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INFINITE FLOCKS OF QUADRATIC CONES–II GENERALIZED FISHER FLOCKS

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Abstract. This article discusses a new representation of the generalized Fisher flocks and shows that there is a unique flock for each full field K of odd or zero characteristic that has a full field quadratic extension. It is also shown that partial flock extensions of 'critical linear subflocks' are completely determined.

1. Introduction

In Jha and Johnson [2], flocks of quadratic cones are considered within $PG(3,K)$, where K is an arbitrary field. When K is infinite, the authors develop a net replacement procedure that is called 'elation-nest replacement' or 'E-nest replacement'. The construction generalizes the q-nest construction given by Baker and Ebert [1], when q is a prime and generalized by [6], for arbitrary odd order q. The translation planes corresponding to flocks of quadratic cones in $PG(3,K)$ admit an elation group E with axis ℓ such that for any line m of $PG(3,K)$ disjoint from ℓ, $Em \cup \ell$ is a regulus. When K is finite isomorphic to $GF(q)$, the order of E is q. In general, such an elation group is said to be 'regulus-inducing'.

In the following, it is assumed that a 'Baer subplane' is always a 2-dimensional vector subspace over the kernel field K that is not a 'line' of the spread in question.

The translation planes constructed by Payne and, by Baker and Ebert are constructed from a Desarguesian affine plane Σ using a regulus-inducing group E and a kernel homology group H of order $(q+1)$. Basically, a Baer subplane π_o of Σ is determined so that $EH\pi_o$ is a partial spread that covers a set of reguli of Σ that are induced using E. If $\mathcal{R}$ denotes the reguli sharing $x=0$ of Σ remaining that are not covered by the images of π_o, then there is a spread $EH\pi_o \cup \mathcal{R}$. In this case, the

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number of reguli in $\mathcal{R}$ is $(q-1)/2$. Payne and Thas [7] have shown that the only finite flocks of quadratic cones that share a linear subflock of $(q-1)/2$ conics are the Fisher flocks and the linear flocks (corresponding to Desarguesian affine plane). More generally, Johnson [4] has shown that, in fact, any non-linear partial flock in $PG(3,q)$ sharing a linear subflock of at least $(q-1)/2$ conics may be uniquely extended to a Fisher flock.

Considering what might be a generalization of having such a maximum linear subflock, we define what we call a 'critical' linear subflock as follows:

DEFINITION 1. Let $\mathcal{P}$ be a linear partial flock of a quadratic cone in $PG(3,K)$ where K is a field. Assume that there is a flock $\mathcal{L}$ in $PG(3,K)$ containing $\mathcal{P}$. Let the partial spreads corresponding to $\mathcal{P}$ and $\mathcal{L}$ be denoted by Π and Σ respectively and note that $\Pi \subseteq \Sigma$. Then, there is a regulus-inducing elation group E with axis ℓ such that Σ is a union of reguli sharing ℓ and each regulus is induced from E. These reguli are called the 'base reguli'. Note that Π is invariant under E so is also a union of reguli sharing ℓ.

We shall say that $\mathcal{P}$ is a 'critical partial flock' if and only if the following two conditions hold:

(i) Every Baer subplane within the affine plane defined by Σ and disjoint from Π intersects each base regulus of $\Sigma - \Pi$ in two components and there is some Baer subplane which is disjoint from Π, and

(ii) if $\mathcal{C}$ is a set of distinct reguli sharing ℓ, invariant under E, that covers $\Sigma - \Pi$, then every Baer subplane within Σ that is disjoint from Π and not in one of the reguli of $\mathcal{C}$ intersects exactly two components of each regulus of $\mathcal{C}$.

REMARK 1. Any linear subset of $(q-1)/2$ reguli in a spread of $PG(3,q)$ that is a union of reguli sharing a component ℓ is critical.

Proof. There are exactly $(q+1)/2$ remaining reguli and a Baer subplane disjoint from Π cannot be a Baer subplane of one of these reguli and therefore shares $0,1$ or 2 lines with each such regulus. However, this implies that there are exactly two shared lines with each regulus.

There are exactly $q(q+1)/2$ components in the remaining reguli so if there is a covering of this set by a set $\mathcal{C}$ of reguli such that $\mathcal{C}$ is invariant under E then there are exactly $(q+1)/2$ reguli in $\mathcal{C}$. If π_o is any Baer subplane of Σ that lies within this set and is not within one of the reguli of $\mathcal{C}$ then π_o has $q+1$ components and cannot be an opposite line of any of the reguli since π_o is disjoint from ℓ. Hence, if π_o is not a line of one of the reguli of $\mathcal{C}$, then π_o shares $0, 1, 2$ components of each. However, since there are but $(q+1)/2$ reguli, it follows that π_o shares exactly two components with each regulus of $\mathcal{C}$. □

In this article, we consider the so-called 'generalized Fisher' planes, defined as those planes of possibly infinite order that may be obtained using infinite E-nest replacement.

In particular, in Jha and Johnson [2], there is an open question as to whether there could be two non-isomorphic generalized Fisher planes arising from different nest replacements using the same field K and quadratic field extension $K[\theta]$, where both K and $K[\theta]$ are full fields of characteristic odd or 0. (In this case, a full field is a field such that the non-zero squares form an index two subgroup of the multiplicative group.)

Furthermore, we consider non-linear partial flocks containing critical linear subflocks and ask whether there is an extension to a flock and whether such extensions are generalized Fisher flocks.

Assuming that critical linear subflocks exist, we are able to show that any partial flock containing a critical linear subflock may be uniquely extended either to a linear flock or to a generalized Fisher flock.

Furthermore, we develop a new representation of generalized Fisher flocks in $PG(3, K)$ using the Galois group of $K[\theta]$ over K, which allows us to prove in general that there is a unique generalized Fisher flock over any full field K of characteristic odd or 0 that admits a quadratic extension full field $K[\theta]$.

2. Representation of generalized Fisher flocks

In this section, we develop a new representation of generalized Fisher flocks and, show that, in fact, there is always a unique generalized Fisher flock when there is at least one in $PG(3, K)$.

We assume that K is a full field of characteristic odd or 0 and that $K[\theta]$ is a full field quadratic extension.

Let $\sigma \in Gal_K K[\theta]$, $\sigma \neq 1$.

LEMMA 1. *All elements of K and of $\{x^{\sigma-1}; x \in K[\theta]\}$ are squares in $K[\theta]$.*

Proof. Let $\{1, e\}$ be a K-basis for $K[\theta]$ such that $e^2 = \gamma$, for γ a non-square in K (since K has odd or 0 characteristic, this is possible).

Then $(e\alpha+\beta)^2=\beta^2+\gamma\alpha^2+2\alpha\beta e$. Hence, if $\alpha\beta=0$ then we obtain either β^2 or $\gamma\alpha^2$ and since we have an index two group of squares in K, it follows that all elements of K are squares in $K[\theta]$. Now $x^{\sigma-1}=x^{\sigma+1}x^{-2}$, implying that $x^{\sigma-1}$ is a square since $x^{\sigma+1}$ is in K and a square in $K[\theta]$ by the previous argument. □

NOTATION 1. Since $x^{\sigma-1}=z^2$, we write $z=x^{(\sigma-1)/2}$, the 'positive square root'.

LEMMA 2. *If α is a non-zero square in K then $\alpha^{(\sigma-1)/2}=1$.*

Proof. If $\alpha=\delta^2$ then $\delta^{2(\sigma-1)/2}=\delta^{\sigma-1}=1$ since $\delta^\sigma=\delta$. □

LEMMA 3. *Under the previous assumptions, let b be in the subgroup of squares in $K[\theta]$. Then*

$(b^{1-\sigma}-1)^{\sigma+1}$ is square in K if -1 is a non-square in K and non-square in K if -1 is a square in K.

Proof. To see this, note that

$$(b^{1-\sigma}-1)^{\sigma+1}=2-(b^{\sigma-1}+b^{1-\sigma})=-(b^{(1-\sigma)/2}-b^{(\sigma-1)/2})^2.$$

We claim that

$$b^{\sigma(\sigma-1)/2}=b^{(1-\sigma)/2}.$$

This is true if and only if

$$b^{\sigma(\sigma-1)/2-(1-\sigma)/2}=1=b^{((\sigma-1)/2)(\sigma+1)}=b^{(\sigma^2-1)/2},$$

which is valid since b is a square in $K[\theta]$.

Then,

$$-(b^{(1-\sigma)/2}-b^{(\sigma-1)/2})^2 \text{ is a square in } K,$$

implies that

$$\begin{aligned}
&(-1)^{(\sigma-1)/2}((b^{(1-\sigma)/2}-b^{(\sigma-1)/2})^2)^{(\sigma-1)/2}\\
=\;&(-1)^{(\sigma-1)/2}(b^{(1-\sigma)/2}-b^{(\sigma-1)/2})^{\sigma-1}\\
=\;&(-1)^{(\sigma-1)/2}(b^{\sigma(1-\sigma)/2}-b^{\sigma(\sigma-1)/2}))/(b^{(1-\sigma)/2}-b^{(\sigma-1)/2})\\
=\;&(-1)^{(\sigma-1)/2}(b^{(\sigma-1)/2}-b^{(1-\sigma)/2})/(b^{(1-\sigma)/2}-b^{\sigma-1)/2})\\
=\;&(-1)^{(\sigma-1)/2}(-1)=(-1)^{(\sigma+1)/2},
\end{aligned}$$

which is a contradiction if -1 is a square in K, since then $(-1)^{(\sigma-1)/2}=1$. Hence, assume that -1 is a non-square in K. Let $\gamma=-1$ so that $e^2=-1$ and $e^{2(\sigma+1)/2}=e^{\sigma+1}=-e=1$. Thus, we have completed the proof of the lemma. □

THEOREM 1. *Let K be a full field of odd or 0 characteristic and let $K[\theta]$ be a quadratic extension of K that is also a full field. Let Σ be the Pappian affine plane coordinatized by $K[\theta]$ and let H be the kernel homology group of squares in Σ.*

Let s be any element of $K[\theta]$ such that $s^{\sigma+1}$ is nonsquare in K if -1 is a non-square in K, and $s^{\sigma+1}$ is square in K if -1 is a square in K. Let E denote the regulus-inducing group and H is the homology group of squares of kernel homologies in Σ. Then,

$$EH(y = x^{\sigma}s) \cup \left\{y = xm; (m+\beta)^{\sigma+1} \neq s^{\sigma+1}\ \forall \beta \in K\right\}$$

is a generalized Fisher conical spread in $PG(3, K)$.

Proof. We now take the group H as the subgroup of squares of the kernel homology group of a Pappian plane Σ coordinatized by $K[\theta]$, and E the regulus-inducing elation group analagous to the finite case. By Johnson [5], any Baer subplane of Σ, the associated Pappian affine plane, disjoint from the axis $x = 0$ of E has the form $y = x^{\sigma}m + xn$ for $m \neq 0$. That is,

$$EH(y = x^{\sigma}s) = \left\{(y = x^{\sigma}sb^{1-\sigma} + x\alpha); b \text{ is a square in } K[\theta], \alpha \in K\right\}.$$

We first claim that this is a partial spread. Since we have an orbit under EH, we only need to check that $y = x^{\sigma}s$ is disjoint from all of the subspaces in the orbit.

Hence, assume that

$$x_o^{\sigma}s = x_o^{\sigma}sb^{1-\sigma} + x_o(\alpha), \text{ for some } x_o \in K[\theta].$$

Then,

$$x_o^{\sigma}s(1 - b^{1-\sigma}) = x_o\alpha.$$

If $x_o \neq 0$ then we have

$$x_o^{\sigma-1}s(1 - b^{1-\sigma}) = \alpha,$$

implying that

$$(s(1 - b^{1-\sigma}))^{1+\sigma} = \alpha^{1+\sigma} = \alpha^2.$$

First assume that -1 is a square in K, so that $s^{1+\sigma}$ is a square in K. Then, by lemma 3 we have $(b^{1-\sigma} - 1)^{\sigma+1}$ is a nonsquare. Hence, this is a contradiction so we have a partial spread. Similarly if -1 is a non-square in K then $(b^{1-\sigma} - 1)^{\sigma+1}$ is a square in K but since $s^{\sigma+1}$ is nonsquare, we have a contradiction and hence a partial spread.

It remains to show that we obtain a spread. Since we have an associated Desarguesian spread Σ, it remains to show that if an element of $EH(y = x^{\sigma}s)$ nontrivially intersects a component $y = xn$ of Σ, then this component is completely covered. Now an element of $EH(y = x^{\sigma}s)$

is a Baer subplane of Σ, H is an index two subgroup of the full kernel homology group H^+ and H^+ acts transitively on the non-zero points of any components. So, it follows that $y = xn$ is at least 'half' covered in the sense that the given subplane π_o of $EH(y = x^\sigma s)$ intersects $y = xn$ in a 1-dimensional K-subpace X and XH is covered by images of intersections of the given subplane under H as $y = xn$ is fixed by H. Now the component $y = xn$ is in a unique orbit Γ of components under the group E. If π_o intersects two components of Γ, say $y = xn$ and $y = x(n + \alpha_o)$ for $\alpha_o \in K$, then there is also a 1-dimensional K-subpace X_{α_o} in π_o on $y = x(n + \alpha_o)$ and a corresponding orbit $X_{\alpha_o}H$ in $y = x(n + \alpha_o)$. Note that E commutes with H. The elation $\tau : (x, y) \longmapsto (x, -x\alpha_o + y)$ maps $X_{\alpha_o}H$ onto $X_{\alpha_o}\tau H$. Since $X_{\alpha_o}\tau$ is a 1 -dimensional K-subspace on $y = xn$, it follows that either XH and $X_{\alpha_o}\tau H$ define the same H-orbit on $y = xn$ or $XH \cup X_{\alpha_o\tau}H = \{(x, y); y = xn; x \neq 0\}$. But, if $XH = X_{\alpha_o}\tau H$, then we do not have a partial spread $EH(y = x^\sigma s)$.

Hence, it remains to show that when an element π_o of $EH(y = x^\sigma s)$ intersects a component $y = xn$ then π_o also intersects $y = x(n + \alpha_o)$ for some $\alpha_o \neq 0$.

Since we have an orbit under EH, we may assume that π_o is $y = x^\sigma s$. Hence, $y = xn$ and $y = x^\sigma s$ intersect nontrivially if and only if

$$x_o n = x_o^\sigma s$$

for $x_o \neq 0$. So,

$$n^{\sigma+1} = s^{\sigma+1}.$$

Now consider when $y = x^\sigma s$ will nontrivially intersect $y = x(n + \alpha)$ for some nonzero $\alpha \in K$. We claim that there is an intersection if and only if

$$s^{\sigma+1} = (n + \alpha)^{\sigma+1},$$

which is certainly necessary. To see that it is sufficient, we note, by Hilbert's Theorem 90, that since $(s/(n + \alpha))^{\sigma+1} = 1$ then $s/(n + \alpha) = v^{1-\sigma}$, for some $v \in K[\theta] - \{0\}$. So,

$$v^\sigma s = v(n + \alpha),$$

which implies that $y = x^\sigma s$ and $y = x(n + \alpha)$ nontrivially intersect.

So, if

$$n^{\sigma+1} = s^{\sigma+1},$$

assume that

$$s^{\sigma+1} = (n + \alpha)^{\sigma+1},$$

but require that this equation implies that $\alpha = 0$. We see that the above equation is equivalent to

$$\alpha^2 + \alpha(n + n^\sigma) = 0.$$

Hence, there are two distinct solutions, 0 and $-(n + n^\sigma)$ for α unless $n + n^\sigma = 0$. Let a basis for $K[\theta]$ be $\{1, e\}$ such that $e^2 = \gamma$, a nonsquare in K. Then $n = e\delta + \rho$ for $\delta, \rho \in K$ and $n^\sigma = -n$ if and only if $\rho = 0$. So, $n^{\sigma+1} = -n^2 = -\gamma\delta^2$. Thus, we arrive at the equation:

$$s^{\sigma+1} = -\gamma\delta^2.$$

But, $s^{\sigma+1}$ is nonsquare or square if and only if -1 is nonsquare or square respectively. If $s^{\sigma+1}$ is nonsquare then $-\gamma$ is square so that $-\gamma\delta^2$ is square in K, a contradiction. Similarly if $s^{\sigma+1}$ is square then $-\gamma$ is nonsquare and $-\gamma\delta^2$ is nonsquare, a contradiction.

Hence, we have that there are two intersections in an E-orbit of components of Σ with an element of $EH(y = x^\sigma s)$ provided there is one. This completes the proof of the theorem. □

3. Uniqueness of generalized Fisher flocks

We begin with a general result on André planes.

LEMMA 4. *Let K be a field and $K[\theta]$ a quadratic field extension of K. Let Σ denote the Pappian plane coordinatized by $K[\theta]$. Let σ denote the involution in $Gal_K K[\theta]$.*

Consider the following André partial spread: $A_\rho = \{y = xn; n^{\sigma+1} = \rho\}$.

(1) *Then, A_ρ is a regulus in $PG(3, K)$ with opposite regulus A_ρ^σ, defined by $A_\rho^\sigma = \{y = x^\sigma n; n^{\sigma+1} = \rho\}$.*

(2) $A_\rho^\sigma = \{y = x^\sigma n_o a^{1-\sigma}; n_o^{\sigma+1} = \rho; \forall a \in K - \{0\}\}$.

Proof. We note that $y = x^\sigma m$ and $y = xn$ such that $m^{\sigma+1} = n^{\sigma+1}$ must intersect in a 1-dimensional K-space (a projective point). Furthermore, note that $(m/n)^{\sigma+1} = 1$ if and only if $mn^{-1} = v^{1-\sigma}$ for some v in $K[\theta]$, by Hilbert's theorem 90, as we have a cyclic extension quadratic extension $K[\theta]$ of K with Galois group over K of order 2. Furthermore, $(v, v^\sigma m) = (v, vn)$ if and only if $v^{1-\sigma} = mn^{-1}$. If $y = xn_o$ is fixed in A_ρ, then $y = xn$ is in A_ρ if and only if $y = xn_o v^{1-\sigma}$ for some v. Hence, every 1-dimensional subspace of $y = x^\sigma m$ lies uniquely on some element $y = xn$ of A_ρ and $y = x^\sigma m$ must intersect each element of A_ρ. This proves part (1).

Now another application of Hilbert's theorem 90 gives the proof to part (2). □

Now assume that we obtain a conical spread obtained via E-nest replacement.

Then, we must have a Baer subplane of the form $y = x^\sigma m + xn$ acting in place of $y = x^\sigma s$ above. The exact same arguement will show that we only obtain a partial spread $EH\{y = x^\sigma m + xn\}$ if and only if $m^{\sigma+1}$ is non-square (respectively, square) in K if and only if -1 is non-square (respectively, non-square) in K.

Now we consider the following mappings that normalize E:

$$\tau_{a,b,\beta} : (x, y) \longmapsto (xa, xb + ya\beta); a, b \in K[\theta]^*, \beta \in K^*.$$

Note that $\tau_{a,0,\beta}$ maps $y = x^\sigma m$ onto $y = x^\sigma m a^{1-\sigma}\beta$. Note that $(ma^{1-\sigma}\beta)^{\sigma+1} = m^{\sigma+1}\ \beta^2$. Thus, since we have a full field, we apply Lemma 4 so show that for a fixed m:

$$\begin{aligned} &\left\{n; n^{\sigma+1} \text{ is square in } K - \{0\}\right\} \\ = &\left\{ma^{1-\sigma}\beta;\ m^{\sigma+1} \text{ is square; } a \in K[\theta]^*,\ \beta \in K - \{0\}\right\}, \\ &\left\{n; n^{\sigma+1} \text{ is nonsquare in } K - \{0\}\right\} \\ = &\left\{ma^{1-\sigma}\beta;\ m^{\sigma+1} \text{ is square; } a \in K[\theta]^*\beta \in K - \{0\}\right\}. \end{aligned}$$

It will now follow that we obtain an isomorphic plane whenever the basic conditions required for a partial spread above are met.

THEOREM 2. *Let K be a full field of odd or 0 characteristic and let $K[\theta]$ be a quadratic extension of K that is also a full field. Σ be the Pappian affine plane coordinatized by $K[\theta]$.*

Then, any two generalized Fisher conical spreads in $PG(3, K)$ are isomorphic.

Proof. The group $GL(2, K[\theta])$ is triply transitive on the components of the spread for Σ. This means that we may assume that in the construction of two generalized Fisher planes, we may assume that we use the same axis $x = 0$, regulus-inducing group E and kernel homology group of squares of Σ in the same form for both planes. The question therefore is merely the choice of the Baer subplane π_o to use to form the partial spread $EH\pi_o$ that induces the spread. But, any two Baer subplanes have the form $y = x^\sigma m_i + xn_i$, for $i = 1, 2$ and $m_i \neq 0$. Clearly, we may apply an appropriate elation with axis $x = 0$ that normalizes EH to allow $n_1 = 0$. Now a partial spread $EH\pi_o$ is obtained if and only if $m_i^{\sigma+1}$ is square or non-square exactly when -1 is square or non-square, respectively. We have shown above that we may apply mappings that normalize EH and map $y = x^\sigma m_1$ onto $y = x^\sigma m_2$. but, then an appropriate elation with axis $x = 0$ will map $y = x^\sigma m_2$ onto $y = x^\sigma m_2 + xn_2$. Hence, any two generalized Fisher planes are isomorphic. □

4. Critical linear subflocks

Assume that $\mathcal{N}$ is a non-linear partial flock in $PG(3, K)$ containing a critical linear subflock $\mathcal{P}$. Let $\mathcal{L}$ denote a linear flock containing $\mathcal{P}$.

LEMMA 5. *There is a unique linear flock containing a critical linear subflock.*

Proof. Suppose there are two such flocks and let Σ and Σ' denote the corresponding Pappian spreads defined by the linear flocks and containing the partial spread Π defined by the critical linear subflock. Let m be a line of $\Sigma' - \Sigma$, so that m becomes a Baer subplane of Σ disjoint from Π. Hence, m intersects each base regulus of $\Sigma - \Pi$ in two components. We are finished unless possibly the critical linear subflock consists of exactly one regulus, which does not occur. Hence, m intersects all but one base reguli of Σ in two components, which cannot be the case. □

Now let $K[\theta]$ denote the quadratic extension field of K coordinatizing the affine plane given by Σ. Assume that K and $K[\theta]$ are full fields of odd or zero characteristic.

Let σ denote the involution in $Gal_K K[\theta]$ and note by Johnson [5] that any Baer subplane disjoint from the elation axis $x = 0$ of E has the form $y = x^\sigma m + xn$, for $m \neq 0$.

By assumption, we may assume that this Baer subplane π_o intersects two components of each of the base reguli of $\Sigma - \Pi$, and this Baer subplane corresponds to a component of the partial spread given by $\mathcal{N} - \pm$.

We see by applying $(x, y) \longmapsto (x, -xn + y)$, we may assume that $n = 0$.

Now $y = x^\sigma m$ intersects $y = xn$ if and only if $m^{\sigma+1} = n^{\sigma+1}$.

Since non-squares exist in K we may choose a basis $\{1, e\}$ such that $e^2 = \gamma$, a non-square. Then, the base regulus defined by $y = xn$ is also defined by $y = xen_1$ for some n_1 in K.

Hence, we must have

$$m^{\sigma+1} = \alpha^2 - \gamma n_1^2$$

has two solutions whenever it has one. Note that $(e\beta+\delta)^{\sigma+1} = \delta^2 - \gamma\beta^2$. There is a solution α if and only if $-\alpha$ is also a solution. Moreover, if $\alpha = 0$ then $m^{\sigma+1}$ cannot be $-\gamma n_1^2$.

Now consider $EH(y = x^\sigma m)$, where H is the kernel subgroup of squares. This is the following set:

$$\{y = x^\sigma b^{1-\sigma} m + x\alpha; \alpha \in K \text{ and } b \text{ a square in } K[\theta]\}.$$

We want to prove that this is a partial spread that covers the base reguli of intersection. Assume that -1 is a square. We note that $m^{\sigma+1}$ cannot be $-\gamma n_1^2$, for any n_1^2, so that in full fields, this implies that $m^{\sigma+1}$ is square. Similarly, if -1 is a square and $m^{\sigma+1}$ cannot be $-\gamma n_1^2$ for any n_1^2, then, for full fields, this implies that $m^{\sigma+1}$ is a square. In the following we show that we obtain a generalized Fisher spread; that $\mathcal{N}$ is a generalized Fisher spread.

Take two components m_1 and m_2 of $\mathcal{N} - \mathcal{P}$ and extend each to two generalized Fisher spreads π_1 and π_2, respectively and note that this is guaranteed possible by the main theorem of Jha and Johnson [2]. Clearly as a set of vectors $EHm_1 = EHm_2$. We wish to show that $\pi_1 = \pi_2$ and contain $\mathcal{N}$; any non-linear extension of a critical partial flock may be uniquely extended to a generalized Fisher flock.

Hence, we may assume that m_2 is not a component of π_1. We note that m_1 and m_2 are both Baer subplanes of Σ and as such define reguli (regulus nets) of Σ. Since $\mathcal{N}$ is a partial flock, it follows that Em_1 and Em_2 are either equal or disjoint (they share only the zero vector). If these two partial spreads are equal then $\pi_1 = \pi_2$. Hence, Em_1 and Em_2 are disjoint partial spreads.

Since m_2 is not in π_1 as a component and since $\mathcal{P}$ is critical, the regulus R_2 intersects two components of each of the reguli of $\pi_1 - \Sigma$ defined by the E-orbits of components, which cannot occur since Em_2 and Em_1 are disjoint.

Note that by property (ii) in the definition of critical subflock, m_2 intersects each regulus of $\pi_1 - \Sigma$ in two components. However, Em_1 union the axis of E is a regulus of $\pi_1 - \Sigma$, implying that m_2 non-trivially intersects Em_1, contradicting the fact that Em_2 and Em_1 are disjoint. Hence, every component of $\mathcal{N} - \mathcal{P}$ is a component of the generalized Fisher spread π_1 obtained by use of a single component m_1. This shows that the partial spread may be extended uniquely to a spread. So, we obtain the following result.

THEOREM 3. *Let K be a full field of characteristic 0 or odd and let $K[\theta]$ be a full field quadratic extension of K.*

If there exists a linear critical partial flock $\mathcal{P}$ of a quadratic cone then any non-linear partial flock extension of $\mathcal{P}$ may be uniquely extended to a generalized Fisher flock.

Finally, we note some examples of full fields admitting quadratic extension full fields. Both of these also appear in Jha and Johnson [3]

EXAMPLE 1. Let P_o be isomorphic to $GF(p)$ where p is an odd prime. Let F be any algebraic field extension of P_o which is not algebraically closed and which is not a series of quadratic extensions of extensions of P_o. Then F is a full field.

EXAMPLE 2. Let F be an ordered field which admits an ordered quadratic extension K such that the positive elements of each field have square roots in the field. Then both F and K are full fields.

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