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# ON THE ORDER OF SPECIALITY OF A SIMPLE, SPECIAL, AND COMPLETE LINEAR SYSTEM ON A CURVE

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Abstract. The order of speciality of an ample invertible sheaf $L$ on a curve is the least integer $m$ so that $L^{\otimes m}$ is nonspecial. There is a reasonable upper bound of the order of speciality for a simple invertible sheaf in terms of its degree and projective dimension. We study the case where it reaches the upper bound. Moreover we formulate Castelnuovo's genus bound involving the order of speciality.

## 1. Introduction

When we have an ample invertible sheaf $L$ on a projective curve $X$ with $h^1(X, L) > 0$, it seems natural to direct our attention to the quantity

$$m(L) := \min\{m | h^1(X, L^{\otimes m}) = 0\}.$$

We propose calling it the *order of speciality* of $L$. In this paper, we study the integer in a restricted situation.

Let $X$ be a projective nonsingular curve of genus $g \geq 3$ over an algebraically closed field of characteristic 0, and $L$ an invertible sheaf on $X$ with $h^1(X, L) > 0$ such that the complete linear system $|L|$ corresponding $H^0(X, L)$ is simple. Here the classical terminology "simple" means that the linear system $|L|$ has no base points and the morphism $\phi_{|L|} : X \to \mathbb{P}^r$ defined by the linear system is birational onto its image $\phi_{|L|}(X)$. In this circumstance, we can see a several properties of the order of speciality $m(L)$ en route to Castelnuovo's theorem.

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A fundamental property of the order of speciality under our situation is the existence of a reasonable upper bound for $m(L)$ in terms of the degree $d$ of $L$ and the projective dimension $r$ of $L$ ([2, Chapter 3], [8, III §2 Theorem 1]). The bound is:

$$m(L) \leq \left\lceil \frac{d-r}{r-1} \right\rceil, \tag{1}$$

where $\lceil \frac{d-r}{r-1} \rceil$ denotes the integer $m_0$ satisfying the inequalities $m_0 - 1 < \frac{d-r}{r-1} \leq m_0$. We will give the details of the matter in the next section. The estimation (1) is obviously true even if $r = 2$, and when $r = 2$, equality holds in (1) if and only if $L$ is very ample, that is, $\phi_{|L|}(X)$ is a nonsingular plane curve of degree $d$. There are two ways to expand this remark. One is to study the curve $\phi_{|L|}(X)$ for an invertible sheaf $L$ next to the extremal case under keeping the condition $r = 2$, which is done in [6]. Another is to study the extremal case itself for $r \geq 3$, which is one of the subjects of this paper. Actually we will analyze the case in Theorem 2.6 for a general $r$ and will give a precise description of it in Theorem 3.2 for $r = 3$.

Another topic is to formulate Castelnuovo's genus bound involving the order of speciality (see, Theorem 2.4 completed by Corollary 2.9).

Those two topics will merge at the last section in order to study the order of speciality of the linear system of hyperplanes of a nonsingular projective curve on a surface of minimal degree.

## 2. Generality

To begin with, we clarify our situation.

SETTING 2.1. *We fix a complete irreducible nonsingular curve $X$ of genus $g > 3$ over an algebraically closed field $k$ of characteristic 0. Let $L$ be an invertible sheaf on $X$ of degree $d$ and of projective dimension $r \geq 3$. Assume that $L$ is generated by $H^0(X, L)$ and the corresponding morphism $\phi_{|L|}$ is birational onto its image. We denote by $Y$ the image $\phi_{|L|}(X)$, which is a curve of degree $d$ in $\mathbb{P}^r$. Furthermore, let $H$ be a hyperplane of $\mathbb{P}^r$ in general position and $\Gamma := Y \cap H$.*

From the exact sequence

$$0 \to \mathcal{I}_\Gamma \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_\Gamma \to 0$$

of $\mathcal{O}_{\mathbb{P}^r}$-modules, we have a natural linear map

$$H^0\left(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)\right) \overset{\alpha_n}{\to} H^0(\mathcal{O}_\Gamma)$$

for each nonnegative integer $n$. The function

$$h_\Gamma(n) = \dim_k \operatorname{Im} \alpha_n$$

is nothing but the Hilbert function of $\Gamma$, which plays an important role in Eisenbud-Harris's approach to the Castelnuovo theory [2], particularly, the following lemma is fundamental.

LEMMA 2.2. *Under Setting* 2.1, *the Hilbert function of* $\Gamma$ *has the property:*

$$h_\Gamma(n_1 + n_2) \geq \min\{d, h_\Gamma(n_1) + h_\Gamma(n_2) - 1\}.$$

*Especially, if* $h_\Gamma(n+1) < d$, *then*

$$h_\Gamma(n+1) \geq n(r-1) + r.$$

*Proof.* The second part of the assertion is obvious from the first part because $h_\Gamma(1) = r$. For the first part, see [2, Corollary 3.5]. □

Now we explain how inequality (1) is justified.

PROPOSITION 2.3. *Under Setting* 2.1, *let* $m = m(L)$. *Then we have*

$$h_\Gamma(m) < d$$

*and*

$$m \leq \left\lceil \frac{d-r}{r-1} \right\rceil.$$

*Proof.* Since $\Gamma$ is a general hyperplane section of $Y$, we may regard $\Gamma$ as a divisor on $X$. Then $\mathcal{O}_X(\Gamma) \simeq L$. Hence we have an exact sequence

$$0 \to L^{\otimes(j-1)} \to L^{\otimes j} \to \mathcal{O}_\Gamma \to 0 \tag{2}$$

of $\mathcal{O}_X$-modules for any integer $j$. From the exact sequence for $j = m$, we have a diagram

$$\begin{array}{ccccccc} H^0(X, L^{\otimes m}) & \xrightarrow{\alpha'_m} & H^0(\mathcal{O}_\Gamma) & \to & H^1(L^{\otimes(m-1)}) & \to & H^1(L^{\otimes m}) \\ \uparrow & & \uparrow \alpha_m & & & & \\ \operatorname{Sym}^m H^0(X, L) & \xrightarrow{\sim} & H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m)), & & & & \end{array}$$

where $\operatorname{Sym}^m$ is the symbol for a $m$th symmetric product, the upper horizontal sequence is exact and the square is commutative. Since $H^1(L^{\otimes(m-1)}) \neq (0)$ and $H^1(L^{\otimes m}) = (0)$ by definition, the linear map $\alpha'_m$ is not surjective. Therefore, by the diagram, we have

$$h_\Gamma(m) = \dim \operatorname{Im} \alpha_m \leq \dim \operatorname{Im} \alpha'_m < d.$$

Next we show (1). Let $m_0 := \lceil \frac{d-r}{r-1} \rceil$. Then we can write as

$$d - r = m_0(r-1) - \eta \tag{3}$$

for an integer $\eta$ with $0 \leq \eta < r-1$. If $m_0 + 1 \leq m$, then $h_\Gamma(m_0+1) \leq h_\Gamma(m) < d$. Hence $h_\Gamma(m_0+1) \geq m_0(r-1)+r$ by Lemma 2.2. Therefore we have $d > m_0(r-1)+r$, which contradicts with (3). □

THEOREM 2.4 (Castelnuovo's bound involving the order of speciality). *Under Setting* 2.1, *let* $m = m(L)$. *Then we have*

$$g \leq \tau(m, d, r), \tag{4}$$

*where*

$$\tau(m,d,r) := m(d-r) - \frac{m(m-1)}{2}(r-1).$$

*If we fix* $d$ *and* $r$ *and regard* $\tau(m,d,r)$ *as a numerical function on*

$$\left\{ m \in \mathbb{Z} | m \leq \left\lceil \frac{d-r}{r-1} \right\rceil \right\},$$

*then it is an increasing function and the maximum value coincides with Castelnuovo's number* $\pi(d,r)$ *in the sense of* [1, p.116].

*Proof.* From the exact sequence (2), we have a diagram

$$\begin{array}{ccccc} 0 \rightarrow & H^0(X, L^{\otimes(j-1)}) & \rightarrow & H^0(X, L^{\otimes j}) & \overset{\alpha'_j}{\rightarrow} & H^0(\mathcal{O}_\Gamma) \\ & & & \uparrow & & \uparrow \alpha_j \\ & & & \mathrm{Sym}^j H^0(X,L) & \overset{\sim}{\rightarrow} & H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(j)) \end{array}$$

with commutative square. Hence

$$h^0(L^{\otimes j}) - h^0(L^{\otimes(j-1)}) = \dim \mathrm{Im}\alpha'_j \geq \dim \mathrm{Im}\alpha_j = h_\Gamma(j). \tag{5}$$

Since $h_\Gamma(m) < d$ by Proposition 2.3, $h_\Gamma(j) < d$ for $j = 1, 2, \ldots, m$. Hence by Lemma 2.2, we have

$$\left\{ \begin{array}{rcl} h^0(L) - h^0(\mathcal{O}_X) & = & r \\ h^0(L^{\otimes 2}) - h^0(L) & \geq & (r-1)+r \\ h^0(L^{\otimes 3}) - h^0(L^{\otimes 2}) & \geq & 2(r-1)+r \\ \cdots & & \cdots \\ h^0(L^{\otimes m}) - h^0(L^{\otimes(m-1)}) & \geq & (m-1)(r-1)+r. \end{array} \right. \tag{6}$$

Adding these expressions together, we have

$$h^0(L^{\otimes m}) \geq \frac{m(m-1)}{2}(r-1) + mr + 1.$$

Since $h^1(L^{\otimes m}) = 0$ by definition, we get the desired bound for $g$ by the Riemann-Roch theorem.

The quadratic function on $x$

$$\tau(x,d,r) = x(d-r) - \frac{x(x-1)}{2}(r-1)$$

takes the maximum at

$$x = \frac{d-r}{r-1} + \frac{1}{2},$$

which is the mean of $\frac{d-r}{r-1}$ and $\frac{d-r}{r-1}+1$. Therefore the numerical function

$$\tau(m,d,r) : \left\{ m \in \mathbb{Z} | m \leq \left\lceil \frac{d-r}{r-1} \right\rceil \right\} \to \mathbb{Z}$$

is an increasing function.

Denoting $d-1 = m'(r-1)+\varepsilon$ with $0 \leq \varepsilon < r-1$, by definition, $\pi(d,r) = \frac{m'(m'-1)}{2}(r-1) + m'\varepsilon$. On the other hand,

$$m' = \begin{cases} m_0 & \text{if } \varepsilon \neq 0 \\ m_0 + 1 & \text{if } \varepsilon = 0, \end{cases}$$

where $m_0 = \lceil \frac{d-r}{r-1} \rceil$. So we can verify $\tau(m_0,d,r) = \pi(d,r)$ by easy calculation. □

REMARK 2.5. Since the geometry of $Y$ is well studied and a little exceptional from our point of view when $L$ is isomorphic to the canonical sheaf $\omega_X$, we exclude frequently the case from our consideration. Under Setting 2.1 with the extra condition $L \not\simeq \omega_X$, we have $d \geq 2r+1$ by Clifford' theorem.

To state our second theorem, we need a notation. Let

$$\lambda := (d-2) - m_0(r-1),$$

where $m_0 = \lceil \frac{d-r}{r-1} \rceil$. Then $\lambda$ is the remainder of the division of $d-2$ by $r-1$, i.e., $0 \leq \lambda < r-1$. In fact, by (3) we have

$$d-2 = m_0(r-1) + r - \eta - 2,$$

and

$$0 \leq r - \eta - 2 < r - 1.$$

THEOREM 2.6. *Under Setting* 2.1 *with the extra condition* $L \not\simeq \omega_X$, *we further assume that*

$$m(L) = \left\lceil \frac{d-r}{r-1} \right\rceil .$$

*If* $d \geq (\lambda+1)r+2$ *and* $m(L)\cdot r \neq d$, *then* $Y$ *lies on a surface of degree* $r-1$, *and the surface is the intersection of all quadrics through* $Y$.

The proof of the theorem will be given after some comments and preliminaries.

The conditions $d \geq (\lambda+1)r+2$ and $m(L)\cdot r \neq d$ in the theorem look untidy. A simpler but weaker statement is as follows.

COROLLARY 2.7. *Under Setting* 2.1, *if* $m(L) = \left\lceil \frac{d-r}{r-1} \right\rceil$ *and* $d > r^2$, *then the conclusion of Theorem* 2.6 *holds.*

*Proof.* Since $d > r^2$, $L \not\simeq \omega_X$. The condition $d \geq (\lambda+1)r+2$ is weaker than $d > r^2$ because $0 \leq \lambda \leq r-2$. Suppose that $d = m(L)r$. Then we have $d \geq \frac{d-r}{r-1}r$, which means $r^2 \geq d$ and contradicts with our assumption. □

The next lemma is a modification of the argument about the Castelnuovo curve to lie on a surface of minimal degree [3, pp.531–532].

LEMMA 2.8. *Under Setting* 2.1 *with the extra condition* $L \not\simeq \omega_X$, *if* $h_\Gamma(2) = 2r-1$, *then the intersection of all quadrics through* $Y$ *is a surface of degree* $r-1$.

*Proof.* Since $Y$ is nondegenerate and linearly normal in $\mathbb{P}^r$, we have a diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & H^0(\mathcal{O}_{\mathbb{P}^r}(1)) & \to & H^0(\mathcal{O}_Y(1)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(\mathcal{I}_Y(2)) & \to & H^0(\mathcal{O}_{\mathbb{P}^r}(2)) & \to & H^0(\mathcal{O}_Y(2)) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(H,\mathcal{I}_\Gamma(2)) & \to & H^0(\mathcal{O}_H(2)) & \to & H^0(\mathcal{O}_\Gamma(2)) & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Here all horizontal sequences, and vertical ones are exact. Hence $H^0(\mathcal{I}_Y(2)) \xrightarrow{\sim} H^0(H,\mathcal{I}_\Gamma(2))$ by the snake lemma, where the snake lives from the southwest to the northeast.

Let $S$ be the intersection of quadrics in $\mathbb{P}^r$ containing $Y$. Since the isomorphism $H^0(\mathcal{I}_Y(2)) \xrightarrow{\sim} H^0(H, \mathcal{I}_\Gamma(2))$ comes from taking the intersection of a quadric with $H$, $S \cap H$ is just the intersection of quadrics in $H = \mathbb{P}^{r-1}$ containing $\Gamma$. On the other hand, since $d \geq 2(r-1)+3$ by Remark 2.5 and $h_\Gamma(2) = 2(r-1)+1$ by the assumption, we can apply Castelnuovo's lemma [3, p.531]; $\Gamma$ is contained in a rational normal curve $D$ (of degree $r-1$) in $H = \mathbb{P}^{r-1}$. Any quadric of $H$ through $\Gamma$ must contain the rational normal curve $D$, because the intersection of the quadric and $D$ contains $\Gamma$ which consists of $d(> 2(r-1))$ points. Since $h_\Gamma(2) = 2r-1$,

$$\begin{aligned} h^0(H, \mathcal{I}_\Gamma(2)) &= h^0(H, \mathcal{O}_H(2)) - (2r-1) \\ &= h^0(H, \mathcal{I}_D(2)). \end{aligned}$$

Hence we have

$$H^0(H, \mathcal{I}_\Gamma(2)) = H^0(H, \mathcal{I}_D(2)),$$

which implies that $S \cap H = D$. Therefore $S$ is an irreducible surface of degree $r-1$ because $H$ is in general position. □

Before giving the proof of Theorem 2.6, we supplement "Castelnuovo's bound involving the order of speciality" with analyzing the case where equality holds in (4).

COROLLARY 2.9. *Under Setting* 2.1 *with the extra condition* $L \not\simeq \omega_X$, *if equality holds in* (4), *then* $Y$ *lies on a surface of degree* $r-1$.

*Proof.* If equality holds in (4), then each equality must hold in (6), specially, $h^0(L^{\otimes 2}) - h^0(L) = 2r-1$. Hence $h_\Gamma(2) = 2r-1$ by (5) and Lemma 2.2. So $Y$ lies on a surface of degree $r-1$ by Lemma 2.8. □

*Proof of Theorem* 2.6. First we consider the case $r = 3$, hence $d = 8$ or $d \geq 10$ by our assumptions. If $Y$ does not lie on any quadric surfaces, then we have

$$g \leq \begin{cases} d^2/6 - d/2 + 1 & (\text{if } d \equiv 0 \bmod 3) \\ d^2/6 - d/2 + 1/3 & (\text{if } d \not\equiv 0 \bmod 3) \end{cases} \tag{7}$$

(see, [2, Theorem 3.13]). On the other hand, since $L^{\otimes(m_0-1)}$ is special, we have $d\left(\lceil \frac{d-3}{2} \rceil - 1\right) \leq 2g-2$, which means

$$g \geq \begin{cases} d^2/4 - 5d/4 + 1 & (\text{if } d \text{ is odd}) \\ d^2/4 - d + 1 & (\text{if } d \text{ is even}). \end{cases} \tag{8}$$

The two bounds (7) and (8) for $g$ contradict.

From now on, we assume that $r \geq 4$. By Lemma 2.8, there is nothing to do when $h_\Gamma(2) = 2r - 1$. We will show that the inequality $h_\Gamma(2) \geq 2r$ never occur in our situation.
*Case* 1. Suppose that $h_\Gamma(2) = 2r$. We want to apply [2, Propositions 3.19 and 3.20] to our $\Gamma \subset H = \mathbb{P}^{r-1}$, which says that

> *if* $d \geq 2r+3$ *and* $h_\Gamma(2) = 2r$, *then* $\Gamma$ *lies on an elliptic normal curve* $E$ *in* $H = \mathbb{P}^{r-1}$.

So we have to show that $d \geq 2r+3$, and prove it by dividing several cases on $\lambda$ and $m_0$. Recall

$$d = m_0(r-1) + \lambda + 2 \tag{9}$$
$$\geq (\lambda+1)r + 2, \tag{10}$$

$m_0 \geq 2$ and $r \geq 4$. The desired inequality obviously holds in the case either $\lambda \geq 2$ or $m_0 \geq 3$. So remaining cases are

(i) $\lambda = 1$ and $m_0 = 2$; and
(ii) $\lambda = 0$ and $m_0 = 2$.

In case (i), we have $d = 2r+1$ by (9), but $d \geq 2r+2$ by (10); so the case is out of our consideration. In case (ii), we have $d = 2r$ by (9), but $d \geq 2r+1$ by Remark 2.5, which is absurd. Therefore it has been established that $\Gamma$ lies on an elliptic normal curve $E$ in $H = \mathbb{P}^{r-1}$.

Let us consider the diagram

$$\begin{array}{ccccccc} H^0(\mathbb{P}^{r-1}, \mathcal{O}(m_0)) & & & & & & \\ \gamma \downarrow & \searrow^{\alpha'_{m_0}} & & & & & \\ H^0(\mathcal{O}_E(m_0)) & \xrightarrow{\beta} & H^0(\mathcal{O}_\Gamma(m_0)) & \to & H^1(\mathcal{O}_E(m_0)(-\Gamma)) & \to & 0, \end{array}$$

where the horizontal sequence is exact and the triangle is commutative. Note that $\gamma$ is surjective because $E$ is projectively normal. Since $h_\Gamma(m_0) < d$, the linear map $\alpha'_{m_0} = \beta \circ \gamma$ is not surjective, and hence neither is $\beta$. Therefore $H^1(\mathcal{O}_E(m_0)(-\Gamma)) \neq 0$, which means that either $\deg \mathcal{O}_E(m_0)(-\Gamma) < 0$ or $\mathcal{O}_E(m_0)(-\Gamma) \simeq \mathcal{O}_E$. Since $rm_0 \neq d$ by our assumption, we have $rm_0 \leq d-1$, which implies $m_0 \leq \lambda + 1$ because of (9). Using (9) again, we have $d \leq (\lambda+1)r + 1$. But it contradicts with our original assumption.
*Case* 2. Suppose that $h_\Gamma(2) \geq 2r+1$. Since $h_\Gamma(m_0) < d$, we have

$$d > h_\Gamma(m_0) \geq \begin{cases} m_0 r & (\text{if } m_0 \text{ is odd}) \\ m_0 r + 1 & (\text{if } m_0 \text{ is even}) \end{cases}$$

by Lemma 2.2. By the same computation as we did at the last part in Case 1, we get a contradiction. So we can exclude this case too for the possibilities. □

## 3. Curves in 3-space

We start a detailed study of our problem for $r = 3$ with an example which shows the assumption $m(L) \cdot r \neq d$ in Theorem 2.6 to be actually necessary.

EXAMPLE 3.1. Let $X$ be a nonsingular curve in $\mathbb{P}^3$ which is a complete intersection of two cubics, and $L$ the invertible sheaf $\mathcal{O}_X(1)$ corresponding to the plane sections. Then we have
(a) the genus of $X$ is 10;
(b) the degree of $L$ is 9 and $h^0(X, L) = 4$;
(c) $L^{\otimes 2} \simeq \omega_X$,
which can be found in [5, IV, 6.4.3]. Hence $m(L) = 3$, which is just $\left\lceil \frac{d-r}{r-1} \right\rceil$ for $d = 9$ and $r = 3$. Moreover, our $d$, $r$ and $m(L)$ fulfill the second condition $d \geq (\lambda + 1)r + 2$ in Theorem 2.6, but fail in the third condition. Since $X$ is a complete intersection of two cubics, it does not lie on any quadric surface.

The next theorem is a complete version of Theorem 2.6 in the case $r = 3$. In the theorem, we do not assume the invertible sheaf $L$ in question not to be canonical.

THEOREM 3.2. *Let $L$ be a simple, special invertible sheaf of degree $d$ on a curve $X$ of genus $g$ with $h^0(X, L) = 4$. If $m(L) = \lceil \frac{d-3}{2} \rceil$, then $Y := \phi_{|L|}(X)$ lies on a quadric surface except in the following cases:*
(a) *$Y$ is a nonsingular curve which is a complete intersection of two cubics* (in this case, $d = 9$ and $g = 10$);
(b) *$X$ is a nonhyperelliptic, nontrigonal curve of genus 5, and $L \simeq \omega_X(-P)$ for a point $P \in X$* (in this case, $d = 7$).
*In those exceptional cases, $Y$ does not actually lie on any quadric surfaces, but lie on a cubic surface.*

*Proof.* By Clifford's theorem, we have $d \geq 6$. Since there is no simple, special invertible sheaf on a hyperelliptic curve, $d = 6$ if and only if $L \simeq \omega_X$; hence in this case $Y$ is a canonical curve of genus 4, which lies on a quadric surface [5, IV, 6.4.2]. On the other hand, if $d > 9$, then $Y$ lies on a quadric surface by Corollary 2.7.

So we have to examine whether $Y$ lies on a quadric surface for $d = 7, 8$ and 9. If $d = 8$, then $m(L) = 3$ and $\lambda = 0$ because $r = 3$; and those integers satisfy the assumption of Theorem 2.6, hence $Y$ lies on a quadric surface.

Assuming $d = 9$, we have $h^1(L^{\otimes 2}) > 0$ because $m(L) = 3$. Hence $g \geq 10$ because $\deg L^{\otimes 2} \leq 2g - 2$. Since $h^1(L^{\otimes 3}) = 0$, we have $h^0(L^{\otimes 3}) \leq 18$ by the Riemann-Roch theorem. Now we consider the exact sequence

$$0 \to I_3 \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(X, L^{\otimes 3}),$$

where $I_3$ is the homogeneous part of degree 3 of the ideal of $Y$ in $\mathbb{P}^3$. Since $h^0(\mathcal{O}_{\mathbb{P}^3}(3)) - h^0(L^{\otimes 3}) \geq 2$, there are two cubic surfaces $S_1$ and $S_2$ such that $Y \subseteq S_1 \cap S_2$. Suppose that $Y$ does not lie on any quadric surface. Then $S_1$ and $S_2$ are irreducible, and $S_1 \cap S_2$ is one-dimensional and of arithmetic genus $p_a(S_1 \cap S_2) = 10$ [5, I, Exercise 7.2]. Since $\deg Y = 9$, we have $Y = S_1 \cap S_2$, and $Y$ is nonsingular because

$$10 \leq g \leq p_a(Y) = p_a(S_1 \cap S_2) = 10.$$

Conversely, a nonsingular curve which is a complete intersection of two cubic surfaces does not lie on any quadric surface.

Finally, we handle the case $d = 7$. In this case, $m(L) = 2$. Suppose that $Y$ does not lie on any quadric surface. Since the natural map $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(X, L^{\otimes 2})$ is injective and $h^1(L^{\otimes 2}) = 0$, we have $g \leq 5$; and equality must hold because $7 \leq \deg L \leq 2g - 2$. Since $\deg \omega_X = 8$ for a nonsingular curve $X$ of genus 5 and $L$ is special of degree 7, there is a point $P \in X$ such that $L \simeq \omega_X(-P)$. Conversely, it is obvious that $m(\omega_X(-P)) = 2$ for a nonsingular curve $X$ of genus 5. In order to complete our proof, we will show the next lemma. □

LEMMA 3.3. *Let $X$ be a nonhyperelliptic curve of genus 5, and $P \in X$. Then:*

(a) *$\omega_X(-P)$ is simple;*
(b) *if the curve $Y := \phi_{|\omega_X(-P)|}(X)$ lies on a quadric surface, then $X$ is trigonal;*
(c) *if $X$ is trigonal, then $Y$ lies on a quadric surface.*

*Proof.* (3.3) Since $X$ is nonhyperelliptic, we have $h^0(X, \omega_X(-P - Q)) = 3$ for any $P, Q \in X$. In particular, the linear system $|\omega_X(-P)|$ is free from base points. If $h^0(X, \omega_X(-P - Q - R)) = 3$ for three points $P, Q, R \in X$, then $\dim |P + Q + R| = 1$ by the Riemann-Roch theorem. Since the number of $g^1_3$'s on $X$ is at most 1, for the fixed point $P$ the number of pairs $\{Q, R\}$ so that $h^0(\omega_X(-P - Q - R)) = 3$ is at most 1, that is, the morphism $X \setminus \{Q, R\} \to Y \setminus \{\phi_{|\omega_X(-P)|}(Q) = \phi_{|\omega_X(-P)|}(R)\}$ induced by $\phi_{|\omega_X(-P)|}$ is an isomorphism.

(3.3) We show the second statement. First we consider the case where $Y$ lies on a nonsingular quadric surface, which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Regarding $Y$ as a divisor on the surface, we can see that $Y$ is linearly equivalent to $al + bm$ for some integers $a, b \in \mathbb{Z}$, where $l = \mathbb{P}^1 \times \mathrm{pt}$, $m = \mathrm{pt} \times \mathbb{P}^1$. Since $7 = \deg Y = a + b$ and $5 \leq p_a(Y) = (a-1)(b-1)$, we have $Y$ is linearly equivalent to $3l + 4m$ or $4l + 3m$, which means that the first projection from $\mathbb{P}^1 \times \mathbb{P}^1$ or the second gives a morphism of degree 3 from $X$ to $\mathbb{P}^1$ via the normalization $X \to Y$.

Next we consider the case where $Y$ lies on a singular irreducible quadric surface $S$, which is a cone over a conic. The blowing-up $\tilde{S} \to S \subset \mathbb{P}^3$ whose center is the vertex of $S$ coincides with $\phi_{|C_2+2f|} : F_2 \to \mathbb{P}^3$, where $F_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$, $C_2$ is the minimal section of the ruled surface $F_2 \xrightarrow{\pi} \mathbb{P}^1$, and $f$ is the divisor class of a fibre of $\pi$. Let $\tilde{Y}$ be the strict transform of $Y$, and $|aC_2 + bf|$ the linear system on $F_2$ in which $\tilde{Y}$ belongs. Then we have $7 = \deg Y = b$ and $5 \leq p_a(\tilde{Y}) = -a^2 + ab - b + 1$. Hence we have that $\tilde{Y}$ is linearly equivalent to $3C_2 + 7f$ or $4C_2 + 7f$. But since the linear system $|4C_2 + 7f|$ has no irreducible member [5, V, 2.18], $\tilde{Y} \in |3C_2 + 7f|$. Hence $\pi|_{\tilde{Y}}$ gives a morphism $X \to \mathbb{P}^1$ of degree 3.

(3.3) It is easy to see the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\phi_{|\omega_X|}} & \phi_{|\omega_X|}(X) & \subset & \mathbb{P}^4 \\ & \phi_{|\omega_X(-P)|} \searrow & & & \downarrow p \\ & & Y & \subset & \mathbb{P}^3 \end{array}$$

is commutative, where $p$ is the projection with center $\phi_{|\omega_X|}(P)$. Since $X$ is trigonal of genus 5, the canonical curve lies on a rational ruled surface $F_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ embedded into $\mathbb{P}^4$ by $|C_1 + 2f|$, where $C_1$ is the minimal section [7]. Since $\deg F_1 = (C_1 + 2f)^2 = 3$ and $\phi_{|\omega_X|}(P) \in F_1$, the surface $\overline{p(F_1)}$, which obviously contains $Y$, is of degree 2. This completes the proof. □

## 4. Curves on a surface of minimal degree

An irreducible nondegenerate surface $S$ of minimal degree $r - 1$ in $\mathbb{P}^r$ is either

(1) a nonsingular scroll; or

(2) a cone over a rational normal curve; or
(3) a Veronese surface

(see, [3, Proposition at p.525]). In this section, we analyze the order of speciality of $\mathcal{O}_Y(1)$ for a nonsingular curve $Y$ on a surface $S$ of minimal degree in $\mathbb{P}^r$ with $h^1(Y, \mathcal{O}_Y(1)) > 0$.

Throughout this section, $F_e$ denotes the rational ruled surface with invariant $e$. To avoid an exceptional situation, we assume that $e > 0$, however, most of the analysis works well even if $e = 0$.

We denote by $C_e$ the minimal section of the ruled structure $\pi : F_e \to \mathbb{P}^1$. Moreover we denote by $f$ the divisor class of a fibre of $\pi$.

In order to compute cohomology of invertible sheaves on $F_e$, it would be convenient to give a summary of fundamental formulas. Since $\operatorname{Pic} F_e = \mathbb{Z}C_e \oplus \mathbb{Z}f$ [5, V, 2.3], each invertible sheaf can be written as $\mathcal{O}_{F_e}(\alpha C_e + \beta f)$ for certain integers $\alpha$ and $\beta$, in particular,

$$\omega_{F_e} \simeq \mathcal{O}_{F_e}(-2C_e - (2+e)f), \tag{11}$$

where $\omega_{F_e}$ is the canonical sheaf on $F_e$ [5, V, 2.10]. By [4, 4.2.7] with [5, V, 2.8],

$$\pi_*\mathcal{O}_{F_e}(\alpha C_e + \beta f) \simeq \begin{cases} S^\alpha(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \otimes \mathcal{O}_{\mathbb{P}^1}(\beta) & \text{if } \alpha \geq 0 \\ 0 & \text{if } \alpha < 0, \end{cases} \tag{12}$$

where the symbol $S^\alpha$ denotes the $\alpha$th symmetric product, and by [5, V, 2.5] with the Riemann-Roch theorem,

$$\chi(\mathcal{O}_{F_e}(\alpha C_e + \beta f)) = -\frac{\alpha(\alpha+1)}{2}e + (\alpha+1)(\beta+1). \tag{13}$$

### 4.1. Curves on a nonsingular scroll

A nonsingular scroll isomorphic to $F_e$ is given by a linear system $|C_e + nf|$ with $n > e$. We identify $F_e$ with $\phi_{|C_e+nf|}(F_e) \subset \mathbb{P}^r$, where

$$r = 2n - e + 1. \tag{14}$$

We consider a nonsingular curve $Y$ on $F_e$ and the invertible sheaf $L := \mathcal{O}_{F_e}(C_e + nf)|_Y$ such that $h^1(Y, L) > 0$. Let $Y \sim aC_e + bf$ in $\operatorname{Pic}(F_e)$. We denote by $g$ the genus of $Y$, and by $d$ the degree of $Y$ in $\mathbb{P}^r$. Then

$$g = (a-1)(b-1) - \frac{a(a-1)}{2}e \tag{15}$$

by (11) with the adjunction formula, and

$$d = (n-e)a + b. \tag{16}$$

LEMMA 4.1. *We have* $b \geq ae$, $a \geq 3$, *and* $b - n \geq e + 2$. *Moreover the linear system, which is special or not, on* $Y$ *coming from the linear system of hyperplanes of* $\mathbb{P}^r$ *is complete, that is,* $H^0(F_e, \mathcal{O}(C_e + nf)) \to H^0(Y, L)$ *is surjective, if and only if* $b - n \geq (a-2)e + 1$.

*Proof.* For the first inequality, see [5, V, 2.18]. To see the remainder of the statements, let us consider the exact sequence

$$0 \to \mathcal{O}_{F_e}(-aC_e - bf) \to \mathcal{O}_{F_e} \to \mathcal{O}_Y \to 0. \tag{17}$$

Tensoring (17) with $\mathcal{O}_{F_e}(C_e + nf)$ gives the exact sequence

$$0 \to \mathcal{O}_{F_e}((1-a)C_e + (n-b)f) \to \mathcal{O}_{F_e}(C_e + nf) \to L \to 0.$$

From (11) with Serre's duality and (12), $h^2(\mathcal{O}_{F_e}(C_e + nf)) = 0$. On the other hand, since $h^0(\mathcal{O}_{F_e}(C_e+nf)) = \chi(\mathcal{O}_{F_e}(C_e+nf))$ by (14) and (13), $h^1(\mathcal{O}_{F_e}(C_e + nf)) = 0$. Hence

$$H^0(\mathcal{O}_{F_e}(C_e+nf)) \to H^0(L) \to H^1(\mathcal{O}_{F_e}((1-a)C_e+(n-b)f)) \to 0 \tag{18}$$

and

$$0 \to H^1(L) \to H^2(\mathcal{O}_{F_e}((1-a)C_e + (n-b)f)) \to 0 \tag{19}$$

are exact. Since $h^2(\mathcal{O}_{F_e}((1-a)C_e+(n-b)f)) = h^0(\mathcal{O}_{F_e}((a-3)C_e+(b-n-e-2)f))$ by Serre's duality, $h^1(L) > 0$ if and only if $h^0(\mathbb{P}^1, S^{a-3}(\mathcal{O}\oplus\mathcal{O}(-e))\otimes\mathcal{O}(b-n-e-2)) > 0$, which means that $a \geq 3$ and $b \geq n+e+2$.

From (18), the linear system on $Y$ is complete if and only if $h^1(\mathcal{O}_{F_e}((1-a)C_e + (n-b)f)) = 0$. Since

$$\begin{aligned} &h^1(\mathcal{O}_{F_e}((1-a)C_e + (n-b)f)) \\ &\quad = h^1(\mathcal{O}_{F_e}((a-3)C_e + (b\text{–}n\text{–}e\text{–}2)f))\text{(by Serre's duality)} \\ &\quad = h^1(\mathbb{P}^1, S^{a-3}(\mathcal{O}\oplus\mathcal{O}(-e))\otimes\mathcal{O}(b\text{–}n\text{–}e\text{–}2))\text{(by [5, V, 2.4] with (12))}, \end{aligned}$$

the vanishing is equivalent to the condition

$$-(a-3)e + b - n - e - 2 \geq -1.$$

This completes the proof. □

THEOREM 4.2. $m(L) = \min\left\{a-1,\, 1 + \left[\dfrac{b-e-2}{n}\right]\right\}$, *where* $[(b-e-2)/n]$ *is the integer part of* $(b-e-2)/n$.

*Proof.* Let $m$ be an integer at least 2. By the similar analysis in the proof of the above lemma, we have that $h^1(L^{\otimes(m-1)}) > 0$ if and only if $a - 1 \geq m$ and $(b-e-2)/n \geq m-1$. □

Now we explain a characterization of extremal curves on a nonsingular scroll in the sense of Castelnuovo's genus bound from our point of view (cf. [1, III 2.5]).

COROLLARY 4.3. *$g = \pi(d,r)$ if and only if*
(a) *$m(L) = a-1$ and $b \leq an+1$; or*
(b) *$m(L) = a-2$ and $\dfrac{d-r}{r-1} = a-2$.*

*Proof.* First we rewrite the genus formula (15) in terms of $a$, $d$ and $r$. From (14) and (16),

$$\begin{aligned} b-1 &= d-(\frac{r+e-1}{2}-e)a-1 \\ &= (d-r)-\frac{r-1}{2}(a-2)+\frac{e}{2}a. \end{aligned} \tag{20}$$

Hence (15) can be expressed as

$$g = (a-1)(d-r) - \frac{(a-1)(a-2)}{2}(r-1). \tag{21}$$

Since $m(L) \leq a-1$, we may write as

$$m(L) = a-1-j \text{ with } j \geq 0.$$

Then, by Theorem 2.4,

$$g \leq (a-1-j)(d-r) - \frac{(a-1-j)(a-2-j)}{2}(r-1), \tag{22}$$

and $g = \pi(d,r)$ if and only if equality holds in (22) and $m(L) = \left\lceil \frac{d-r}{r-1} \right\rceil$. Taking account of (21), equality holds in (22) if and only if

$$j\left((r-1)j + 2(d-r) - (2a-3)(r-1)\right) = 0. \tag{23}$$

Suppose that $g = \pi(d,r)$. Then since $a-1-j = m(L) = \left\lceil \frac{d-r}{r-1} \right\rceil$, there is an integer $\eta$ with $0 \leq \eta < r-1$ such that

$$d-r = (a-1-j)(r-1)-\eta.$$

Hence (23) can be expressed as

$$j\left((r-1)(1-j)-2\eta\right) = 0.$$

Hence $j=0$ or "$j=1$ and $\eta=0$." In the latter case, the condition means that $m(L) = a-2 = \frac{d-r}{r-1}$. In order to handle the former case, we should note the identity

$$(d-r) = (a-1)(r-1) + (b-an-1), \tag{24}$$

which comes from (20) and (14). So we have $b - an - 1 \leq 0$ because $\left\lceil \frac{d-r}{r-1} \right\rceil = a - 1$.

Conversely, in each of the cases, (23) obviously holds, and $m(L) = \left\lceil \frac{d-r}{r-1} \right\rceil$ in case (b). In case (a), $\left\lceil \frac{d-r}{r-1} \right\rceil \leq a - 1$ by (24). Since $m(L) \leq \left\lceil \frac{d-r}{r-1} \right\rceil$ by Theorem 2.6 and $m(L) = a - 1$ by our assumption, we have $m(L) = \left\lceil \frac{d-r}{r-1} \right\rceil$ in case (a) too. □

REMARK 4.4. As we will see later, any nonsingular curve $Y$ on a cone over a rational normal curve in $\mathbb{P}^r$ or a Veronese surface in $\mathbb{P}^5$ with $h^1(Y, \mathcal{O}_Y(1)) > 0$ is extremal in the sense of Castelnuovo's genus bound. However, we can find a nonsingular scroll $F_e \subset \mathbb{P}^r$ and a nonsingular curve $Y$ on the surface such that the linear system $|L|$ is complete, $h^1(L) > 0$, $m(L) = \left\lceil \frac{d-r}{r-1} \right\rceil$ but $g < \pi(d,r)$. For example, consider the ruled surface $F_2$ with the linear system $|C_2 + 4f|$. Then there is a nonsingular curve $Y$ so that $Y \sim 4C_2 + 9f$. Then we have $r = 7$, $d = 17$, $m(L) = 2$, $g = 12$ and $\pi(d,r) = 14$.

### 4.2. Curves on a cone over a rational normal curve

A cone $S \subset \mathbb{P}^{e+1}$ $(e \geq 2)$ over a rational normal curve of degree $e$ in $\mathbb{P}^e$ is the image of the morphism from the rational ruled surface $F_e$ by means of the linear system $|C_e + ef|$. Let $Y$ be a nonsingular curve on $S$, and $L = \mathcal{O}_{\mathbb{P}^{e+1}}(1)|_Y$. We identify $Y$ with its strict transform on $F_e$. Then $L$ can be identified with $\mathcal{O}_{F_e}(C_e + ef)|_Y$, and $Y$ can be regarded as a divisor on $F_e$. Let $aC_e + bf \sim Y$. Since $Y$ is nonsingular on $S$, the intersection number $(Y.C_e)$ on $F_e$ is either 0 or 1. So, in the former case, $d = ae$ and $g = (ae-2)(a-1)/2$, where $d$ is the degree of $Y$ in $\mathbb{P}^{e+1}$ and $g$ is the genus of $Y$; and in the latter case, $d = ae+1$ and $g = ae(a-1)/2$. In particular, we have $\left\lceil \frac{d-r}{r-1} \right\rceil = a - 1$ in each case. Hence, as we can see easily, $\pi(d,r) = \tau(a-1, d, r) = g$, and hence $m(L) = a - 1$.

PROPOSITION 4.5. *Under the above setting, we have*

(a) $H^0(F_e, \mathcal{O}_{F_e}(C_e + ef)) \to H^0(Y, L)$ *is surjective,*
(b) $h^1(L) > 0$ *if and only if* $a \geq 3$.

*Proof.* These can be proved by the similar idea of the proof of Lemma 4.1. So we omit it. □

### 4.3. Curves on the Veronese surface

Let $Y$ be a nonsingular plane curve of degree $e \geq 4$ and $L = \mathcal{O}_Y(s)$ with $1 \leq s \leq e-3$. Then it is easy to see that

$$m(L) = \left[\frac{e-3}{s}\right] + 1.$$

Therefore any nonsingular curve $Y$ on the Veronese surface with $h^1(\mathcal{O}_{\mathbb{P}^5}(1)|_Y) > 0$ is extremal. In fact,

$$m(\mathcal{O}_{\mathbb{P}^5}(1)|_Y) = \begin{cases} \frac{e-1}{2} & \text{if } e \text{ is odd} \\ \frac{e-2}{2} & \text{if } e \text{ is even,} \end{cases}$$

each of which coincides with $\left[\frac{d-r}{r-1}\right]$ for $d = 2e$ and $r = 5$, and

$$\tau\left(\frac{e-1}{2}, 2e, 5\right) = \tau\left(\frac{e-2}{2}, 2e, 5\right) = \frac{(e-1)(e-2)}{2} = \text{the genus of } Y.$$

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