $0 \in U$ and $\theta|_{\partial U} = \{0\}$). Thus there exists a continuous function $\mu : Y \to [0,1]$ with $\mu(A) = 1$ and $\mu(Y \backslash U) = 0$. Define a map J_0 by

$$J_0(x) = \mu(x)\,\theta(r(x)) \quad \text{for} \quad x \in Y.$$

Note $\theta r \in \mathcal{A}(Y,Y)$ so $J_0 \in \mathfrak{A}_c^\kappa(Y,Y)$ from (3.4). Theorem 2.3 implies that there exists $x \in Y$ with $x \in \mu(x)\,\theta(r(x))$. If $x \in Y\backslash U$ then $\mu(x) = 0$, a contradiction since $0 \in U$. Thus $x \in U$ and so $x \in \mu(x)\,\theta(x)$. As a result $x \in A$, so $\mu(x) = 1$. Consequently $x \in \theta(x)$. □

Of course we can obtain a nonlinear alternative of Leray–Schauder type by combining Theorems 3.1 and 3.3. In fact, we can obtain a more general result.

THEOREM 3.4. *Let E, Y, U and r be as above. Suppose $F \in \mathcal{A}(\overline{U},Y)$ satisfies* (3.4) *and assume the following conditions hold*:

(3.5) $$\begin{cases} \text{for any continuous function } \mu : \overline{U} \to [0,1] \text{ and} \\ \text{any map } G \in \mathcal{A}(\overline{U},Y) \text{ we have } \mu G \in \mathcal{A}(\overline{U},Y) \end{cases}$$

and

(3.6) $$x \notin \lambda F(x) \quad \text{for every } x \in \partial U \text{ and } \lambda \in (0,1].$$

Then F is essential in $\mathcal{A}_{\partial U}(\overline{U},Y)$.

Proof. Let $\Phi \in \mathcal{A}_{\partial U}(\overline{U},Y)$ with $\Phi|_{\partial U} = F|_{\partial U}$. We must show Φ has a fixed point in U. Let

$$D = \{x \in \overline{U} : \ x \in \lambda\,\Phi(x) \ \text{ for some } \ \lambda \in [0,1]\}.$$

Now $D \neq \emptyset$ is compact and $D \cap \partial U = \emptyset$ (note (3.6) with $\Phi|_{\partial U} = F|_{\partial U}$ and that $0 \in U$). Thus there exists a continuous function $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define a map R by $R(x) = \mu(x)\,\Phi(x)$. Now (3.5) guarantees that $R \in \mathcal{A}(\overline{U},Y)$. Also R is compact with $R|_{\partial U} = \{0\}$. Now since $R \in \mathcal{A}_{\partial U}(\overline{U},Y)$ and since the zero map is essential in $\mathcal{A}_{\partial U}(\overline{U},Y)$ (Theorem 3.3) there exists $x \in U$ with $x \in R(x)$. Thus $x \in D$ and so $\mu(x) = 1$, i.e., $x \in \Phi(x)$. □

4. Quasi–equilibrium theorem

We begin this section by expressing Theorem 2.4 as an equilibrium theorem. Then a general result will be deduced from our main theorem.

THEOREM 4.1. *Let E and Y be Hausdorff topological vector spaces, Q a subset of E, $G : Q \to k(Q)$ (nonempty compact subsets of Q) and*

$T : Q \to 2^C$ where C is a subset of Y. In addition assume the following conditions hold:

(4.1) $f : Q \times C \times Q \to \mathbf{R}$ *is a upper semicontinuous function,*

(4.2) G *and* T *are compact maps,*

(4.3) $Q \times C$ *is an Schauder admissible subset of* $E \times Y$,

and

(4.4) $$F \in \mathfrak{A}_c^\kappa(Q \times C, Q \times C);$$

here $F(x, y) = \Phi(x, y) \times T(x)$ *for* $(x, y) \in Q \times C$ *with*

$$\Phi(x, y) = \{w \in G(x) : \ f(x, y, w) = M(x, y)\}$$

and $M(x, y) = \max_{w \in G(x)} f(x, y, w)$. *Then there exist* $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$, *and* $y_0 \in T(x_0)$ *with*

$$f(x_0, y_0, z) \le f(x_0, y_0, x_0) \ \text{ for all } \ z \in G(x_0).$$

If in addition

(4.5) $$f(x, y, x) \le 0 \ \text{ for all } \ (x, y) \in Q \times C,$$

then there exists $(x_0, y_0) \in Q \times C$ *such that* $x_0 \in G(x_0)$, $y_0 \in T(x_0)$, *and*

$$f(x_0, y_0, z) \le 0 \ \text{ for all } \ z \in G(x_0).$$

Proof. Notice $\Phi(x, y)$ is nonempty (and compact) for each $(x, y) \in Q \times C$. As a result $F : Q \times C \to 2^{Q \times C}$ and also F is compact since $F(Q \times C) \subseteq G(Q) \times T(Q)$. Now Theorem 2.4 guarantees that there exists $(x_0, y_0) \in Q \times C$ with $(x_0, y_0) \in \Phi(x_0, y_0) \times T(x_0)$. That is, there exists $(x_0, y_0) \in Q \times C$ with $x_0 \in G(x_0)$, $y_0 \in T(x_0)$ and $f(x_0, y_0, x_0) = M(x_0, y_0)$ (i.e., $f(x_0, y_0, z) \le f(x_0, y_0, x_0)$ for all $z \in G(x_0)$), so we are finished the first part. For the second part assume (4.5) holds, and so the result is immediate from the first part. □

Next we consider a subclass $\mathcal{D}$ of $\mathfrak{A}_c^\kappa$. If X and Y are subsets of Hausdorff topological vector spaces then we say $F \in \mathcal{D}(X, Y)$ if $F \in \mathfrak{A}_c^\kappa(X, Y)$ and is upper semicontinuous with nonempty compact values and satisfies Property (C) (to be specified in the examples considered). Also we assume for subsets X_1 and X_2 of Hausdorff topological vector spaces

(4.6) $$\begin{cases} \text{if } F_1 \in \mathcal{D}(X_1 \times X_2, X_1) \text{ and } F_2 \in \mathcal{D}(X_1, X_2) \\ \text{then } F_3 \in \mathfrak{A}_c^\kappa(X_1 \times X_2, X_1 \times X_2); \end{cases}$$

here $F_3(x,y) = F_1(x,y) \times F_2(x)$. A typical example of a class $\mathcal{D}$ is the acyclic maps $\mathbb{V}$ (i.e., Property (C) means the map is acyclic valued).

THEOREM 4.2. *Let E and Y be Hausdorff topological vector spaces, Q a subset of E, $G : Q \to k(Q)$ and $T : Q \to k(C)$ where C is a subset of Y. Suppose* (4.1), (4.2), (4.3), (4.6) *hold and in addition assume the following conditions are satisfied*:

$$G : Q \to 2^Q \quad \text{is upper semicontinuous} \tag{4.7}$$

$$\begin{cases} M : Q \times C \to Q \quad \text{is lower semicontinuous} \\ (\text{ here } M(x,y) = \max_{w \in G(x)} f(x,y,w) \,) \end{cases} \tag{4.8}$$

(4.9)
$T \in \mathfrak{A}_c^\kappa(Q,C)$ *is upper semicontinuous and satisfies Property* (C)

and

$$\Phi \in \mathfrak{A}_c^\kappa(Q \times C, Q) \quad \text{and satisfies Property } (C); \tag{4.10}$$

here

$$\Phi(x,y) = \{w \in G(x) : \ f(x,y,w) = M(x,y)\}.$$

Then there exists $(x_0,y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with*

$$f(x_0,y_0,z) \le f(x_0,y_0,x_0) \quad \text{for all} \quad z \in G(x_0).$$

If in addition (4.5) *holds, then there exists* $(x_0,y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with* $f(x_0,y_0,z) \le 0$ *for all* $z \in G(x_0)$.

Proof. The result follows from Theorem 4.1 once we show (4.4) holds. First we show Φ is upper semicontinuous. To show this it suffices (note Φ is compact) to show Φ is closed. Let $\{(x_\alpha, y_\alpha, w_\alpha)\}$ be a net in *graph* (Φ) with $(x_\alpha, y_\alpha, w_\alpha) \to (x,y,w)$. From (4.8) it follows that

$$f(x,y,w) \ge \limsup f(x_\alpha, y_\alpha, w_\alpha) \ge \liminf M(x_\alpha, y_\alpha) \ge M(x,y).$$

Also $w_\alpha \in G(x_\alpha)$ together with $x_\alpha \to x$, $w_\alpha \to w$ and G upper semicontinuous (so G is closed) implies $w \in G(x)$ and $f(x,y,w) \ge M(x,y)$. Consequently $f(x,y,w) = M(x,y)$, so $(x,y,w) \in$ *graph* (Φ). Thus Φ is upper semicontinuous with nonempty, compact values, so this together with (4.10) implies $\Phi \in \mathcal{D}(Q \times C, Q)$. Also (4.2) and (4.9) guarantees that $T \in \mathcal{D}(Q,C)$. As a result $F \in \mathfrak{A}_c^\kappa(Q \times C, Q \times C)$ from (4.6); here $F(x,y) = \Phi(x,y) \times T(x)$. Thus (4.4) holds. □

For the motivation and some related results in this section, the reader can refer to [4, 10, 11, 13, 14, 16].

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Ravi P. Agarwal
Department of Mathematical Sciences
Florida Institute of Technology
Melbourne, Florida 32901-6975, U.S.A.
E-mail: matravip@nus.edu.sg

Donal O'Regan
Department of Mathematics
National University of Ireland
Galway, Ireland
E-mail: donal.oregan@nuigalway.ie

Sehie Park
National Academy of Sciences, Republic of Korea
and
School of Mathematical Sciences
Seoul National University
Seoul 151-747, Korea
E-mail: shpark@math.snu.ac.kr