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AN ANALOGUE OF HILBERT'S INEQUALITY AND ITS EXTENSIONS

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ABSTRACT. In this paper, we obtain an extension of an analogue of Hilbert's inequality involving series of nonnegative terms. The integral analogies of the main results are also given.

1. Introduction

Hilbert's double series theorem [2, p.226] was proved first by Hilbert in his lectures on integral equations. The determination of the constant, the integral analogue, the extension, other proofs of the whole or of parts of the theorems and generalizations in different directions have been given by several authors (cf. [2, Chap. 9]). Recently, B. G. Pachpatte [11] established a new inequalities similar to those of Hilbert. A representative sample is the following.

THEOREM 1.1. *Let* $p > 1,\ q > 1$ *be constants and let* $1/p + 1/q = 1$. *Let* $a(s) : N_m \to R, b(t) : N_n \to R$, *and* $a(0) = b(0) = 0$. *Then*

$$\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{|a(s)||b(t)|}{qs^{p-1}+pt^{q-1}}$$
$$\leq M(p,q,m,n)\Big(\sum_{s=1}^{m}(m-s+1)|\nabla a(s)|^p\Big)^{1/p}$$
$$\times\Big(\sum_{t=1}^{n}(n-t+1)|\nabla b(t)|^q\Big)^{1/q}$$

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for $m, n \in N$, *where*

$$M(p,q,m,n) = \frac{1}{pq} m^{(p-1)/p} n^{(q-1)/q}$$

for $m, n \in N$.

The purpose of the present paper is to derive an extension of the inequality given in Theorem 1.1 and its integral analogue. In addition, we obtain some new Hilbert type inequalities. These inequalities improve the results obtained by B. G. Pachpatte in [11]. The author would like to thank the editor and the referee for their useful comments.

2. Main results

In what follows we denote by R the set of real numbers. Let $N = \{1, 2, ...\}$, $N_0 = \{0, 1, 2, ...\}$, $N_\alpha = \{1, 2, ..., \alpha\}$, $\alpha \in N$. We define the operator ∇ by $\nabla u(t) = u(t) - u(t-1)$ for any function u defined on N_0. For a function $v(s,t) : N_0 \times N_0 \to R$, we define the operators $\nabla_1 v(s,t) = v(s,t) - v(s-1,t), \nabla_2 v(s,t) = v(s,t) - v(s,t-1)$, and $\nabla_2\nabla_1 v(s,t) = \nabla_2(\nabla_1 v(s,t)) = \nabla_1(\nabla_2 v(s,t))$. Let $I = [0,\infty), I_0 = (0,\infty), I_\beta = [0,\beta), \beta \in I_0$, denote the subintervals of R. For any function $u : I \to R$, we denote by u' the derivatives of u, and for the function $u(s,t) : I \times I \to R$, we denote the partial derivatives $(\partial/\partial s)$ $u(s,t)$, $(\partial/\partial t)$ $U(s,t)$, and $(\partial^2/\partial s\partial t)$ $u(s,t)$ by $D_1u(s,t)$, $D_2u(s,t)$, and $D_2D_1u(s,t) = D_1D_2u(s,t)$, respectively. Our main results are given in the following theorems:

THEOREM 2.1. *Let* $p > 1$, $q > 1$ *be constants. Let* $a(s) : N_m \to R, b(t) : N_n \to R$, *and* $a(0) = b(0) = 0$. *Then*

$$\sum_{s=1}^{m}\sum_{t=1}^{n} \frac{|a(s)||b(t)|}{qs^{(p-1)(p+q)/pq} + pt^{(q-1)(p+q)/pq}}$$
$$\leq M(p,q,m,n)\Big(\sum_{s=1}^{m}(m-s+1)|\nabla a(s)|^p\Big)^{1/p}$$
$$\times \Big(\sum_{t=1}^{n}(n-t+1)|\nabla b(t)|^q\Big)^{1/q}$$

for $m, n \in N$, *where*

$$M(p,q,m,n) = \frac{1}{p+q} m^{(p-1)/p} n^{(q-1)/q}.$$

Proof. From the hypotheses of Theorem 2.1, it is easy to observe that the following identities hold

$$a(s) = \sum_{\tau=1}^{s} \nabla a(\tau), \tag{1}$$

$$b(t) = \sum_{\sigma=1}^{s} \nabla b(\sigma) \tag{2}$$

for $s \in N_m, t \in N_n$. From (1) and (2) and using Hölder's inequality with indices $p, p/(p-1)$ and $q, q/(q-1)$, respectively, we have

$$|a(s)| \leq (s)^{(p-1)/p} \left(\sum_{\tau=1}^{s} |\nabla a(\tau)|^p \right)^{1/p}, \tag{3}$$

$$|b(t)| \leq (t)^{(q-1)/q} \left(\sum_{\sigma=1}^{s} |\nabla b(\sigma)|^q \right)^{1/q} \tag{4}$$

for $s \in N_m, t \in N_n$. Using the inequality of means [8, p. 15]

$$\left(\prod_{i=1}^{n} s_i^{\omega_i} \right)^{1/\Omega_n} \leq \left(\frac{1}{\Omega_n} \sum_{i=1}^{n} \omega_i s_i^r \right)^{1/r}$$

for $r > 0, \omega_i > 0, \sum_{i=1}^{n} \omega_i = \Omega_n$, we observe that

$$(s_1^{\omega_1} s_2^{\omega_2})^{r/(\omega_1+\omega_2)} \leq \frac{1}{\omega_1 + \omega_2} \left(\omega_1 s_1^r + \omega_2 s_2^r \right). \tag{5}$$

Let $s_1 = s^{p-1}, s_2 = t^{q-1}, \omega_1 = 1/p, \omega_2 = 1/q$ and $r = \omega_1 + \omega_2$, from (3) and (4), we have

$$\begin{aligned}
&|a(s)||b(t)| \\
&\leq (s)^{(p-1)/p}(t)^{(q-1)/q} \left(\sum_{\tau=1}^{s} |\nabla a(\tau)|^p \right)^{1/p} \left(\sum_{\sigma=1}^{t} |\nabla b(\sigma)|^q \right)^{1/q} \\
&\leq \frac{pq}{p+q} \left(\frac{s^{(p-1)(p+q)/pq}}{p} + \frac{t^{(q-1)(p+q)/pq}}{q} \right) \left(\sum_{\tau=1}^{s} |\nabla a(\tau)|^p \right)^{1/p} \\
&\quad \times \left(\sum_{\sigma=1}^{t} |\nabla b(\sigma)|^q \right)^{1/q}
\end{aligned}$$

for $s \in N_m, t \in N_n$. We observe that for $s \in N_m, t \in N_n$,

$$\frac{|a(s)||b(t)|}{F(p,q,s,t)} \leq \frac{1}{p+q}\left(\sum_{\tau=1}^{s}|\nabla a(\tau)|^p\right)^{1/p}\left(\sum_{\sigma=1}^{t}|\nabla b(\sigma)|^q\right)^{1/q}, \tag{6}$$

where $F(p,q,s,t) = qs^{(p-1)(p+q)/pq} + pt^{(q-1)(p+q)/pq}$. Taking the sum on both sides of (6) first over t from 1 to n and then over s from 1 to m of the resulting inequality and using Hölder's inequality with indices $p, p/(p-1)$ and $q, q/(q-1)$ and interchanging the order of summations, we observe that

$$\begin{aligned}
&\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{|a(s)||b(t)|}{qs^{(p-1)(p+q)/pq}+pt^{(q-1)(p+q)/pq}}\\
&\leq \frac{1}{p+q}\left\{\sum_{s=1}^{m}\left(\sum_{\tau=1}^{s}|\nabla a(\tau)|^p\right)^{1/p}\right\}\left\{\sum_{t=1}^{n}\left(\sum_{\sigma=1}^{t}|\nabla b(\sigma)|^q\right)^{1/q}\right\}\\
&\leq \frac{1}{p+q}(m)^{(p-1)/p}\left\{\sum_{s=1}^{m}\left(\sum_{\tau=1}^{s}|\nabla a(\tau)|^p\right)^{1/p}\right\}(n)^{(q-1)/q}\\
&\quad\times\left\{\sum_{t=1}^{n}\left(\sum_{\sigma=1}^{t}|\nabla b(\sigma)|^q\right)^{1/q}\right\}\\
&= M(p,q,m,n)\left(\sum_{s=1}^{m}(m-s+1)|\nabla a(s)|^p\right)^{1/p}\\
&\quad\times\left(\sum_{t=1}^{n}(n-t+1)|\nabla b(t)|^q\right)^{1/q}.
\end{aligned}$$

The proof of Theorem 2.1 is complete. □

REMARK 1. In Theorem 2.1, setting $1/p+1/q=1$, we have Theorem 1.1. From the inequality (5), we obtain

$$(s_1^{\omega_1}s_2^{\omega_2}) \leq \frac{1}{\omega_1+\omega_2}\left(\omega_1 s_1^{\omega_1+\omega_2}+\omega_2 s_2^{\omega_1+\omega_2}\right). \tag{7}$$

If we apply the elementary inequality (7) on the right-hand sides of result inequality in Theorem 2.1, then we get the following inequality

$$\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{|a(s)||b(t)|}{qs^{(p-1)(p+q)/pq}+pt^{(q-1)(p+q)/pq}}$$
$$\leq \frac{pq}{(p+q)^2}m^{(p-1)/p}n^{(q-1)/q}\left\{\frac{1}{p}\left(\sum_{s=1}^{m}(m-s+1)|\nabla a(s)|^p\right)^{(p+q)/pq}\right.$$
$$\left.+\frac{1}{q}\left(\sum_{t=1}^{n}(n-t+1)|\nabla b(t)|^q\right)^{(p+q)/pq}\right\}.$$

An integral analogue of Theorem 2.1 is established in the following theorem.

THEOREM 2.2. *Let $p>1$, $q>1$ be constants. Let $f(s)$ and $g(t)$ be real-valued continuous functions defined on I_x and I_y, respectively, and let $f(0)=g(0)=0$. Then*

$$\int_0^x\int_0^y\frac{|f(s)||g(t)|}{qs^{(p-1)(p+q)/pq}+pt^{(q-1)(p+q)/pq}}\,ds\,dt$$
$$\leq M(p,q,x,y)\left(\int_0^x(x-s)|f'(s)|^p\,ds\right)^{1/p}\left(\int_0^y(y-t)|g'(t)|^q\,dt\right)^{1/q}$$

for $x,y\in I_0$, where

$$K(p,q,x,y)=\frac{1}{p+q}x^{(p-1)/p}y^{(q-1)/q}.$$

Proof. From the hypotheses of Theorem 2.2, we have the following identities

$$f(s)=\int_0^s f'(\tau)\,d\tau, \tag{8}$$

$$g(t)=\int_0^t g'(\sigma)\,d\sigma \tag{9}$$

for $s\in I_x, t\in I_y$. From (8) and (9) and using Hölder's integral inequality with indices $p, p/(p-1)$ and $q, q/(q-1)$, respectively, we have

$$|f(s)|\leq (s)^{(p-1)/p}\left(\int_0^s|f'(\tau)|^p\,d\tau\right)^{1/p}, \tag{10}$$

$$|g(t)|\leq (t)^{(q-1)/q}\left(\int_0^t|g'(\sigma)|^q\,d\sigma\right)^{1/q} \tag{11}$$

for $s \in I_x, t \in I_y$. From (10) and (11) and using the inequality (5), we observe that

$$
\begin{aligned}
&|f(s)||g(t)| \\
&\le (s)^{(p-1)/p}(t)^{(q-1)/q}\left(\int_0^s |f'(\tau)|^p \, d\tau\right)^{1/p}\left(\int_0^t |g'(\sigma)|^q \, d\sigma\right)^{1/q} \\
&\le \frac{pq}{p+q}\left(\frac{s^{(p-1)(p+q)/pq}}{p} + \frac{t^{(q-1)(p+q)/pq}}{q}\right)\left(\int_0^s |f'(\tau)|^p \, d\tau\right)^{1/p} \\
&\quad \times \left(\int_0^t |g'(\sigma)|^q \, d\sigma\right)^{1/q}
\end{aligned}
$$

for $s \in I_x, t \in I_y$. From above inequality, we observe that

$$
\frac{|f(s)||g(t)|}{F(p,q,s,t)} \le \frac{1}{p+q}\left(\int_0^s |f'(\tau)|^p \, d\tau\right)^{1/p}\left(\int_0^t |g'(\sigma)|^q \, d\sigma\right)^{1/q}, \tag{12}
$$

where $F(p,q,s,t) = qs^{(p-1)(p+q)/pq} + pt^{(q-1)(p+q)/pq}$. Integrating both sides of (12) over t from 0 to y first and then integrating the resulting inequality over s from 0 to x and using Hölder's inequality with indices $p, p/(p-1)$ and $q, q/(q-1)$, we observe that

$$
\begin{aligned}
&\int_0^x \int_0^y \frac{|f(s)||g(t)|}{qs^{(p-1)(p+q)/pq} + pt^{(q-1)(p+q)/pq}} \, ds \, dt \\
&\le \frac{1}{p+q}\left\{\int_0^x \left(\int_0^s |f'(\tau)|^p \, d\tau\right)^{1/p} ds\right\}\left\{\int_0^y \left(\int_0^t |g'(\sigma)|^q \, d\sigma\right)^{1/q} dt\right\} \\
&\le \frac{1}{p+q}(x)^{(p-1)/p}\left\{\int_0^x \left(\int_0^s |f'(\tau)|^p \, d\tau\right) ds\right\}^{1/p} \\
&\quad \times (y)^{(q-1)/q}\left\{\int_0^y \left(\int_0^t |g'(\sigma)|^q \, d\sigma\right) dt\right\}^{1/q} \\
&= M(p,q,x,y)\left(\int_0^x (x-s)|f'(s)|^p \, ds\right)^{1/p}\left(\int_0^y (y-t)|g'(t)|^q \, dt\right)^{1/q}.
\end{aligned}
$$

The proof of Theorem 2.2 is complete. □

REMARK 2. By applying the elementary inequality (7) on the right-hand sides of result inequality in Theorem 2.2, we get the following inequality

$$\int_0^x \int_0^y \frac{|f(s)||g(t)|}{qs^{(p-1)(p+q)/pq} + pt^{(q-1)(p+q)/pq}}\, ds\, dt$$
$$\leq \frac{pq}{(p+q)^2} m^{(p-1)/p} n^{(q-1)/q} \left\{ \frac{1}{p} \left(\int_0^x (x-s)|f'(s)|^p\, ds \right)^{(p+q)/pq} + \frac{1}{q} \left(\int_0^y (y-t)|g'(t)|^q\, dt \right)^{(p+q)/pq} \right\}.$$

In the following theorems we establish the two independent variable versions of the inequalities given in Theorems 2.1 and 2.2.

THEOREM 2.3. *Let $p > 1, q > 1$ be constants. Let $a(s,t) : N_x \times N_y \to R, b(k,r) : N_z \times N_w \to R$, and $a(0,t) = b(0,t) = 0, a(s,0) = b(s,0) = 0$. Then*

$$\sum_{s=1}^{x} \sum_{t=1}^{y} \left(\sum_{k=1}^{z} \sum_{r=1}^{w} \frac{|a(s,t)||b(k,r)|}{qs^{(p-1)(p+q)/pq} + pt^{(q-1)(p+q)/pq}} \right)$$
$$\leq L(p,q,x,y,z,w) \left(\sum_{s=1}^{x} \sum_{t=1}^{y} (x-s+1)(y-t+1)|\nabla_2 \nabla_1 a(s,t)|^p \right)^{1/p}$$
$$\times \left(\sum_{k=1}^{z} \sum_{r=1}^{w} (z-k+1)(w-r+1)|\nabla_2 \nabla_1 b(k,r)|^q \right)^{1/q}$$

for $x, y, z, w \in N$, where

$$L(p,q,x,y,z,w) = \frac{1}{p+q} (xy)^{(p-1)/p} (zw)^{(q-1)/q}.$$

Proof. From the hypotheses of Theorem 2.3, it is easy to observe that the following identities hold

$$a(s,t) = \sum_{\xi=1}^{s} \sum_{\eta=1}^{t} \nabla_2 \nabla_1 a(\xi, \eta), \tag{13}$$

$$b(k,r) = \sum_{\sigma=1}^{k} \sum_{\tau=1}^{r} \nabla_2 \nabla_1 b(\sigma, \tau) \tag{14}$$

for $(s,t) \in N_x \times N_y, (k,r) \in N_z \times N_w$. From (13) and (14) and using Hölder's inequality with indices $p, p/(p-1)$ and $q, q/(q-1)$, respectively, we have

$$|a(s,t)| \le (st)^{(p-1)/p}\left(\sum_{\xi=1}^{s}\sum_{\eta=1}^{t}|\nabla_2\nabla_1 a(\xi,\eta)|^p\right)^{1/p}, \tag{15}$$

$$|b(k,r)| \le (kr)^{(q-1)/q}\left(\sum_{\sigma=1}^{k}\sum_{\tau=1}^{r}|\nabla_2\nabla_1 b(\sigma,\tau)|^q\right)^{1/q} \tag{16}$$

for $(s,t) \in N_x \times N_y, (k,r) \in N_z \times N_w$. From (15) and (16) and using the inequality (5), for $(s,t) \in N_x \times N_y, (k,r) \in N_z \times N_w$. it is easy to observe that

$$\begin{aligned}&\frac{|a(s,t)||b(k,r)|}{F(p,q,s,t)}\\ &\le \frac{1}{p+q}\left(\sum_{\xi=1}^{s}\sum_{\eta=1}^{t}|\nabla_2\nabla_1 a(\xi,\eta)|^p\right)^{1/p}\left(\sum_{\sigma=1}^{k}\sum_{\tau=1}^{r}|\nabla_2\nabla_1 b(\sigma,\tau)|^q\right)^{1/q},\end{aligned}$$

where $F(p,q,s,t) = qs^{(p-1)(p+q)/pq} + pt^{(q-1)(p+q)/pq}$. Taking the sum on both sides of above inequality first over r from 1 to w and then over k from 1 to z and taking the sum on both sides of the resulting inequality first over t from 1 to y and then over s from 1 to x and and using Hölder's inequality with indices $p, p/(p-1)$ and $q, q/(q-1)$ and interchanging the order of summations, we observe that

$$\begin{aligned}&\sum_{s=1}^{x}\sum_{t=1}^{y}\left(\sum_{k=1}^{z}\sum_{r=1}^{w}\frac{|a(s,t)||b(k,r)|}{(qs^{(p-1)(p+q)/pq} + pt^{(q-1)(p+q)/pq}}\right)\\ &\le \frac{1}{p+q}\sum_{s=1}^{x}\sum_{t=1}^{y}\left(\sum_{\xi=1}^{s}\sum_{\eta=1}^{t}|\nabla_2\nabla_1 a(\xi,\eta)|^p\right)^{1/p}\\ &\quad\times\sum_{k=1}^{z}\sum_{r=1}^{w}\left(\sum_{\sigma=1}^{k}\sum_{\tau=1}^{r}|\nabla_2\nabla_1 b(\sigma,\tau)|^q\right)^{1/q}\\ &\le \frac{1}{p+q}(xy)^{(p-1)/p}\left\{\sum_{s=1}^{x}\sum_{t=1}^{y}\left(\sum_{\xi=1}^{s}\sum_{\eta=1}^{t}|\nabla_2\nabla_1 a(\xi,\eta)|^p\right)\right\}^{1/p}\\ &\quad\times(zw)^{(q-1)/q}\left\{\sum_{k=1}^{z}\sum_{r=1}^{w}\left(\sum_{\sigma=1}^{k}\sum_{\tau=1}^{r}|\nabla_2\nabla_1 b(\sigma,\tau)|^q\right)\right\}^{1/q}\end{aligned}$$

$$= L(p,q,x,y,z,w)\left(\sum_{s=1}^{x}\sum_{t=1}^{y}(x-s+1)(y-t+1)|\nabla_2\nabla_1 a(s,t)|^p\right)^{1/p}$$
$$\times\left(\sum_{k=1}^{z}\sum_{r=1}^{w}(z-k+1)(w-r+1)|\nabla_2\nabla_1 b(k,r)|^q\right)^{1/q}$$

The proof of Theorem 2.3 is complete. □

THEOREM 2.4. *Let $p>1, q>1$ be constants. Let $f(s,t)$ and $g(k,r)$ be real-valued continuous functions defined on $I_x \times I_y$ and $I_z \times I_w$, respectively, and let $f(0,t)=g(0,t)=0, f(s,0)=g(s,0)=0$. Then*

$$\int_0^x\int_0^y\left(\int_0^z\int_0^w \frac{|f(s,t)||g(k,r)|}{qs^{(p-1)(p+q)/pq}+pt^{(q-1)(p+q)/pq}}\,dk\,dr\right)ds\,dt$$
$$\le L(p,q,x,y,z,w)\left(\int_0^x\int_0^y (x-s)(y-t)|D_2D_1f(s,t)|^p\,ds\,dt\right)^{1/p}$$
$$\times\left(\int_0^z\int_0^w (z-k)(w-r)|D_2D_1g(k,r)|^q\,dk\,dr\right)^{1/q}$$

for $x,y,z,w\in I_0$, where

$$L(p,q,x,y,z,w)=\frac{1}{p+q}(xy)^{(p-1)/p}(zw)^{(q-1)/q}.$$

Proof. From the hypotheses of Theorem 2.4, we have the following identities

$$f(s,t)=\int_0^s\int_0^t D_2D_1f(\xi,\eta)\,d\xi\,d\eta, \tag{17}$$

$$g(k,r)=\int_0^k\int_0^r D_2D_1g(\sigma,\tau)\,d\sigma\,d\tau \tag{18}$$

for $(s,t)\in I_x\times I_y, (k,r)\in I_z\times I_w$. From (17) and (18) and using Hölder's integral inequality with indices $p, p/(p-1)$ and $q, q/(q-1)$, respectively, we have

$$|f(s,t)|\le (st)^{(p-1)/p}\left(\int_0^s\int_0^t |D_2D_1f(\xi,\eta)|^p\,d\xi\,d\eta\right)^{1/p}, \tag{19}$$

$$|g(k,r)|\le (kr)^{(q-1)/q}\left(\int_0^k\int_0^r |D_2D_1g(\sigma,\tau)|\,d\sigma\,d\tau\right)^{1/q} \tag{20}$$

for $(s,t) \in I_x \times I_y, (k,r) \in I_z \times I_w$. From (19) and (20) and using the inequality (5), for $(s,t) \in I_x \times I_y, (k,r) \in I_z \times I_w$. it is easy to observe that

$$\frac{|f(s,t)||g(k,r)|}{F(p,q,s,t,)} \leq \frac{1}{p+q}\left(\int_0^s \int_0^t |D_2D_1f(\xi,\eta)|^p \, d\xi \, d\eta\right)^{1/p} \times \left(\int_0^k \int_0^r |D_2D_1g(\sigma,\tau)|^q \, d\sigma \, d\tau\right)^{1/q},$$

where $F(p,q,s,t,) = qs^{(p-1)(p+q)/pq} + pt^{(q-1)(p+q)/pq}$. Integrating both sides of above inequality first over r from 0 to w and then over k from 0 to z and integrating both sides of the resulting inequality over t from 0 to y and over s from 0 to x and using Hölder's inequality with indices $p, p/(p-1)$ and $q, q/(q-1)$ and Fubini's theorem, we observe that

$$\begin{aligned}
&\int_0^x \int_0^y \left(\int_0^z \int_0^w \frac{|f(s,t)||g(k,r)|}{qs^{(p-1)(p+q)/pq} + pt^{(q-1)(p+q)/pq}} \, dk \, dr\right) ds \, dt \\
&\leq \frac{1}{p+q} \int_0^x \int_0^y \left(\int_0^s \int_0^t |D_2D_1f(\xi,\eta)|^p \, d\xi \, d\eta\right)^{1/p} ds \, dt \\
&\quad \times \int_0^z \int_0^w \left(\int_0^k \int_0^r |D_2D_1g(\sigma,\tau)|^q \, d\sigma \, d\tau\right)^{1/q} dk \, dr \\
&\leq \frac{1}{p+q}(xy)^{(p-1)/p}\left\{\int_0^x \int_0^y \left(\int_0^s \int_0^t |D_2D_1f(\xi,\eta)|^p \, d\xi \, d\eta\right)^{1/p} ds \, dt\right\}^{1/p} \\
&\quad \times (zw)^{(q-1)/q}\left\{\int_0^z \int_0^w \left(\int_0^k \int_0^r |D_2D_1g(\sigma,\tau)|^q \, d\sigma \, d\tau\right)^{1/q} dk \, dr\right\}^{1/q} \\
&= L(p,q,x,y,z,w)\left(\int_0^x \int_0^y (x-s)(y-t)|D_2D_1f(s,t)|^p \, ds \, dt\right)^{1/p} \\
&\quad \times \left(\int_0^z \int_0^w (z-k)(w-r)|D_2D_1g(k,r)|^q \, dk \, dr\right)^{1/q}.
\end{aligned}$$

The proof of Theorem 2.4 is complete. □

REMARK 3. By applying the elementary inequality (7) in the right-hand sides of main inequalities in Theorem 2.3 and 2.4, we get the following inequalities

$$\sum_{s=1}^{x}\sum_{t=1}^{y}\Big(\sum_{k=1}^{z}\sum_{r=1}^{w}\frac{|a(s,t)||b(k,r)|}{qs^{(p-1)(p+q)/pq}+pt^{(q-1)(p+q)/pq}}\Big)$$
$$\leq L_1\Big\{\frac{1}{p}\Big(\sum_{s=1}^{x}\sum_{t=1}^{y}(x-s+1)(y-t+1)|\nabla_2\nabla_1 a(s,t)|^p\Big)^{(p+q)/pq}$$
$$+\frac{1}{q}\Big(\sum_{k=1}^{z}\sum_{r=1}^{w}(z-k+1)(w-r+1)|\nabla_2\nabla_1 b(k,r)|^q\Big)^{(p+q)/pq}\Big\}$$

and

$$\int_0^x\int_0^y\Big(\int_0^z\int_0^w\frac{|f(s,t)||g(k,r)|}{qs^{(p-1)(p+q)/pq}+pt^{(q-1)(p+q)/pq}}\,dr\,dt\Big)\,ds\,dt$$
$$\leq L_1\Big\{\frac{1}{p}\Big(\int_0^x\int_0^y(x-s)(y-t)|D_2D_1f(s,t)|^p\,ds\,dt\Big)^{(p+q)/pq}$$
$$+\frac{1}{q}\Big(\int_0^z\int_0^w(z-k)(w-r)|D_2D_1g(k,r)|^q\,dk\,dr\Big)^{(p+q)/pq}\Big\},$$

where $L_1=pq(xy)^{(p-1)/p}(zw)^{(q-1)/q}/(p+q)^2$.

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