J. Korean Math. Soc. **39** (2002), No. 4, pp. 621–651

$(f, 2)$–ROTATIONAL EXTENDED STEINER TRIPLE SYSTEMS WITH $f = 2$ AND $f = 3$

CHUNG JE CHO

ABSTRACT. An extended Steiner triple system of order v, denoted by $ESTS(v)$, is said to be (f,k)*–rotational* if it admits an automorphism consisting of exactly f fixed elements and k cycles of length $\frac{v-f}{k}$. In this paper, we obtain a necessary and sufficient condition for the existence of $(f, 2)$–rotational extended Steiner triple systems with $f = 2$ and $f = 3$.

1. Introduction

An *extended Steiner triple system of order* v, denoted $ESTS(v)$, is an ordered pair $(V, \mathfrak{B})$ where V is a set of v elements and $\mathfrak{B}$ is a set of triples of (not necessarily distinct) elements of V, called *blocks*, such that every unordered pair of (not necessarily distinct) elements of V occurs in exactly one block of $\mathfrak{B}$.

In an $ESTS(v)$, there are three types of blocks of the forms

$$\{a, a, a\}, \qquad \{a, a, b\}, \qquad \{a, b, c\},$$

where a, b and c are distinct elements. If there is a block of the form $\{a, a, a\}$, then such an element a is called an *idempotent*, but the element b which forms a block $\{b, b, c\}$ with $b \neq c$ is a *nonidempotent*. We will denote by $ESTS(v, \rho)$ an extended Steiner triple system of order v with ρ idempotents. In 1972, Johnson and Mendelsohn [6] obtained a necessary condition for the existence of an $ESTS(v, \rho)$, and, in 1978, Bennett and Mendelsohn [1] showed the necessary condition was also sufficient.

Received October 15, 2001. Revised March 11, 2002.

2000 Mathematics Subject Classification: 05B05, 05B07, 05B10.

Key words and phrases: extended Steiner triple system, automorphism of an extended Steiner triple system.

This research was supported by the KISTEP in 2000-2001.

THEOREM 1.1 [1,6]. *Let $0 \leq \rho \leq v$. Then there exists an $ESTS(v,\rho)$ if and only if*

(i) $v \equiv 0 \pmod 3$ *and* $\rho \equiv 0 \pmod 3$, *or*
(ii) $v \equiv 1, 2 \pmod 3$ *and* $\rho \equiv 1 \pmod 3$, *but*
(iii) *when v is even,* $\rho \leq \frac{v}{2}$, *and*
(iv) *when* $\rho = v-1$, $v = 2$.

An *automorphism* of an $ESTS(v,\rho)$ $(V,\mathfrak{B})$ is a permutation of V which preserves the blocks of $\mathfrak{B}$. An $ESTS(v,\rho)$ is said to be (f,k)–*rotational* if it admits an automorphism consisting of exactly f fixed elements and k cycles of length $\frac{v-f}{k}$. Solutions to the existence problem of an (f,k)–rotational $ESTS(v,\rho)$ were first given when $f=0$, $k=1$, as the existence of cyclic extended triple systems [2], when $f=0$, $k \geq 2$, as that of k–regular extended triple systems [3] which were completely determined for $k=2,3$ and partially for $k=4$, when $f=1$, $k \geq 1$, as that of k–rotational extended triple systems [2] which were completely determined for $k=1,2$ and partially for $k=3$.

If $(V,\mathfrak{B})$ is an $ESTS(v,\rho)$ with an automorphism α, then the cyclic group generated by α, $<\alpha>$, acts on the set V. So the whole blocks $\mathfrak{B}$ are partitioned into disjoint orbits of blocks under the cyclic group $<\alpha>$. We say that a set of blocks which are taken from each of the orbits exactly one is called a *set of starter blocks* for the system under the automorphism α.

An (A,k)-*system* is a set of ordered pairs $\{(a_r,b_r)|r=1,2,\ldots,k\}$ that partition the set $\{1,2,\ldots,2k\}$ with the property that $b_r - a_r = r$ for $r=1,2,\ldots,k$; and there exists an (A,k)-system if and only if $k \equiv 0$ or $1 \pmod 4$ [8, 9]. A (B,k)-*system* is a set of ordered pairs $\{(a_r,b_r)|r=1,2,\ldots,k\}$ that partition the set $\{1,2,\ldots,2k-1,2k+1\}$ with the property that $b_r - a_r = r$ for $r=1,2,\ldots,k$; and there exists a (B,k)-system if and only if $k \equiv 2$ or $3 \pmod 4$ [7, 8]. A (C,k)-*system* is a set of ordered pairs $\{(a_r,b_r)|r=1,2,\ldots,k\}$ that partition the set $\{1,2,\ldots,k,k+2,\ldots,2k+1\}$ with the property that $b_r - a_r = r$ for $r=1,2,\ldots,k$; and there exists a (C,k)-system if and only if $k \equiv 0$ or 3 (mod 4) [8]. A (D,k)-*system* is a set of ordered pairs $\{(a_r,b_r)|r=1,2,\ldots,k\}$ that partition the set $\{1,2,\ldots,k,k+2,\ldots,2k,2k+2\}$ with the property that $b_r - a_r = r$ for $r=1,2,\ldots,k$; and there exists a (D,k)-system if and only if $k \equiv 1$ or $2 \pmod 4$ and $k \neq 1$ [8].

In this paper, we obtain a necessary and sufficient condition for the existence of $(f,2)$–rotational extended Steiner triple systems with $f=2$ and $f=3$.

2. (2, 2)–rotational extended Steiner triple systems

Suppose that there exists a $(2,1)$–rotational $ESTS(v,\rho)$ whose element-set is $V = \{\infty_1, \infty_2\} \cup Z_{v-2}$ and with an automorphism

$$\alpha = (\infty_1)(\infty_2)(0\ 1\ \cdots\ v-3).$$

Then either $\{\infty_1, \infty_1, \infty_1\}$ and $\{\infty_1, \infty_2, \infty_2\}$, or $\{\infty_2, \infty_2, \infty_2\}$ and $\{\infty_2, \infty_1, \infty_1\}$ must be blocks. So there must exist starter blocks of the forms

$$\{\infty_1, 0, a\} \text{ and } \{\infty_2, 0, b\}$$

for some $a \neq b$ in Z_{v-2} and then at least one of the pairs $\{\infty_1, 0\}$ and $\{\infty_2, 0\}$ occur twice. Thus there is no $(2,1)$–rotational extended Steiner triple systems.

In a $(2,k)$–rotational $ESTS(v,\rho)$, if $\{a,a,a\}$ is a block then the length of its orbit is either 1 or $\frac{v-2}{k}$; so we have the following lemma.

LEMMA 2.1. *If there exists a $(2,k)$–rotational $ESTS(v,\rho)$, then*

$$\rho = 1 \text{ or } 1 + n \cdot \frac{v-2}{k}$$

for $n = 0, 1, \ldots, k$.

REMARK 2.2. If there exists a $(2,2)$–rotational $ESTS(v,\rho)$, then

$$\rho = 1, \ \frac{v}{2}, \ \text{ or } v-1.$$

But if $\rho = v-1$, v must be 2 (by Theorem 1.1); so if $\rho = v-1$ then $v = 2$ and $\rho = 1$.

LEMMA 2.3. *A necessary condition for the existence of a $(2,2)$–rotational $ESTS(v,\rho)$ is*

(i) $\rho = 1$ *and* $v \equiv 2, 4 \pmod 6$, *or*

(ii) $\rho = \frac{v}{2}$ *and* $v \equiv 0, 2 \pmod 6$.

Proof. (i) If $\rho = 1$, then, by Theorem 1.1, $v \equiv 1$ or $2 \pmod 3$. But $\frac{v-2}{2}$ must be an integer. Thus v satisfies both $v \equiv 1, 2 \pmod 3$ and $v \equiv 0 \pmod 2$ and hence $v \equiv 2$ or $4 \pmod 6$.

(ii) If $\rho = \frac{v}{2}$ then, by Theorem 1.1, $\frac{v}{2} \equiv 0$ or $1 \pmod 3$ and hence $v \equiv 0$ or $2 \pmod 6$. □

Now, we will construct $(2,2)$–rotational $ESTS(v,\rho)$s. We assume that our $(2,2)$–rotational $ESTS(v,\rho)$s have the element-set,

$$V=\{\infty_1,\infty_2\}\cup\left(Z_{\frac{v-2}{2}}\times\{1,2\}\right),$$

and the corresponding automorphism is

$$\alpha=(\infty_1)(\infty_2)\left(0_1\ 1_1\ \cdots\ \left(\frac{v-2}{2}-1\right)_1\right)\left(0_2\ 1_2\ \cdots\ \left(\frac{v-2}{2}-1\right)_2\right),$$

where we write for brevity x_i instead of (x,i).

LEMMA 2.4. *There exists a $(2,2)$–rotational $ESTS(v,1)$ for $v=8$ and 20.*

Proof. If $v=8$, then the following triples

$$\{\infty_1,\infty_1,\infty_1\},\quad \{\infty_1,\infty_2\infty_2\},\quad \{\infty_1,0_1,0_2\},$$
$$\{\infty_2,0_1,1_2\},\quad \{0_1,0_1,2_2\},\quad \{0_1,1_1,2_1\},\quad \{0_2,0_2,1_2\}$$

form a set of starter blocks for a $(2,2)$–rotational $ESTS(8,1)$.

If $v=20$, the following triples

$$\{\infty_1,\infty_1,\infty_1\},\quad \{\infty_1,\infty_2,\infty_2\},$$
$$\{0_1,1_1,3_1\},\quad \{0_1,0_1,4_1\},\quad \{0_2,0_2,0_1\},$$
$$\{0_2,1_2,2_1\},\quad \{0_2,2_2,7_1\},\quad \{0_2,4_2,8_1\}$$
$$\{\infty_1,0_2,6_1\},\quad \{\infty_2,0_2,3_1\},\quad \{0_2,3_2,6_2\}$$

form a set of starter blocks of a $(2,2)$–rotational $ESTS(20,1)$. □

LEMMA 2.5. *If $v\equiv 8 \pmod{12}$, then there exists a $(2,2)$–rotational $ESTS(v,1)$.*

Proof. Let $v=12t+8=2(6t+3)+2$ and let $t\geq 0$ be an arbitrary integer. The cases $t=0$ and 1 have been treated in Lemma 2.4.

If $t\geq 2$, then the following triples

$$\{\infty_1,\infty_1,\infty_1\},\qquad \{\infty_1,\infty_2,\infty_2\},$$
$$\{0_1,r_1,(b_r+t)_1\},\qquad r=1,2,\ldots,t,$$

where $\{(a_r,b_r)|r=1,2,\dots,t\}$ is a (C,t)-system if $t\equiv 0$, 3 (mod 4), or a (D,t)-system if $t\equiv 1$, 2 (mod 4), and

$$\begin{array}{ll}\{0_2,(2t+1)_2,(4t+2)_2\}, & \{0_2,0_2,(3t+1)_2\},\\ \{0_1,(2t+1)_1,(-c_{2t+1})_2\}, & \\ \{\infty_1,0_2,(c_{3t+1})_1\}, & \{\infty_2,0_2,(d_{3t+1})_1\},\\ \{0_2,r_2,(d_r)_1\}, & r=1,2,\dots,2t,2t+2,\dots,3t,\end{array}$$

where $\{(c_r,d_r)|r=1,2,\dots,3t+1\}$ is an $(A,3t+1)$-system if $t\equiv 0$, 1 (mod 4), or a $(B,3t+1)$-system if $t\equiv 2$, 3 (mod 4), and $6t+3$ should be read as 0, and

$$\begin{array}{ll}\{0_1,0_1,0_2\} & \text{if } t\equiv 0 \text{ or } 1 \pmod 4, \text{ or}\\ \{(6t+2)_1,(6t+2)_1,0_2\} & \text{if } t\equiv 2 \text{ or } 3 \pmod 4,\end{array}$$

form a set of starter blocks for a $(2,2)$–rotational $ESTS(v,1)$. □

LEMMA 2.6. *If $v\equiv 4$ (mod 12), then there exists a $(2,2)$–rotational $ESTS(v,1)$.*

Proof. Let $v=12t+4=2(6t+1)+2$ and let $t\geq 0$ be an arbitrary integer. If $t=0$, the following triples

$$\{\infty_1,\infty_1,\infty_1\},\{\infty_1,\infty_2,\infty_2\},\{\infty_1,0_1,0_2\},\{\infty_2,0_1,0_1\},\{\infty_2,0_2,0_2\}$$

form a set of starter blocks for a $(2,2)$–rotational $ESTS(4,1)$.

If $t\geq 1$, then the following triples

$$\begin{array}{ll}\{\infty_1,\infty_1,\infty_1\}, & \{\infty_1,\infty_2,\infty_2\},\\ \{0_1,r_1,(b_r+t)_1\}, & r=1,2,\dots,t,\end{array}$$

where $\{(a_r,b_r)|r=1,2,\dots,t\}$ is an (A,t)-system if $t\equiv 0$, 1 (mod 4), or a (B,t)-system if $t\equiv 2$, 3 (mod 4), and

$$\begin{array}{ll}\{0_2,0_2,(3t)_2\}, & \\ \{\infty_1,0_2,(c_{3t})_1\}, & \{\infty_2,0_2,(d_{3t})_1\},\\ \{0_2,r_2,(d_r)_1\}, & r=1,2,\dots,3t-1,\end{array}$$

where $\{(c_r, d_r)|r = 1, 2, \ldots, 3t\}$ is an $(A, 3t)$-system if $t \equiv 0$, 3 (mod 4), or a $(B, 3t)$-system if $t \equiv 1$, 2 (mod 4), and $6t + 1$ should be read as 0, and

$$\begin{array}{ll} \{0_1, 0_1, 0_2\} & \text{if } t \equiv 0,\ 3 \pmod 4, \text{ or} \\ \{(6t)_1, (6t)_1, 0_2\} & \text{if } t \equiv 1,\ 2 \pmod 4, \end{array}$$

form a set of starter blocks for a (2, 2)–rotational $ESTS(v, 1)$. □

LEMMA 2.7. *If* $v \equiv 2$ *or* 26 (mod 48), *then there exists a* (2, 2)–*rotational* $ESTS(v, 1)$.

Proof. Let $v = 12t + 2$ and let $t \equiv 0 \pmod 2$. If $t = 0$, then the triples

$$\{\infty_1, \infty_1, \infty_1\}, \quad \{\infty_1, \infty_2, \infty_2\}$$

are starter blocks for a (2, 2)–rotational $ESTS(2, 1)$.

If $t \geq 2$ is an even integer, then the following triples

$$\begin{array}{lll} \{\infty_1, \infty_1, \infty_1\}, & \{\infty_1, \infty_2, \infty_2\}, & \{\infty_2, 0_1, 0_2\}, \\ \{\infty_1, 0_1, (3t)_1\}, & \{\infty_1, 0_2, (3t)_2\}, & \{0_1, (2t)_1, (4t)_1\}, \end{array}$$

and

$$\begin{array}{ll} \left\{0_1, 0_1, \left(\frac{5t}{2}\right)_1\right\}, & \\ \left\{0_1, (2r-1)_1, \left(\frac{t}{2} + t - 1 + r\right)_1\right\}, & r = 1, 2, \ldots, \frac{t}{2}, \\ \{0_1, (2r)_1, (3t - r)_1\}, & r = 1, 2, \ldots, \frac{t}{2} - 1, \\ \{0_2, r_2, (b_r)_1\}, & r = 1, 2, \ldots, 3t - 1, \end{array}$$

where $\{(a_r, b_r)|r = 1, 2, \ldots, 3t-1\}$ is an $(A, 3t-1)$-system if $t \equiv 2$ (mod 4), or a $(B, 3t - 1)$-system if $t \equiv 0$ (mod 4), and

$$\begin{array}{ll} \{0_2, 0_2, (6t-1)_1\} & \text{if } t \equiv 2 \pmod 4, \text{ or} \\ \{0_2, 0_2, (6t-2)_1\} & \text{if } t \equiv 0 \pmod 4, \end{array}$$

form a set of starter blocks for a (2, 2)–rotational $ESTS(v, 1)$. □

LEMMA 2.8. *If* $v \equiv 14$ *or* $38 \pmod{48}$, *then there exists a* $(2,2)$–*rotational* $ESTS(v,1)$.

Proof. Let $v = 12t + 2$ and let $t \equiv 1 \pmod 2$. Then the following triples

$$\{\infty_1,\infty_1,\infty_1\},\quad \{\infty_1,\infty_2,\infty_2\},\quad \{\infty_2,0_1,0_2\},$$
$$\{\infty_1,0_1,(3t)_1\},\quad \{\infty_1,0_2,(3t)_2\},\quad \{0_1,(2t)_1,(4t)_1\},$$

and

$$\left\{0_1,0_1,\left(\frac{3t-1}{2}\right)_1\right\},$$
$$\left\{0_1,(2r)_1,\left(\frac{t+1}{2}+t-1+r\right)_1\right\}, r = 1,2,\dots,\frac{t-1}{2},$$
$$\{0_1,(2r-1)_1,(3t-r)_1\},\qquad r = 1,2,\dots,\frac{t-1}{2},$$
$$\{0_2,r_2,(b_r)_1\},\qquad r = 1,2,\dots,3t-1,$$

where $\{(a_r,b_r)|r = 1,2,\dots,3t-1\}$ is an $(A,3t-1)$-system if $t \equiv 3 \pmod 4$, or a $(B,3t-1)$-system if $t \equiv 1 \pmod 4$, and

$$\{0_2,0_2,(6t-1)_1\} \text{ if } t \equiv 3 \pmod 4, \text{ or}$$
$$\{0_2,0_2,(6t-2)_1\} \text{ if } t \equiv 1 \pmod 4,$$

form a set of starter blocks for a $(2,2)$–rotational $ESTS(v,1)$. □

LEMMA 2.9. *There exists a* $(2,2)$–*rotational* $ESTS(10,1)$.

Proof. The following triples

$$\{\infty_1,\infty_1,\infty_1\},\quad \{\infty_1,\infty_2,\infty_2\},\quad \{\infty_1,0_1,2_1\},\quad \{\infty_1,0_2,2_2\},$$
$$\{\infty_2,0_1,0_2\},\quad \{0_1,0_1,1_1\},\quad \{0_2,1_2,2_1\},\quad \{0_2,0_2,3_1\}$$

form a set of starter blocks of a $(2,2)$–rotational $ESTS(10,1)$. □

LEMMA 2.10. *If* $v \equiv 10 \pmod{12}$, *then there exists a* $(2,2)$–*rotational* $ESTS(v,1)$.

Proof. Let $v = 12t + 10$ and let t be any nonnegative integer. The case $t = 0$ has been treated in Lemma 2.9. If $t \geq 1$, then the following triples

$$\{\infty_1, \infty_1, \infty_1\}, \qquad \{\infty_1, \infty_2, \infty_2\}, \qquad \{\infty_2, 0_1, 0_2\},$$
$$\{\infty_1, 0_1, (3t+2)_1\}, \quad \{\infty_1, 0_2, (3t+2)_2\},$$

and

$$\{0_1, 0_1, (3t+1)_1\} \quad \text{if } t \equiv 0,\ 1 \pmod 4, \text{ or}$$
$$\{0_1, 0_1, (3t)_1\} \qquad \text{if } t \equiv 2,\ 3 \pmod 4,$$

and

$$\{0_1, r_1, (b_r + t)_1\}, \quad r = 1, 2, \ldots, t,$$

where $\{(a_r, b_r) | r = 1, 2, \ldots, t\}$ is an (A, t)-system if $t \equiv 0, 1 \pmod 4$, or a (B, t)-system if $t \equiv 2, 3 \pmod 4$, and

$$\{0_2, r_2, (b_r)_1\}, \qquad r = 1, 2, \ldots, 3t+1,$$

where $\{(a_r, b_r) | r = 1, 2, \ldots, 3t+1\}$ is an $(A, 3t+1)$-system if $t \equiv 0, 1 \pmod 4$, or a $(B, 3t+1)$-system if $t \equiv 2, 3 \pmod 4$, and

$$\{0_2, 0_2, (6t+3)_1\} \quad \text{if } t \equiv 0, 1 \pmod 4, \text{ or}$$
$$\{0_2, 0_2, (6t+2)_1\} \quad \text{if } t \equiv 2, 3 \pmod 4,$$

form a set of starter blocks for a $(2,2)$–rotational $ESTS(v, 1)$. □

Lemmas 2.3, 2.5, 2.6, 2.7, 2.8 and 2.10 together yield the following theorem.

THEOREM 2.11. *There exists a $(2,2)$–rotational $ESTS(v, 1)$ if and only if $v \equiv 2$ or $4 \pmod 6$.*

LEMMA 2.12. *There exists a $(2,2)$–rotational $ESTS(8, 4)$*

Proof. The following triples

$$\begin{array}{lll} \{\infty_1,\infty_1,\infty_1\}, & \{\infty_1,\infty_2,\infty_2\}, & \\ \{0_1,0_1,0_1\}, & \{0_1,2_2,2_2\}, & \{0_1,1_1,2_1\}, \\ \{\infty_1,0_1,0_2\}, & \{\infty_2,0_1,1_2\}, & \{0_2,1_2,2_2\} \end{array}$$

form a set of starter blocks of a $(2,2)$–rotational $ESTS(8,4)$. □

LEMMA 2.13. *There exists no $(2,2)$–rotational $ESTS(20,10)$*

Proof. If there exists a $(2,2)$–rotational $ESTS(20,10)$, by counting of the number of orbits, there must exist two orbits of length 3. Consequently, there exists a cyclic Steiner triple system of order 9, which is non-existing system. □

LEMMA 2.14. *If $v \equiv 8 \pmod{12}$ and $v \neq 20$, then there exists a $(2,2)$–rotational $ESTS\left(v,\frac{v}{2}\right)$.*

Proof. Let $v = 12t+8 = 2(6t+3)+2$. The case $t=0$ has been treated in Lemma 2.12. By Lemma 2.13, $t \neq 1$. Let $t \geq 2$ be an integer. Then the following triples

$$\begin{array}{lll} \{\infty_1,\infty_1,\infty_1\}, & \{\infty_1,\infty_2,\infty_2\}, & \{0_1,0_1,0_1\} \\ \{0_1,r_1,(b_r+t)_1\}, & r=1,2,\ldots,t, & \end{array}$$

where $\{(a_r,b_r)|r=1,2,\ldots,t\}$ is a (C,t)-system if $t \equiv 0,\ 3 \pmod 4$, or a (D,t)-system if $t \equiv 1,\ 2 \pmod 4$, and

$$\begin{array}{ll} \{0_1,(2t+1)_1,(4t+2)_1\}, & \{\infty_1,0_2,(c_{2t+1})_1\}, \\ \{\infty_2,0_2,(d_{2t+1})_1\}, & \{0_2,(2t+1)_2,(4t+2)_2\}, \\ \{\{0_2,r_2,(d_r)_1\}, & r=1,2,\ldots,2t,2t+2,\ldots,3t+1\}, \end{array}$$

where $\{(c_r,d_r)|r=1,2,\ldots,3t+1\}$ is an $(A,3t+1)$-system if $t \equiv 0$ or $1 \pmod 4$, or a $(B,3t+1)$-system if $t \equiv 2$ or $3 \pmod 4$, and $6t+3$ should be read as 0, and

$$\begin{array}{ll} \{0_1,0_2,0_2\} & \text{if } t \equiv 0,\ 1 \pmod 4, \text{ or} \\ \{(6t+2)_1,0_2,0_2\} & \text{if } t \equiv 2,\ 3 \pmod 4, \end{array}$$

form a set of starter blocks for a $(2,2)$–rotational $ESTS\left(v,\frac{v}{2}\right)$. □

LEMMA 2.15. *There exists no* $(2,2)$*–rotational* $ESTS(12,6)$.

Proof. Suppose that there exists a $(2,2)$–rotational $ESTS(12,6)$. Then we may say that it has starter blocks of the forms

$$\begin{array}{lll} \{\infty_1,\infty_1,\infty_1\}, & \{\infty_1,\infty_2,\infty_2\}, & \{a_1,a_1,a_1\}, \\ \{\infty_1,a_1,b_2\}, & \{\infty_2,x_1,y_2\}, & \{u_1,v_1,w_2\}, \\ \{a_1,b_1,c_2\}, & \{x_2,y_2,z_1\}, & \{r_2,s_2,t_1\}, \\ \{a_2,a_2,x_1\}. & & \end{array}$$

We see that we need 11 orbits of the form $\{a_1,b_2\}$. But there are only 5 such orbits. □

LEMMA 2.16. *If* $v \equiv 0 \pmod{12}$ *and* $v \neq 12$*, then there is a* $(2,2)$*–rotational* $ESTS\left(v,\frac{v}{2}\right)$.

Proof. Let $v = 12t = 2(6t-1)+2$. By Lemma 2.15, $t \neq 1$. Let $t \geq 2$ be an integer. Then the following triples

$$\{\infty_1,\infty_1,\infty_1\},\{\infty_1,\infty_2,\infty_2\},\{0_1,0_1,0_1\}$$

and

$$\begin{array}{ll} \{0_2,1_2,(3t-1)_2\} & \text{if } t \equiv 1,\ 2 \pmod 4, \text{ or} \\ \{0_2,2_2,(3t-1)_2\} & \text{if } t \equiv 0,\ 3 \pmod 4 \end{array}$$

and

$$\{0_1,r_1,(b_r+t-1)_1\},\ \ r=1,2,\ldots,t-1,$$

where $\{(a_r,b_r)|r=1,2,\ldots,t-1\}$ is an $(A,t-1)$-system if $t \equiv 1$, 2 (mod 4), or a $(B,t-1)$-system if $t \equiv 0$, 3 (mod 4), and let $\{(c_r,d_r)|r = 1,2,\ldots,3t-1\}$ be an $(A,3t-1)$-system if $t \equiv 2$, 3 (mod 4), or a $(B,3t-1)$-system if $t \equiv 0$, 1 (mod 4) (if $d_r = 6t-1$ then d_r should be read as 0), then

$$\{0_1,(3t-1)_1,(d_{3t-1})_2\}$$

and

$$\begin{array}{ll} \{0_1,(3t-2)_1,(d_{3t-2})_2\} & \text{if } t \equiv 1,\ 2 \pmod 4, \text{ or} \\ \{0_1,(3t-3)_1,(d_{3t-3})_2\} & \text{if } t \equiv 0,\ 3 \pmod 4 \end{array}$$

and

$$\{0_2, r_2, (-c_r)_1\}, \quad r = 2, 3, \ldots, 3t-3 \quad \text{if } t \equiv 2,\ 3 \pmod 4, \text{ or}$$
$$r = 1, 3, \ldots, 3t-4, 3t-2 \quad \text{if } t \equiv 0,\ 1 \pmod 4$$

and

$$\{\infty_2, 0_1, 0_2, \}, \{0_1, (c_1)_2, (c_1)_2\}, \{\infty_1, 0_1, (d_1)_2\} \text{ if } t \equiv 0,\ 1 \pmod 4$$

or

$$\{\infty_2, 0_1, (6t-2)_2, \}, \{0_1, (c_2)_2, (c_2)_2\}, \{\infty_1, 0_1, (d_2)_2\} \text{ if } t \equiv 2,\ 3 (\text{mod } 4),$$

form a set of starter blocks for a $(2, 2)$–rotational $ESTS\left(v, \frac{v}{2}\right)$. □

LEMMA 2.17. *There exists a* $(2, 2)$*–rotational* $ESTS(6, 3)$.

Proof. The following triples

$$\begin{array}{lll} \{\infty_1, \infty_1, \infty_1\}, & \{\infty_1, \infty_2, \infty_2\}, & \\ \{0_1, 0_1, 0_1\}, & \{0_1, 1_2, 1_2\}, & \\ \{\infty_1, 0_1, 1_1\}, & \{\infty_1, 0_2, 1_2\}, & \{\infty_2, 0_1, 0_2\} \end{array}$$

form a set of starter blocks of a $(2, 2)$–rotational $ESTS(6, 3)$. □

LEMMA 2.18. *If* $v \equiv 6$ *or* $18 \pmod{48}$, *then there exists a* $(2, 2)$*–rotational* $ESTS\left(v, \frac{v}{2}\right)$.

Proof. Let $v = 12t + 6 = 2(6t + 2) + 2$ and let $t \equiv 0$ or $1 \pmod 4$. The case $t = 0$ has been treated in Lemma 2.16. Let $t \geq 2$. Then the following triples

$$\begin{array}{lll} \{\infty_1, \infty_1, \infty_1\}, & \{\infty_1, \infty_2, \infty_2\}, & \{0_1, 0_1, 0_1\}, \\ \{\infty_1, 0_1, (3t+1)_1\}, & \{\infty_1, 0_2, (3t+1)_2\}, & \{\infty_2, 0_1, 0_2\}, \end{array}$$

and

$$\{0_1, r_1, (b_r + t)_1\}, \quad r = 1, 2, \ldots, t,$$

where $\{(a_r, b_r)|r = 1, 2, \ldots, t\}$ is an (A, t)-system, and

$$\{0_2, r_2, (b_r)_1\}, \qquad r = 1, 2, \ldots, 3t,$$

where $\{(a_r, b_r)|r = 1, 2, \ldots, 3t\}$ is an $(A, 3t)$-system if $t \equiv 0 \pmod 4$, or a $(B, 3t)$-system if $t \equiv 1 \pmod 4$, and

$$\begin{array}{ll} \{0_2, 0_2, (6t+1)_1\} & \text{if } t \equiv 0 \pmod 4, \text{ or} \\ \{0_2, 0_2, (6t)_1\} & \text{if } t \equiv 1 \pmod 4, \end{array}$$

form a set of starter blocks for a $(2,2)$–rotational $ESTS\left(v, \frac{v}{2}\right)$. □

DEFINITION 2.19. A $(R_1, 3k)$-*system* is a set $\{(a_r, b_r)|r=1, 2, \ldots, 3k\}$ such that
(i) $\{a_r, b_r|r = 1, 2, \ldots, 3k\} = Z_{6k+2} \setminus \{0, \frac{9k+4}{2}\}$, and
(ii) $b_r - a_r = r + 1$ for $r = 1, 2, \ldots, 3k - 1$ and $b_{3k} - a_{3k} = 3k - 2$.

LEMMA 2.20. *If $k \equiv 2 \pmod 4$, then there exists a $(R_1, 3k)$-system.*

Proof. If $k \equiv 2 \pmod 4$, form ordered pairs

$$\begin{array}{ll} (r, 3k+2-r), & r = 1, 2, \ldots, \frac{3k}{2}, \\ (3k+1+2r, 6k-2r), & r = 1, 2, \ldots, \frac{3k-6}{4}, \\ (3k+2+2r, 6k+3-2r), & r = 1, 2, \ldots, \frac{3k-2}{4}, \\ \left(\frac{3k+2}{2}, \frac{9k}{2}\right), (3k+2, 6k). & \end{array}$$

□

DEFINITION 2.21. A $(R_2, 3k)$-*system* is a set $\{(a_r, b_r)|r=1, 2, \ldots, 3k\}$ such that
(i) $\{a_r, b_r|r = 1, 2, \ldots, 3k\} = Z_{6k+2} \setminus \{0, \frac{9k-3}{2}\}$, and
(ii) $b_r - a_r = r + 1$ for $r = 1, 2, \ldots, 3k - 1$ and $b_{3k} - a_{3k} = 3k - 3$.

LEMMA 2.22. *If $k \equiv 3 \pmod 4$ and $k \neq 3$, then there exists a $(R_2, 3k)$-system.*

Proof. Case 1. $k \equiv 7 \pmod 8$:

$$\begin{array}{ll} (r, 3k+1-r), & r = 1, 2, \ldots, \frac{3k-1}{2} \\ (3k+8+4r, 6k-4r), & r = 0, 1, \ldots, \frac{3k-13}{8}, \\ \left(\frac{3k+1}{2}, \frac{9k+1}{2}\right), (6k+1, 3k+4), \left(\frac{9k-1}{2}, \frac{9k+5}{2}\right). & \\ (3k+r, 6k-r), & r = 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, \ldots, \frac{3k-7}{2}. \end{array}$$

Case 2. $k \equiv 3 \pmod 8$ and $k \neq 3$:

$$\begin{array}{ll} (r, 3k+1-r), & r = 1, 2, \ldots, \frac{3k-1}{2}, \\ (3k+8+4r, 6k-4r), & r = 0, 1, \ldots, \frac{3k-17}{8}, \\ \left(\frac{3k+1}{2}, \frac{9k+1}{2}\right), (6k+1, 3k+4), \left(\frac{9k+3}{2}, \frac{9k+9}{2}\right), & \\ (3k+r, 6k-r), & r = 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, \ldots, \frac{3k-5}{2}. \end{array} \quad \square$$

LEMMA 2.23. *There exists a* $(2,2)$*-rotational* $ESTS(42, 21)$.

Proof. The following triples

$$\begin{array}{lll} \{\infty_1, \infty_1, \infty_1\}, & \{\infty_1, \infty_2, \infty_2\}, & \{0_1, 0_1, 0_1\}, \\ \{\infty_1, 0_1, 7_1\}, & \{\infty_1, 0_2, 7_2\}, & \{\infty_2, 0_1, 0_2\}, \\ \{0_1, 1_1, 4_1\}, & \{0_1, 2_1, 7_1\}, & \\ \{0_1, 6_1, 8_2\}, & \{0_1, 8_1, 9_2\}, & \{0_1, 9_1, 14_2\}, \\ \{0_2, 1_2, 9_2\}, & \{0_1, 16_2, 16_2\}, & \\ \{0_2, 2_2, 5_1\}, & \{0_2, 3_2, 17_1\}, & \{0_2, 4_2, 13_1\}, \\ \{0_2, 5_2, 7_1\}, & \{0_2, 6_2, 16_1\}, & \{0_2, 7_2, 8_1\}, \end{array}$$

form a set of starter blocks of a $(2,2)$-rotational $ESTS(42, 21)$. $\square$

LEMMA 2.24. *If* $v \equiv 30$ *or* $42 \pmod{48}$, *then there exists a* $(2,2)$–*rotational* $ESTS\left(v, \frac{v}{2}\right)$.

Proof. Let $v = 12k + 6 = 2(6k+2) + 2$ and let $k \equiv 2$ or $3 \pmod 4$. The case $k = 3$ has been treated in Lemma 2.23. Let $k \neq 3$. Then the following triples

$$\begin{array}{lll} \{\infty_1, \infty_1, \infty_1\}, & \{\infty_1, \infty_2, \infty_2\}, & \{0_1, 0_1, 0_1\}, \\ \{\infty_1, 0_1, (3k+1)_1\}, & \{\infty_1, 0_2, (3k+1)_2\}, & \{\infty_2, 0_1, 0_2\}, \end{array}$$

and

$$\{0_1, r_1, (b_r + k - 1)_1\}, \quad r = 1, 2, \ldots, k-1,$$

where $\{(a_r, b_r) | r = 1, 2, \ldots, k-1\}$ is an $(A, k-1)$-system if $k \equiv 2 \pmod 4$, or a $(B, k-1)$-system if $k \equiv 3 \pmod 4$, and

$$\begin{array}{ll} \{0_1, (3k-2)_1, (b_{3k})_2\} & \text{if } k \equiv 2 \pmod 4, \text{ or} \\ \{0_1, (3k-3)_1, (b_{3k})_2\} & \text{if } k \equiv 3 \pmod 4 \end{array}$$

and

$$\{0_2, 1_2, (3k)_2\},$$
$$\{0_1, (3k-1)_1, (b_{3k-2})_2\}, \qquad \{0_1, (3k)_1, (b_{3k-1})_2\},$$
$$\{0_2, (r+1)_2, (-a_r)_1\}, \qquad r = 1, 2, \ldots, 3k-3,$$

where $\{(a_r, b_r) | r = 1, 2, \ldots, 3k\}$ is an $(R_1, 3k)$-system if $k \equiv 2 \pmod 4$, or a $(R_2, 3k)$-system if $k \equiv 3 \pmod 4$, and

$$\left\{0_1, \left(\frac{9k+4}{2}\right)_2, \left(\frac{9k+4}{2}\right)_2\right\} \quad \text{if } k \equiv 2 \pmod 4, \text{ or}$$
$$\left\{0_1, \left(\frac{9k-3}{2}\right)_2, \left(\frac{9k-3}{2}\right)_2\right\} \quad \text{if } k \equiv 3 \pmod 4,$$

form a set of starter blocks for a (2, 2)–rotational $ESTS\left(v, \frac{v}{2}\right)$. □

DEFINITION 2.25. A *$(S_1, k-1)$-system* is a set $\{(a_r, b_r) | r = 1, 2, \ldots, k-1\}$ such that
(i) $\{a_r, b_r | r = 1, 2, \ldots, k-1\} = \{1, 2, \ldots, k, k+2, \ldots, 2k-1\}$, and
(ii) $b_r - a_r = r$ for $r = 1, 2, \ldots, k-1$.

LEMMA 2.26. *If $k \equiv 2$ or 3 (mod 4), then there exists a $(S_1, k-1)$-system.*

Proof. Case 1. $k = 4t + 2$.
$t = 0$: (1, 2).
$t = 1$: (10, 11), (2, 4), (6, 9), (1, 5), (3, 8).
$t \geq 2$:
$(r, 4t+2-r), \qquad r = 1, 2, \ldots, 2t,$
$(4t+3+r, 8t+4-r), \quad r = 1, 2, \ldots, t-1,$
$(5t+2+r, 7t+3-r), \quad r = 1, 2, \ldots, t-1,$
$(2t+1, 6t+2), (4t+2, 6t+3), (7t+3, 7t+4).$
Case 2. $k = 4t + 3$.
$t = 0$: (1, 2), (3, 5).
$t = 1$: (11, 12), (3, 5), (10, 13), (2, 6), (4, 9), (1, 7).
$t \geq 2$:
$(r, 4t+4-r), \qquad r = 1, 2, \ldots, 2t+1,$

$(4t+4+r, 8t+5-r)$, $r = 1, 2, \ldots, t-1$,
$(5t+3+r, 7t+4-r)$, $r = 1, 2, \ldots, t-1$,
$(2t+2, 6t+3), (6t+4, 8t+5), (7t+4, 7t+5)$. □

DEFINITION 2.27. A $(S_2, k-1)$-*system* is a set $\{(a_r, b_r) | r = 1, 2, \ldots, k-1\}$ such that

(i) $\{a_r, b_r | r = 1, 2, \ldots, k-1\} = \{1, 2, \ldots, k, k+2, \ldots, 2k-2, 2k\}$,

and

(ii) $b_r - a_r = r$ for $r = 1, 2, \ldots, k-1$.

LEMMA 2.28. *If* $k \equiv 0$ *or 1* (mod 4), *then there exists a* $(S_2, k-1)$-*system.*

Proof. Case 1. $k = 4t$.

$t = 1$: (2, 3), (6, 8), (1, 4).

$t = 2$: (1, 2), (14, 16), (3, 6), (8, 12), (5, 10), (7, 13), (4, 11).

$t \geq 3$:

$(r, 4t+1-r)$, $r = 1, 2, \ldots, 2t-1$,
$(4t+1+r, 8t-3-r)$, $r = 1, 2, \ldots, t-2$,
$(5t-1+r, 7t-3-r)$, $r = 1, 2, \ldots, t-3$,
$(2t, 6t-2), (2t+1, 6t-3), (6t-1, 8t-3), (8t-2, 8t), (7t-3, 7t-2)$.

Case 2. $k = 4t+1$.

$t = 1$: (2, 3), (8, 10), (4, 7), (1, 5).

$t \geq 2$:

$(r, 4t+2-r)$, $r = 1, 2, \ldots, 2t$,
$(4t+4+r, 8t+1-r)$, $r = 1, 2, \ldots, 2t-2$,
$(2t+1, 4t+4), (4t+3, 8t+2)$. □

DEFINITION 2.29. A $(S_3, 3k)$-*system* is a set $\{(a_r, b_r) | r = 1, 2, \ldots, 2k-1, 2k+1, \ldots, 3k\}$ such that

(i) $\{a_r, b_r | r = 1, 2, \ldots, 2k-1, 2k+1, \ldots, 3k\}$
$$= \left\{1, 2, \ldots \frac{3k}{2}, \frac{3k+4}{2} \ldots, 6k-1\right\}, \text{ and}$$

(ii) $b_r - a_r = r$ for $r = 1, 2, \ldots, 2k-1, 2k+1, \ldots, 3k-1$, and $b_{3k} - a_{3k} = 3k-1$.

LEMMA 2.30. *If $k \equiv 2 \pmod 4$, then there exists a $(S_3, 3k)$-system.*

Proof. If $k \equiv 2 \pmod 4$, form ordered pairs

$(r, 3k+1-r)$, $\quad r = 1, 2, \ldots, \frac{3k-2}{2}$,

$(3k+r, 6k-r)$, $\quad r = 1, 2, \ldots, \frac{k-2}{2}$ $(k > 2)$,

$\left(\frac{9k-2}{2} - r, \frac{9k-2}{2} + r\right)$, $\quad r = 1, 2, \ldots, k-1$,

$\left(\frac{3k}{2}, \frac{9k-2}{2}\right), \left(\frac{11k-2}{2}, \frac{11k}{2}\right)$. □

DEFINITION 2.31. A $(S_4, 3k)$-*system* is a set $\{(a_r, b_r) | r = 1, 2, \ldots, 2k-1, 2k+1, \ldots, 3k\}$ such that

(i) $\{a_r, b_r | r = 1, 2, \ldots, 2k-1, 2k+1, \ldots, 3k\}$
$$= \left\{1, 2, \ldots \frac{3k-1}{2}, \frac{3k+3}{2} \ldots, 6k-1\right\}, \text{ and}$$

(ii) $b_r - a_r = r$ for $r = 1, 2, \ldots, 2k-1, 2k+1, \ldots, 3k-1$, and $b_{3k} - a_{3k} = 3k-1$.

LEMMA 2.32. *If $k \equiv 3 \pmod 4$, then there exists a $(S_4, 3k)$-system.*

Proof. If $k \equiv 3 \pmod 4$, form ordered pairs

$(r, 3k-r)$, $\quad r = 1, 2, \ldots, \frac{3k-3}{2}$,

$(3k+r, 6k-1-r)$, $\quad r = 1, 2, \ldots, \frac{k-3}{2}$ $(k > 3)$,

$\left(\frac{9k-3}{2} - r, \frac{9k-3}{2} + r\right)$, $\quad r = 1, 2, \ldots, k-1$,

$\left(\frac{3k-1}{2}, \frac{9k-3}{2}\right), (3k, 6k-1), \left(\frac{11k-3}{2}, \frac{11k-1}{2}\right)$. □

DEFINITION 2.33. A $(S_5, 3k)$-*system* is a set $\{(a_r, b_r) | r = 1, 2, \ldots, 2k-1, 2k+1, \ldots, 3k\}$ such that

(i) $\{a_r, b_r | r = 1, 2, \ldots, 2k-1, 2k+1, \ldots, 3k\}$
$$= \left\{1, 2, \ldots, \frac{3k-2}{2}, \frac{3k+2}{2} \ldots, 6k-1\right\}, \text{ and}$$

(ii) $b_r - a_r = r$ for $r = 1, 2, \ldots, 2k-1, 2k+1, \ldots, 3k-1$, and $b_{3k} - a_{3k} = 3k-2$.

LEMMA 2.34. *If $k \equiv 0 \pmod 4$, then there exists a $(S_5, 3k)$-system.*

Proof. If $k \equiv 0 \pmod 4$, form ordered pairs

$$\begin{array}{ll} (r, 3k+1-r), & r = 1,2,\ldots,\frac{3k-2}{2}, \\ (3k+r, 6k-1-r), & r = 1,2,\ldots,\frac{k-4}{2}\ (k>4), \\ \left(\frac{9k-2}{2}-r, \frac{9k-2}{2}+r\right), & r = 1,2,\ldots,k-1, \\ \left(\frac{3k+2}{2}, \frac{9k-2}{2}\right), (3k+1, 6k-1), \left(\frac{11k-2}{2}, \frac{11k}{2}\right). & \end{array}$$

□

DEFINITION 2.35. A $(S_6, 3k)$-*system* is a set $\{(a_r, b_r) | r = 1,2,\ldots, 2k-1, 2k+1,\ldots,3k\}$ such that

(i) $\{a_r, b_r | r = 1,2,\ldots,2k-1,2k+1,\ldots,3k\}$
$= \left\{1,2,\ldots\frac{3k-3}{2}, \frac{3k+1}{2},\ldots,6k-1\right\}$, and

(ii) $b_r - a_r = r$ for $r = 1,2,\ldots,2k-1,2k+1,\ldots,3k-1$, and
$b_{3k} - a_{3k} = 3k-2$.

LEMMA 2.36. *If $k \equiv 1 \pmod 4$, then there exists a $(S_6, 3k)$-system.*

Proof. If $k \equiv 1 \pmod 4$, form ordered pairs

$$\begin{array}{ll} (r, 3k-r), & r = 1,2,\ldots,\frac{3k-3}{2}\ (k>1), \\ (3k-1+r, 6k-1-r), & r = 1,2,\ldots,\frac{k-1}{2}\ (k>1), \\ \left(\frac{9k-3}{2}-r, \frac{9k-3}{2}+r\right), & r = 1,2,\ldots,k-1\ (k>1), \\ \left(\frac{3k+1}{2}, \frac{9k-3}{2}\right), \left(\frac{11k-3}{2}, \frac{11k-1}{2}\right). & \end{array}$$

□

LEMMA 2.37. *If $v \equiv 2 \pmod{12}$, then there exists a $(2,2)$–rotational $ESTS(v, \frac{v}{2})$.*

Proof. Let $v = 12k+2$ and let k be a nonnegative integer. The case $k=0$ is trivial. If $k \geq 1$, then the following triples

$$\begin{array}{ll} \{\infty_1, \infty_1, \infty_1\}, & \{\infty_1, \infty_2, \infty_2\}, \\ \{0_1, 0_1, 0_1\}, & \{\infty_2, 0_1, 0_2\}, \\ \{\infty_1, 0_1, (3k)_1\}, & \{\infty_1, 0_2, (3k)_2\}, \\ \{0_1, (2k)_1, (4k)_1\}, & \{0_2, (2k)_2, (4k)_2\}, \\ \{0_1, r_1, (b_r+k-1)_1\}, & r = 1,2,\ldots,k-1, \end{array}$$

where $\{(a_r, b_r) | r = 1, 2, \ldots, k-1\}$ is either a $(S_1, k-1)$-system if $k \equiv 2$ or 3 (mod 4), or a $(S_2, k-1)$-system if $k \equiv 0$ or 1 (mod 4), and

$$\begin{array}{ll} \left\{0_2, 0_2, \left(\frac{3k}{2}\right)_1\right\} & \text{if } k \equiv 0 \pmod 4, \text{ or} \\ \left\{0_2, 0_2, \left(\frac{3k-1}{2}\right)_1\right\} & \text{if } k \equiv 1 \pmod 4, \text{ or} \\ \left\{0_2, 0_2, \left(\frac{3k+2}{2}\right)_1\right\} & \text{if } k \equiv 2 \pmod 4, \text{ or} \\ \left\{0_2, 0_2, \left(\frac{3k+1}{2}\right)_1\right\} & \text{if } k \equiv 3 \pmod 4 \end{array}$$

and

$$\begin{array}{ll} \{0_2, (a_{3k})_1, (b_{3k})_1\}, & \\ \{0_2, r_2, (b_r)_1\}, & r = 1, 2, \ldots, 2k-1, 2k+1, \ldots, 3k-1, \end{array}$$

where $\{(a_r, b_r) | r = 1, 2, \ldots, 2k-1, 2k+1, \ldots, 3k\}$ is a $(S_t, 3k)$-system such that

$$\begin{array}{ll} t = 3 & \text{if } k \equiv 2 \pmod 4, \text{ or} \\ t = 4 & \text{if } k \equiv 3 \pmod 4, \text{ or} \\ t = 5 & \text{if } k \equiv 0 \pmod 4, \text{ or} \\ t = 6 & \text{if } k \equiv 1 \pmod 4, \end{array}$$

form a set of starter blocks of a (2, 2)–rotational $ESTS\left(v, \frac{v}{2}\right)$. □

Lemmas 2.3, 2.12 through 2.37 together yield the following theorem.

THEOREM 2.38. *There exists a* (2, 2)*–rotational* $ESTS\left(v, \frac{v}{2}\right)$ *if and only if* $v \equiv 0$ *or* 2 (mod 6) *and* $v \neq 12, 20$.

Now, we can conclude the following theorem.

THEOREM 2.39. *There exists a* (2, 2)*–rotational* $ESTS(v, \rho)$ *if and only if*

(i) $\rho = 1$ *and* $v \equiv 2, 4 \pmod 6$, *or*

(ii) $\rho = \frac{v}{2}$ *and* $v \equiv 0, 2 \pmod 6$ *and* $v \neq 12, 20$.

3. (3, 2)-rotational extended Steiner triple systems

As before $(2,2)$–rotational $ESTS(v,\rho)$s, we assume that our $(3,2)$–rotational $ESTS(v,\rho)$s have the element-set

$$V = \{\infty_1, \infty_2, \infty_3\} \cup (Z_{\frac{v-3}{2}} \times \{1,2\})$$

and the corresponding automorphism is

$$\alpha = (\infty_1)(\infty_2)(\infty_3)\left(0_1\ 1_1 \cdots \left(\frac{v-3}{2}-1\right)_1\right)\left(0_2\ 1_2 \cdots \left(\frac{v-3}{2}-1\right)_2\right),$$

where we also write for brevity x_i instead of (x,i).

By an elementary argument, we have the following necessary condition for the existence of a $(3,k)$–rotational $ESTS(v,\rho)$.

LEMMA 3.1. *If there exists a $(3,k)$–rotational $ESTS(v,\rho)$, then*

$$\rho = 0 \text{ or } 3 + n \cdot \frac{v-3}{k}$$

for each $n = 0, 1, \cdots, k$.

REMARK 3.2. If there exists a $(3,2)$–rotational $ESTS(v,\rho)$, then ρ is

$$0, \quad 3, \quad \frac{v+3}{2} \quad \text{or} \quad v.$$

LEMMA 3.3. *A necessary condition for the existence of a $(3,2)$–rotational $ESTS(v,\rho)$ is*

(i) $\rho = v$ *and* $v \equiv 1$, *3* (mod 6), $v \neq 13, 21$, *or*
(ii) $\rho = \frac{v+3}{2}$ *and* $v \equiv 3$, *5* (mod 6), $v \neq 5$, *or*
(iii) $\rho = 0, 3$ *and* $v \equiv 3$ (mod 6).

Proof. (i) This follows from the existence of a $(3,2)$–rotational Steiner triple systems [4, 5].

(ii) If $\rho = \frac{v+3}{2}$, then

$$\frac{v+3}{2} \equiv 0 \text{ or } 1 \pmod 3$$

since the existence of $ESTS(v,\rho)$ implies $\rho = 0$ or 1 (mod 3). Thus $v \equiv 3$ or 5 (mod 6). If $v = 5$, then $\rho = \frac{5+3}{2} = 4 = v - 1$; so v must be 2.

(iii) If $\rho = 0$ or 3, then $v \equiv 0$ (mod 3). But $\frac{v-3}{2}$ is an integer; so $v \equiv 3$ (mod 6). □

The following theorem is a consequence of the existence of a $(3,2)$–rotational Steiner triple system of order v [4, 5].

THEOREM 3.4. *There exists a* $(3,2)$*–rotational* $ESTS(v,v)$ *if and only if* $v \equiv 1$ *or 3* (mod 6), $v \neq 13,\ 21$.

LEMMA 3.5. *There exists a* $(3,2)$*–rotational* $ESTS(9,3)$.

Proof. The following triples

$$\begin{array}{lll} \{\infty_1, \infty_2, \infty_3\}, & \{0_1, 0_1, 1_1\}, & \{0_2, 0_2.1_2\}, \\ \{\infty_1, \infty_1, \infty_1\}, & \{\infty_2, \infty_2, \infty_2\}, & \{\infty_3, \infty_3, \infty_3\}, \\ \{\infty_1, 0_1, 0_2\}, & \{\infty_2, 0_1, 1_2\}, & \{\infty_3, 0_1, 2_2\} \end{array}$$

form a set of starter blocks of a $(3,2)$–rotational $ESTS(9,3)$. □

LEMMA 3.6. *If* $v \equiv 9$ (mod 12), *then there exists a* $(3,2)$*–rotational* $ESTS(v,3)$.

Proof. Let $v = 12t + 9 = 2(6t+3) + 3$ and let t be a nonnegative integer. The case $t = 0$ has been treated in Lemma 3.5. Let $t \geq 1$. Then the following triples

$$\{\infty_1, \infty_2, \infty_3\}, \{\infty_i, \infty_i, \infty_i\},\ \ i = 1,2,3,$$

and if $t \equiv 0,\ 1$ (mod 4), then

$$\{\infty_1, 0_1, 0_2\}, \{0_i, 0_i, (3t+1)_i\},\ \ i = 1,2,$$

or if $t \equiv 2,\ 3$ (mod 4), then

$$\{\infty_1, (6t+2)_1, 0_2\}, \{0_1, 0_1, (3t)_1\}, \{0_2, 0_2, (3t+1)_2\},$$

and

$$\{0_1, r_1, (b_r + t)_1\},\ \ r = 1, 2, \ldots, t,$$

where $\{(a_r, b_r) | r = 1, 2, \ldots, t\}$ is an (A,t)-system if $t \equiv 0,\ 1$ (mod 4) or a (B,t)-system if $t \equiv 2,\ 3$ (mod 4), and

$$\begin{array}{ll} \{0_2, r_2, (d_r)_1\}, & r = 1, 2, \ldots, 3t \\ \{\infty_2, 0_2, (d_{3t+1})_1\}, & \{\infty_3, 0_2, (c_{3t+1})_1\}, \end{array}$$

where $\{(c_r, d_r) | r = 1, 2, \ldots, 3t+1\}$ is an $(A, 3t+1)$-system if $t \equiv 0$, 1 (mod 4) or a $(B, 3t+1)$-system if $t \equiv 2,\ 3$ (mod 4), and here $6t+3$ should be regarded as 0, form a set of starter blocks for a $(3,2)$–rotational $ESTS(v,3)$. □

DEFINITION 3.7. Let t be a positive integer. A $(Z_{6t}\setminus\{0, 3t, \frac{9t-1}{2}, 6t-1\}, 3t-2)$*-system* is a set of ordered pairs of integers $\{(a_r, b_r) \mid r = 1, 2, \ldots, 3t-2\}$ such that

(i) $\{a_r, b_r \mid r = 1, 2, \ldots, 3t-2\} = Z_{6t} \setminus \{0, 3t, \frac{9t-1}{2}, 6t-1\}$, and
(ii) $b_r - a_r = r$, for $r = 1, 2, \ldots, 3t-2$.

LEMMA 3.8. *If* $t \equiv 1$ *or 3* (mod 4), *then there exists a* $(Z_{6t} \setminus \{0, 3t, \frac{9t-1}{2}, 6t-1\}, 3t-2)$*-system.*

Proof. Let $t \equiv 3$ (mod 4). Then the following ordered pairs form a $(Z_{6t} \setminus \{0, 3t, \frac{9t-1}{2}, 6t-1\}, 3t-2)$-system:

$(r, 3t-r)$, $\quad r = 1, 2, \ldots, \frac{3t-1}{2}$,
$(3t+r, 6t-1-r)$, $\quad r = 1, 2, \ldots, \frac{3t-3}{2}$. □

DEFINITION 3.9. Let t be a positive integer. A $(Z_{6t} \setminus \{0, 3t, 6t-2, 6t-1\}, 3t-2)$*-system* is a set of ordered pairs of integers $\{(a_r, b_r) \mid r = 1, 2, \ldots, 3t-2\}$ such that

(i) $\{a_r, b_r \mid r = 1, 2, \ldots, 3t-2\} = Z_{6t} \setminus \{0, 3t, 6t-2, 6t-1\}$, and
(ii) $b_r - a_r = r$, for $r = 1, 2, \ldots, 3t-2$.

LEMMA 3.10. *If* $t \equiv 0$ (mod 4) *and* $t \geq 4$, *then there exists a* $(Z_{6t} \setminus \{0, 3t, 6t-2, 6t-1\}, 3t-2)$*-system.*

Proof. Let $t \equiv 0$ (mod 4) and if $t = 4$, then the following ordered pairs form a $(Z_{24} \setminus \{0, 12, 22, 23\}, 3t-2)$-system:

$$(16,17),\ (5,7),\ (18,21),\ (4,8),\ (14,19),$$
$$(3,9),\ (13,20),\ (2,10),\ (6,15),\ (1,11).$$

If ≥ 8, then the following ordered pairs form a $(Z_{6t} \setminus \{0, 3t, 6t-2, 6t-1\}, 3t-2)$–system:

$(r, 3t-r)$, $\quad r = 1, 2, \ldots, \frac{3t-1}{2}$,
$(3t+r, 6t-3-r)$, $\quad r = 1, 2, \ldots, t-2$,
$(4t-1+r, 5t-2-r)$, $\quad r = 1, 2, \ldots, \frac{t-4}{4}$,
$\left(\frac{9t-6}{2} - r, \frac{9t-4}{2} + r\right)$, $\quad r = 1, 2, \ldots, \frac{t-8}{4}$,
$\left(\frac{3t}{2}, \frac{9t-6}{2}\right), \left(4t-1, \frac{9t-4}{2}\right), (5t-2, 6t-3), \left(\frac{19t-12}{4}, \frac{19t-8}{4}\right)$. □

DEFINITION 3.11. Let t be a positive integer. A $(Z_{6t} \setminus \{0, \frac{3t}{2}, 3t, 6t-1\}, 3t-2)$-*system* is a set of ordered pairs of integers $\{(a_r, b_r) \mid r = 1, 2, \ldots, 3t-2\}$ such that

(i) $\{a_r, b_r \mid r = 1, 2, \ldots, 3t-2\} = Z_{6t} \setminus \{0, \frac{3t}{2}, 3t, 6t-1\}$, and
(ii) $b_r - a_r = r$, for $r = 1, 2, \ldots, 3t-2$.

LEMMA 3.12. *If* $t \equiv 2 \pmod 4$, *then there exists a* $(Z_{6t} \setminus \{0, \frac{3t}{2}, 3t, 6t-1\}, 3t-2)$-*system.*

Proof. Let $t \equiv 2 \pmod 4$. Then the following ordered pairs form a $(Z_{6t} \setminus \{0, \frac{3t}{2}, 3t, 6t-1\}, 3t-2)$-system:

$$\begin{array}{ll} (r, 3t-r), & r = 1, 2, \ldots, \frac{3t-2}{2}, \\ (3t+r, 6t-1-r), & r = 1, 2, \ldots, \frac{3t-2}{2}. \end{array}$$

□

LEMMA 3.13. *There exists a* $(3,2)$*–rotational* $ESTS(15,3)$.

Proof. The following triples

$$\begin{array}{lll} \{\infty_1, \infty_2, \infty_3\}, & \{0_2, 1_2, 2_1\}, & \{0_2, 0_2, 2_2\}, \\ \{\infty_1, \infty_1, \infty_1\}, & \{\infty_2, \infty_2, \infty_2\}, & \{\infty_3, \infty_3, \infty_3\}, \\ \{0_2, 0_1, 4_1\}, & \{0_1, 0_1, 1_1\}, & \{\infty_1, 0_1, 3_1\}, \\ \{\infty_1, 0_2, 3_2\}, & \{\infty_2, 0_2, 3_1\}, & \{\infty_3, 0_2, 5_1\} \end{array}$$

form a set of starter blocks of a $(3,2)$–rotational $ESTS(15,3)$. □

LEMMA 3.14. *If* $v \equiv 3 \pmod{12}$, *then there exists a* $(3,2)$*–rotational* $ESTS(v,3)$.

Proof. Let $v = 2 \cdot 6t + 3$ and let t be a nonnegative integer. The case $t = 0$ is trivial and the case $v = 15$ has been treated in Lemma 3.13. Let $t \geq 2$. Then the following triples

$$\begin{array}{ll} \{\infty_1, \infty_2, \infty_3\}, & \{0_2, (3t)_1, (6t-1)_1\}, \\ \{\infty_i, \infty_i, \infty_i\}, & i = 1, 2, 3, \\ \{\infty_1, 0_i, (3t)_i\}, & i = 1, 2, \\ \{\infty_2, 0_2, 0_1\}, & \{0_2, 0_2, (3t-1)_2\} \end{array}$$

and

$$\left\{\infty_3, 0_2, \left(\frac{9t-1}{2}\right)_1\right\} \quad \text{if } t \equiv 1,\ 3 \pmod 4, \text{ or}$$
$$\left\{\infty_3, 0_2, \left(\frac{3t}{2}\right)_1\right\} \quad \text{if } t \equiv 2 \pmod 4, \text{ or}$$
$$\{\infty_3, 0_2, (6t-2)_1\} \quad \text{if } t \equiv 0 \pmod 4,$$

and

$$\{0_1, 0_1, (3t-2)_1\} \quad \text{if } t \equiv 1,\ 2 \pmod 4, \text{ or}$$
$$\{0_1, 0_1, (3t-3)_1\} \quad \text{if } t \equiv 0,\ 3 \pmod 4$$

and

$$\{0_1, r_1, (b_r + t - 1)_1\},\ r = 1, 2, \ldots, t-1,$$

where $\{(a_r, b_r) | r = 1, 2, \ldots, t-1\}$ is a $(A, t-1)$-system if $t \equiv 1$, $2 \pmod 4$ or a $(B, t-1)$-system if $t \equiv 0$, $3 \pmod 4$ and

$$\{0_2, r_2, (d_r)_1\},\ r = 1, 2, \ldots, 3t-2,$$

where $\{(c_r, d_r) | r = 1, 2, \ldots, 3t-2\}$ is a $(Z_{6t} \setminus \{0, 3t, \frac{9t-1}{2}, 6t-1\}, 3t-2)$-system if $t \equiv 1$, $3 \pmod 4$, or a $(Z_{6t} \setminus \{0, \frac{3t}{2}, 3t, 6t-1\}, 3t-2)$-system if $t \equiv 2 \pmod 4$, or a $(Z_{6t} \setminus \{0, 3t, 6t-2, 6t-1\}, 3t-2)$-system if $t \equiv 0 \pmod 4$, form a set of starter blocks for a $(3,2)$–rotational $ESTS(v, 3)$. □

Now we can conclude the following theorem.

THEOREM 3.15. *There exists a* $(3,2)$*–rotational* $ESTS$ $(v, 3)$ *if and only if* $v \equiv 3 \pmod 6$.

In a $(3,2)$–rotational $ESTS(v,3)$, if the blocks $\{\infty_i, \infty_i, \infty_i\}, i = 1, 2, 3$, and $\{\infty_1, \infty_2, \infty_3\}$ are replaced by the blocks $\{\infty_1, \infty_1, \infty_2\}$, $\{\infty_2, \infty_2, \infty_3\}$ and $\{\infty_3, \infty_3, \infty_1\}$, we get a $(3,2)$–rotational $ESTS(v, 0)$. Thus we have the following theorem.

THEOREM 3.16. *There exists a* $(3,2)$*–rotational* $ESTS$ $(v, 0)$ *if and only if* $v \equiv 3 \pmod 6$.

DEFINITION 3.17. A *hooked* $(Z_{6t+3} \setminus \{\frac{9}{2}(t-1)+6\}, 3t+1)$*-system* is a set of ordered pairs of integers $\{(a_r, b_r) \mid r = 1, 2, \ldots, 3t+1\}$ such that

(i) $\{a_r, b_r | r = 1, 2, \ldots, 3t+1\} = Z_{6t+3} \setminus \{\frac{9}{2}(t-1)+6\}$, and
(ii) $b_r - a_r = r$ for $r = 1, 2, \ldots, 3t+1$.

LEMMA 3.18. *If t is an odd positive integer, then there exists a hooked* $(Z_{6t+3} \setminus \{\frac{9}{2}(t-1)+6\}, 3t+1)$*-system.*

Proof. For each odd positive integer t, the following ordered pairs form a hooked $(Z_{6t+3} \setminus \{\frac{9}{2}(t-1)+6\}, 3t+1)$-system:

$$(r, 3t-r), \qquad r = 0, 1, \ldots, \tfrac{3t+1}{2} - 1,$$
$$(3t+r, 6t+3-r), \qquad r = 1, 2, \ldots, \tfrac{3t+1}{2}.$$ □

DEFINITION 3.19. A *hooked* $(Z_{6t+3} \setminus \{\frac{9t}{2}+2\}, 3t+1)$*-system* is a set of ordered pairs of integers $\{(a_r, b_r) \mid r = 1, 2, \cdots, 3t+1\}$ such that

(i) $\{a_r, b_r | r = 1, 2, \ldots, 3t+1\} = Z_{6t+3} \setminus \{\frac{9t}{2}+2\}$, and
(ii) $b_r - a_r = r$ for $r = 1, 2, \ldots, 3k+1$.

LEMMA 3.20. *If t is an even positive integer, then there exists a hooked* $(Z_{6t+3} \setminus \{\frac{9t}{2}+2\}, 3t+1)$*-system.*

Proof. For each even positive integer t, the following ordered pairs form a hooked $(Z_{6t+3} \setminus \{\frac{9t}{2}+2\}, 3t+1)$-system:

$$(r, 3t+1-r), \qquad r = 0, 1, \ldots, \tfrac{3t}{2},$$
$$(3t+1+r, 6t+3-r), \qquad r = 1, 2, \ldots, \tfrac{3t}{2}.$$ □

LEMMA 3.21. *There exists a $(3,2)$–rotational $ESTS(9,6)$.*

Proof. The following triples

$$\{\infty_1, \infty_2, \infty_3\}, \quad \{\infty_i, \infty_i, \infty_i\}, \quad i = 1, 2, 3,$$
$$\{0_1, 0_1, 0_1\}, \quad \{0_1, 1_1, 2_1\}, \quad \{0_2, 1_2, 2_2\},$$
$$\{\infty_1, 0_1, 0_2\}, \quad \{\infty_2, 0_1, 1_2\}, \quad \{\infty_3, 0_1, 2_2\}$$

form a set of starter blocks of a $(3,2)$–rotational $ESTS(9,6)$. □

LEMMA 3.22. *If $v \equiv 9 \pmod{12}$, then there exists a $(3,2)$–rotational* $ESTS\left(v, \frac{v+3}{2}\right)$.

Proof. Let $v = 2(6t + 3) + 3$. The case $t = 0$ has been treated in Lemma 3.21. If $t \geq 1$ is an integer, then the following triples:

$$
\begin{array}{ll}
\{\infty_1, \infty_2, \infty_3\}, & \\
\{\infty_i, \infty_i, \infty_i\}, & i = 1, 2, 3, \\
\{0_2, 0_2, 0_2\}, & \{0_2, (2t + 1)_2, (4t + 2)_2\}, \\
\{0_1, r_1, (b_r + t)_1\}, & r = 1, 2, \ldots, t,
\end{array}
$$

where $\{(a_r, b_r) | r = 1, 2, \ldots, t\}$ is an (A, t)-system if $t \equiv 0$ or $1 \pmod 4$, or a (B, t)-system if $t \equiv 2$ or $3 \pmod 4$, and

$$
\begin{array}{ll}
\left\{\infty_1, 0_2, \left(\frac{9}{2}(t - 1) + 6\right)_1\right\} & \text{if } t \text{ is odd, or} \\
\left\{\infty_1, 0_2, \left(\frac{9}{2}t + 2\right)_1\right\} & \text{if } t \text{ is even,}
\end{array}
$$

and

$$
\begin{array}{ll}
\{0_2, r_2, (b_r)_1\}, & r = 1, 2, \ldots, 2t, 2t + 2, \ldots, 3t + 1, \\
\{\infty_2, 0_2, (a_{2t+1})_1\}, & \{\infty_3, 0_2, (b_{2t+1})_1\},
\end{array}
$$

where $\{(a_r, b_r) | r = 1, 2, \ldots, 3t+1\}$ is a hooked $(Z_{6t+3} \setminus \{\frac{9}{2}(t-1)+6\}, 3t+1)$-system if t is odd, or a hooked $(Z_{6t+3} \setminus \{\frac{9t}{2} + 2\}, 3t + 1)$-system if t is even, and finally,

$$
\begin{array}{ll}
\{0_1, 0_1, (3t + 1)_1\} & \text{if } t \equiv 0 \text{ or } 1 \pmod 4, \text{ or} \\
\{0_1, 0_1, (3t)_1\} & \text{if } t \equiv 2 \text{ or } 3 \pmod 4,
\end{array}
$$

form a set of starter blocks for a $(3, 2)$–rotational $ESTS\left(v, \frac{v+3}{2}\right)$. □

DEFINITION 3.23. Let t be a positive integer. A $(Z_{6t} \setminus \{0, \frac{9t}{2}\}, 3t-1)$-*system* is a set of ordered pairs of integers $\{(a_r, b_r) \mid r = 1, 2, \ldots, 3t - 1\}$ such that

(i) $\{a_r, b_r \mid r = 1, 2, \ldots, 3t - 1\} = Z_{6t} \setminus \{0, \frac{9t}{2}\}$, and
(ii) $b_r - a_r = r$, for $r = 1, 2, \ldots, 3t - 1$.

LEMMA 3.24. *If $t \equiv 0 \pmod 2$, then there exists a $(Z_{6t} \setminus \{0, \frac{9t}{2}\}, 3t-1)$-system.*

Proof. Let $t \geq 2$ be an even integer. Then the following ordered pairs form a $(Z_{6t} \setminus \{0, \frac{9t}{2}\}, 3t-1)$-system:

$$\begin{array}{ll} (r, 3t+1-r), & r = 1, 2, \ldots, \frac{3t}{2}, \\ (3t+r, 6t-r), & r = 1, 2, \ldots, \frac{3t-2}{2}. \end{array}$$ □

DEFINITION 3.25. Let $t \geq 2$ be a positive integer. A *$(K, t-1)$-system* is a set of ordered pairs of integers $\{(a_r, b_r) \mid r = 1, 2, \ldots, t-1\}$ such that

(i) $\{a_r, b_r \mid r = 1, 2, \ldots, t-1\} = \{1, 2, \ldots, t, t+2, \ldots, \frac{3t}{2}, \frac{3t}{2}+2, \ldots, 2t\}$, and

(ii) $b_r - a_r = r$, for $r = 1, 2, \ldots, t-1$.

LEMMA 3.26. *If $t \equiv 0 \pmod 2$ and $t \geq 4$, then there exists a $(K, t-1)$-system.*

Proof. Let $t \geq 4$ be an even integer. Then the following ordered pairs form a $(K, t-1)$-system:

$$\begin{array}{ll} (r, t+1-r), & r = 1, 2, \ldots, \frac{t}{2}, \\ (t+1+r, 2t-r), & r = 1, 2, \ldots, \frac{t}{2}. \end{array}$$ □

DEFINITION 3.27. Let t be a positive integer. A *$(Z_{6t} \setminus \{0, \frac{9t-1}{2}\}, 3t-1)$-system* is a set of ordered pairs of integers $\{(a_r, b_r) \mid r = 1, 2, \ldots, 3t-1\}$ such that

(i) $\{a_r, b_r \mid r = 1, 2, \ldots, 3t-1\} = Z_{6t} \setminus \{0, \frac{9t-1}{2}\}$, and

(ii) $b_r - a_r = r$, for $r = 1, 2, \ldots, 3t-1$.

LEMMA 3.28. *If $t \equiv 1 \pmod 2$, then there exists a $(Z_{6t} \setminus \{0, \frac{9t-1}{2},\}, 3t-1)$-system.*

Proof. Let t be an odd integer. Then the following ordered pairs form a $(Z_{6t} \setminus \{0, \frac{9t-1}{2}\}, 3t-1)$-system:

$$\begin{array}{ll} (r, 3t-r), & r = 1, 2, \ldots, \frac{3t-1}{2}, \\ (3t-1+r, 6t-r), & r = 1, 2, \ldots, \frac{3t-1}{2}. \end{array}$$ □

DEFINITION 3.29. Let $t \geq 2$ be a positive integer. A $(L, t-1)$-*system* is a set of ordered pairs of integers $\{(a_r, b_r) \mid r = 1, 2, \ldots, t-1\}$ such that

(i) $\{a_r, b_r \mid r = 1, 2, \ldots, t-1\} = \{1, 2, \ldots, \frac{t-1}{2}, \frac{t+3}{2}, \ldots, 2t\}$, and
(ii) $b_r - a_r = r$, for $r = 1, 2, \ldots, t-1$.

LEMMA 3.30. *If* $t \equiv 1 \pmod 2$ *and* $t \geq 3$, *then there exists a* $(L, t-1)$-*system.*

Proof. Let $t \geq 3$ be an odd integer. Then the following ordered pairs form a $(L, t-1)$-system:

$$(r, t+1-r), \qquad r = 1, 2, \ldots, \tfrac{t-1}{2},$$
$$(t+1+r, 2t-r), \qquad r = 1, 2, \ldots, \tfrac{t-1}{2}.$$

□

LEMMA 3.31. *There exists a* $(3,2)$*–rotational* $ESTS(15, 9)$.

Proof. The following triples form a set of starter blocks for a (3, 2)–rotational $ESTS(15, 9)$:

$$\{\infty_1, \infty_2, \infty_3\}, \quad \{\infty_i, \infty_i, \infty_i\}, \quad i = 1, 2, 3,$$
$$\{\infty_1, 0_1, 3_1\}, \quad \{\infty_1, 0_2, 3_2\}, \quad \{\infty_2, 0_2, 0_1\},$$
$$\{\infty_3, 0_2, 4_1\}, \quad \{0_1, 0_1, 1_1\}, \quad \{0_2, 0_2, 0_2\},$$
$$\{0_2, 1_2, 2_1\}, \quad \{0_2, 2_2, 5_1\}, \quad \{0_1, 2_1, 4_1\}.$$

□

LEMMA 3.32. *If* $v \equiv 3 \pmod{12}$, *then there exists a* $(3,2)$*–rotational* $ESTS\left(v, \frac{v+3}{2}\right)$.

Proof. Let $v = 2 \cdot 6t + 3$ and let t be a nonnegative integer. The case $t = 0$ is trivial and the case $t = 1$ has been treated in Lemma 3.31. Let $t \geq 2$. Then the following triples:

$$\{\infty_1, \infty_2, \infty_3\}, \quad \{0_1, (2t)_1, (4t)_1\}, \quad \{\infty_1, 0_1, (3t)_1\},$$
$$\{\infty_1, 0_2, (3t)_2\}, \quad \{\infty_i, \infty_i, \infty_i\}, \quad i = 1, 2, 3,$$
$$\{0_2, 0_2, 0_2\}, \quad \{\infty_2, 0_2, 0_1\},$$

and

$$\left\{0_1, 0_1, \left(\frac{5t}{2}\right)_1\right\} \qquad \text{if } t \equiv 0 \pmod 2, \text{ or}$$
$$\left\{0_1, 0_1, \left(\frac{3t-1}{2}\right)_1\right\} \qquad \text{if } t \equiv 1 \pmod 2)$$

and

$$\{0_1, r_1, (b_r + t - 1)_1\}, \quad r = 1, 2, \ldots, t-1,$$

where $\{(a_r, b_r) | r = 1, 2, \ldots, t-1\}$ is an $(K, t-1)$-system if $t \equiv 0 \pmod 2$ or a $(L, t-1)$-system if $t \equiv 1 \pmod 2$, and

$$\{0_2, r_2, (b_r)_1\}, \qquad r = 1, 2, \ldots, 3t-1,$$

where $\{(a_r, b_r) | r = 1, 2, \ldots, 3t-1\}$ is a $(Z_{6t} \setminus \{0, \frac{9t}{2}\}, 3t-1)$-system if $t \equiv 0 \pmod 2$ or a $(Z_{6t} \setminus \{0, \frac{9t-1}{2}\}, 3t-1)$-system if $t \equiv 1 \pmod 2$, and finally,

$$\left\{\infty_3, 0_2, \left(\frac{9t}{2}\right)_1\right\} \qquad \text{if } t \equiv 0 \pmod 2, \text{ or}$$
$$\left\{\infty_3, 0_2, \left(\frac{9t-1}{2}\right)_1\right\} \qquad \text{if } t \equiv 1 \pmod 2,$$

form a set of starter blocks for a $(3, 2)$–rotational $ESTS\left(v, \frac{v+3}{2}\right)$. □

LEMMA 3.33. *If* $v \equiv 5 \pmod{12}$ *and* $v \neq 5$, *then there exists a* $(3, 2)$*–rotational* $ESTS\left(v, \frac{v+3}{2}\right)$.

Proof. Let $v = 2(6t+1) + 3$. If $t \geq 1$, the following triples:

$$\{\infty_1, \infty_2, \infty_3\}, \qquad \{0_1, 0_1, 0_1\},$$
$$\{\infty_i, \infty_i, \infty_i\}, \qquad i = 1, 2, 3,$$
$$\{0_2, 0_2, (3t)_2\},$$
$$\{0_1, r_1, (b_r + t)_1\}, \quad r = 1, 2, \ldots, t,$$

where $\{(a_r, b_r) | r = 1, 2, \ldots, t\}$ is an (A, t)-system if $t \equiv 0$ or $1 \pmod 4$, or a (B, t)-system if $t \equiv 2$ or $3 \pmod 4$, and

$$\begin{array}{ll} \{0_2, r_2, (b_r)_1\}, & r = 1, 2, \ldots, 3t-1 \\ \{\infty_2, 0_2, (a_{3t})_1\}, & \{\infty_3, 0_2, (b_{3t})_1\}, \end{array}$$

where $\{(a_r, b_r) | r = 1, 2, \ldots, 3t\}$ is an $(A, 3t)$-system if $t \equiv 1$ or $2 \pmod 4$, or a $(B, 3t)$-system if $t \equiv 0$ or $3 \pmod 4$, here, $6t+1$ is treated as 0, and finally,

$$\begin{array}{ll} \{\infty_1, 0_2, (6t)_1\} & \text{if } t \equiv 1 \text{ or } 2 \pmod 4, \text{ or} \\ \{\infty_1, 0_2, 0_1\} & \text{if } t \equiv 0 \text{ or } 3 \pmod 4, \end{array}$$

form a set of starter blocks for a $(3, 2)$–rotational $ESTS\left(v, \frac{v+3}{2}\right)$. □

LEMMA 3.34. *There exists a $(3, 2)$–rotational $ESTS(11, 7)$.*

Proof. The following triples form a set of starter blocks for a $(3, 2)$–rotational $ESTS(11, 7)$:

$$\begin{array}{lll} \{\infty_1, \infty_2, \infty_3\}, & \{\infty_i, \infty_i, \infty_i\}, & i = 1, 2, 3, \\ \{\infty_1, 0_1, 2_1\}, & \{\infty_2, 0_2, 0_1\}, & \{\infty_3, 0_2, 3_1\}, \\ \{0_1, 0_1, 1_1\}, & \{0_2, 0_2, 0_2\}, & \{\infty_1, 0_2, 2_2\}, \\ \{0_2, 1_2, 2_1\}. & & \end{array}$$

□

LEMMA 3.35. *If $v \equiv 11 \pmod{12}$, then there exists a $(3, 2)$–rotational $ESTS\left(v, \frac{v+3}{2}\right)$.*

Proof. Let $v = 2(6t+4)+3$. The case $v = 11$ has been treated in Lemma 3.34. If $t \geq 1$, the following triples:

$$\begin{array}{ll} \{\infty_1, \infty_2, \infty_3\}, & \\ \{\infty_i, \infty_i, \infty_i\}, & i = 1, 2, 3, \\ \{\infty_1, 0_1, (3t+2)_1\}, & \{\infty_1, 0_2, (3t+2)_2\}, \\ \{\infty_2, 0_2, 0_1\}, & \{0_2, 0_2, 0_2\}, \\ \{0_1, r_1, (b_r+t)_1\}, & r = 1, 2, \ldots, t, \end{array}$$

where $\{(a_r, b_r)|r = 1, 2, \ldots, t\}$ is an (A, t)-system if $t \equiv 0$ or $1 \pmod 4$, or a (B, t)-system if $t \equiv 2$ or $3 \pmod 4$, and

$$\{0_2, r_2, (b_r)_1\}, \qquad r = 1, 2, \ldots, 3t + 1,$$

where $\{(a_r, b_r)|r = 1, 2, \ldots, 3t + 1\}$ is an $(A, 3t + 1)$-system if $t \equiv 0$ or $1 \pmod 4$, or a $(B, 3t + 1)$-system if $t \equiv 2$ or $3 \pmod 4$, and finally,

$$\{\infty_3, 0_2, (6t + 3)_1\} \quad \text{if } t \equiv 0 \text{ or } 1 \pmod 4, \text{ or}$$
$$\{\infty_3, 0_2, (6t + 2)_1\} \quad \text{if } t \equiv 2 \text{ or } 3 \pmod 4,$$

form a set of starter blocks for a $(3, 2)$–rotational $ESTS\left(v, \frac{v+3}{2}\right)$. □

Now, we can have the following theorems.

THEOREM 3.36. *There exists a $(3, 2)$-rotational $ESTS\left(v, \frac{v+3}{2}\right)$ if and only if $v \equiv 3$ or 5 (mod 6) and $v \neq 5$.*

THEOREM 3.37. *A necessary and sufficient condition for the existence of a $(3, 2)$–rotational $ESTS(v, \rho)$ is*
(i) $\rho = v$ *and* $v \equiv 1$, 3 (mod 6), $v \neq 13, 21$, or
(ii) $\rho = \frac{v+3}{2}$ *and* $v \equiv 3$, 5 (mod 6), $v \neq 5$, *or*
(iii) $\rho = 0, 3$ *and* $v \equiv 3$ (mod 6).

References

[1] F. E. Bennett and N. S. Mendelsohn, *On the existence of extended triple systems*, Util. Math. **14** (1978), 249–267.
[2] C. J. Cho, *Rotational Extended Triple Systems*, Kyungpook Math. J. **31** (1991), 219–234.
[3] ———, *Regular Extended Triple Systems*, Kyungpook Math. J. **32** (1992), 115–124.
[4] R. B. Gardner, *Steiner triple systems with near-rotational automorphisms*, J. Combin. Theory Ser. A **61** (1992), 322–327.
[5] Z. Jiang, *$(F, 2)$-Rotational Steiner Triple Systems*, Combinatorics advances, 181–206, Math. Appl. 329, Kluwer Acad. Publ., 1995.
[6] D. M. Johnson and N.S. Mendelsohn, *Extended triple systems*, Aequationes Math. **3** (1972), 291–298.
[7] E. S. O'Keefe, *Verification of conjecture of Th. Skolem*, Math. Scand. **9** (1961), 80–82.
[8] A. Rosa, *Poznámka o cycklických Steinerových systémoch trojíc*, Math.–Fzy. Čas **16** (1966), 285–290.

[9] Th. Skolem, *On certain distributions of integers in pairs with given differences*, Math. Scand. **5** (1957), 57–68.

Department of Mathematics and Statistics
Sookmyung Women's University
Seoul 140-742, Korea
E-mail: cjcho@sookmyung.ac.kr