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CONVERGENCE THEOREMS FOR ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPS

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ABSTRACT. We prove that the results of Chang, Park and Cho in [1] concerning the iterative approximation of asymptotically pseudocontractive maps with bounded ranges can be extended to a certain class of asymptotically pseudocontractive maps whose ranges need not be bounded.

1. Introduction

Let E be an arbitrary real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where E^* denotes the dual space of E and $\langle .,.\rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single-valued. In the sequel we shall denote single-valued normalized duality mapping by j.

Let K be a nonempty subset of E. A mapping $T : K \to K$ is called *asymptotically pseudocontractive* with sequence $\{k_n\} \subseteq [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ (see e.g., [1], [9]) if for all $x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \le k_n \|x-y\|^2 \qquad (1)$$

for all $n \in \mathbf{N}$. As shown in [9], this class of mapping is more general than the important class of asymptotically nonexpansive mappings (i.e., mappings $T : K \to K$ such that $\|T^n x - T^n y\| \le k_n\|x-y\|$ for all $x, y \in K$, for some sequence $\{k_n\}$ with $\lim k_n = 1$ and for all $n \in \mathbf{N}$).

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T is called *uniformly L-Lipschizian* if $||T^n x - T^n y|| \leq L||x - y||$ for all $x, y \in K$, $n \in \mathbf{N}$ and for some $L > 0$.

The class of asymptotically pseudocontractive maps was introduced by Schu [9]. In [9], Schu also introduced the *modified Ishikawa iteration method* $\{x_n\}$ generated from an arbitrary $x_0 \in E$ by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T^n y_n,\ n \geq 1, \\ y_n &= (1 - \beta_n)x_n + \beta_n T^n x_n,\ n \geq 1, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$. If $\beta_n = 0$ for all n, then the above scheme reduces to the *modified Mann iteration method* also introduced by Schu in [9]. Using the modified Ishikawa iteration method, Schu [9] proved a strong convergence theorem for the iterative approximation of fixed points (when they exist) of asymptotically pseudocontractive maps in Hilbert spaces.

THEOREM CPC ([1], Theorem 2.1). *Let E be a real Banach space and let D be a nonempty closed convex subset of E. Let $T : D \to D$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a Lipschitz constant $L \geq 1$, and a sequence $\{k_n\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ satisfying the conditions*

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$,
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Let $\{x_n\}$ be the modified Ishikawa iteration method. If $F(T) := \{x \in D : Tx = x\} \neq \emptyset$, $T(D)$ is bounded and for any $q \in F(T)$, there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$(2) \qquad \langle T^n x_{n+1} - q, j(x_{n+1} - q)\rangle \leq k_n ||x_{n+1} - q||^2 - \phi(||x_{n+1} - q||)$$

for all $n \geq 1$, where $j(x_{n+1} - q) \in J(x_{n+1} - q)$ is such that

$$\langle T^n x_{n+1} - T^n q, j(x_{n+1} - q)\rangle \leq k_n ||x_{n+1} - q||^2 \text{ for all } n \geq 1,$$

then the sequence $\{x_n\}$ converges strongly to the fixed point q of T.

The assumption that $T(D)$ is bounded in Theorem CPC appears restrictive since it is not satisfied by many nonexpansive mappings and even many strict contraction maps.

It is our purpose in this paper to complement Theorem CPC by dropping the condition that $T(D)$ is bounded. We impose the condition that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ which is used in the original result of Schu [9], and which has been used by several authors (see for example [2], [5-11]). It is clear that the condition $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ is satisfied by all nonexpansive maps. Our results readly show that if T has a bounded range, then the condition $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ could be dropped. Our proof is short and of independent interest since the convergence of both the modified Mann and Ishikawa iteration methods with errors in the sense of Xu [13] and Liu [4] are obtained from the convergence of a simple perturbed modified Mann iteration method.

2. Main results

LEMMA 1. *Let E be a real Banach space and let $T : E \to E$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a nonempty fixed point set $F(T)$ and a sequence $\{k_n\} \subseteq [1, \infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$ be a real sequence satisfying the conditions*

(i) $0 < \alpha_n < 1$ *for all* $n \geq 1$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$.

Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in E$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n + u_n, \; n \geq 1, \tag{3}$$

where $\{u_n\} \subseteq E$ is such that

$$||u_n|| \leq \rho_n + \gamma_n ||x_n - p||$$

for some $p \in F(T)$ and the sequences $\{\rho_n\}$ and $\{\gamma_n\}$ are such that $\sum_{n=1}^{\infty} \rho_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. If there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x_{n+1} - p, j(x_{n+1} - p)\rangle \leq k_n ||x_{n+1} - p||^2 - \phi(||x_{n+1} - p||), \tag{4}$$

then $\{x_n\}$ converges strongly to p.

Proof. It follows from (4) that

$$\begin{aligned}
&\langle (I - T^n)x_{n+1} - (I - T^n)p, j(x_{n+1} - p)\rangle \\
&\geq \phi(||x_{n+1} - p||) - (k_n - 1)||x_{n+1} - p||^2 \\
&\geq \frac{\phi(||x_{n+1} - p||)||x_{n+1} - p||^2}{1 + \phi(||x_{n+1} - p||) + ||x_{n+1} - p||^2} - (k_n - 1)||x_{n+1} - p||^2 \\
&= r(x_{n+1}, p)||x_{n+1} - p||^2,
\end{aligned}$$

where $r(x_{n+1}, p) = \frac{\phi(||x_{n+1} - p||)}{1+\phi(||x_{n+1}-p||)+||x_{n+1}-p||^2} - (k_n - 1)$. Thus

$$\langle (I - T^n)x_{n+1} - r_n(x_{n+1}, p)x_{n+1} - [(I - T^n)p - r(x_{n+1}, p)p], j(x_{n+1} - p)\rangle \geq 0$$

so that from Lemma 1 of Kato [3]

$$\begin{aligned} \|x_{n+1}-p\| \le \|x_{n+1}-p+\lambda\{(I-T^n)x_{n+1}-r_n(x_{n+1},p)x_{n+1} \\ -[(I-T^n)p-r(x_{n+1},p)p]\}\| \end{aligned} \tag{5}$$

for all $\lambda > 0$. From (3) we obtain

$$\begin{aligned} x_n &= (1+\alpha_n)x_{n+1}+\alpha_n[(I-T^n)x_{n+1}-r(x_{n+1},p)x_{n+1}] \\ &\quad -(1-r(x_{n+1},p))\alpha_n x_n+(2-r(x_{n+1},p))\alpha_n^2(x_n-T^n x_n) \\ &\quad +\alpha_n(T^n x_{n+1}-T^n x_n)-[1+(2-r(x_{n+1},p))\alpha_n]u_n. \end{aligned}$$

Furthermore,

$$p=(1+\alpha_n)p+\alpha_n[(I-T^n)p-r(x_{n+1},p)p]-(1-r(x_{n+1},p))\alpha_n p,$$

so that

$$\begin{aligned} x_n-p = {} & (1+\alpha_n)(x_{n+1}-p) \\ & +\alpha_n\Big[(I-T^n)x_{n+1}-r(x_{n+1},p)x_{n+1}-((I-T^n)p \\ & \qquad -r(x_{n+1},p)p)\Big]-(1-r(x_{n+1},p))\alpha_n(x_n-p) \\ & +(2-r(x_{n+1},p))\alpha_n^2(x_n-T^n x_n)+\alpha_n(T^n x_{n+1}-T^n x_n) \\ & -[1+(2-r(x_{n+1},p))\alpha_n]u_n. \end{aligned}$$

Hence

$$\begin{aligned} \|x_n-p\| \ge {} & (1+\alpha_n)\Big\|x_{n+1}-p+\frac{\alpha_n}{(1+\alpha_n)}\Big[(I-T^n)x_{n+1} \\ & \qquad -r(x_{n+1},p)x_{n+1}-((I-T^n)p-r(x_{n+1},p)p)\Big]\Big\| \\ & -(1-r(x_{n+1},p))\alpha_n\|x_n-p\| \\ & -(2-r(x_{n+1},p))\alpha_n^2\|x_n-T^n x_n\| \\ & -\alpha_n\|T^n x_{n+1}-T^n x_n\|-[1+(2-r(x_{n+1},p))\alpha_n]\|u_n\| \\ \ge {} & (1+\alpha_n)\|x_{n+1}-p\|-(1-r(x_{n+1},p))\alpha_n\|x_n-p\| \\ & -(2-r(x_{n+1},p))\alpha_n^2\|x_n-T^n x_n\|-\alpha_n\|T^n x_{n+1}-T^n x_n \\ & -[1+(2-r(x_{n+1},p))\alpha_n]\|u_n\|, \quad \text{(using (5))}. \end{aligned}$$

Hence

$$\begin{aligned}\|x_{n+1}-p\| &\leq \frac{[1+(1-r(x_{n+1},p))\alpha_n]}{(1+\alpha_n)}\|x_n-p\| + 2\alpha_n^2\|x_n-T^nx_n\| \\ &\quad + \alpha_n\|T^nx_{n+1}-T^nx_n\| + [1+(2-r(x_{n+1},p))\alpha_n]\|u_n\| \\ &\leq [1+(1-r(x_{n+1},p))\alpha_n][1-\alpha_n+\alpha_n^2]\|x_n-p\| \\ &\quad + 2\alpha_n^2\|x_n-T^nx_n\| + \alpha_n\|T^nx_{n+1}-T^nx_n\| + 3\|u_n\| \\ &\leq [1-r(x_{n+1},p)\alpha_n+\alpha_n^2]\|x_n-p\| + 2\alpha_n^2\|x_n-T^nx_n\| \\ &\quad + \alpha_n\|T^nx_{n+1}-T^nx_n\| + 3\|u_n\|.\end{aligned}$$

Observe that $||x_n - T^nx_n|| \leq (1+L)||x_n-p||$ and

$$\begin{aligned}&||T^nx_{n+1}-T^nx_n|| \\ &\leq L(1+L)\alpha_n||x_n-p|| + L||u_n|| \\ &\leq [L(1+L)\alpha_n + L\gamma_n]||x_n-p|| + L\rho_n,\end{aligned}$$

so that

(6)

$$\begin{aligned}&||x_{n+1}-p|| \\ &\leq [1+\alpha_n(k_n-1)+\alpha_n^2(1+2(1+L)+L(1+L))+L\alpha_n\gamma_n]||x_n-p|| \\ &\quad - \Big[\frac{\phi(||x_{n+1}-p||)\alpha_n}{1+\phi(||x_{n+1}-p||)+||x_{n+1}-p||^2}\Big]||x_n-p|| + L\alpha_n\rho_n + 3||u_n|| \\ &\leq [1+\alpha_n(k_n-1)+\alpha_n^2(1+2(1+L)+L(1+L)) \\ &\quad + L\alpha_n\gamma_n + 3\gamma_n]||x_n-p|| \\ &\quad - \Big[\frac{\phi(||x_{n+1}-p||)\alpha_n}{1+\phi(||x_{n+1}-p||)+||x_{n+1}-p||^2}\Big]||x_n-p|| + L\rho_n + 3\rho_n \\ &= [1+\delta_n]||x_n-p|| \\ &\quad - \Big[\frac{\phi(||x_{n+1}-p||)\alpha_n}{1+\phi(||x_{n+1}-p||)+||x_{n+1}-p||^2}\Big]||x_n-p|| + (3+L)\rho_n,\end{aligned}$$

where $\delta_n = \alpha_n(k_n-1)+\alpha_n^2(1+2(1+L)+L(1+L))+(L\alpha_n+3)\gamma_n$. Observe that condition (iii) implies that $\sum_{n=1}^{\infty}\delta_n < \infty$. Since $\sum_{n=1}^{\infty}\rho_n < \infty$, it follows from (6) that $\{||x_n-p||\}$ is bounded. Let $||x_n-p|| \leq D$ for all $n \geq 1$. Then it follows from (6) that

$$||x_{n+1}-p|| \leq ||x_n-p|| + D\delta_n + (3+L)\rho_n = ||x_n-p|| + \lambda_n,$$

where $\lambda_n = D\delta_n + (3+L)\rho_n$. Since $\sum_{n=0}^{\infty}\lambda_n < \infty$, it follows from Lemma 1 of [12] that $\lim_{n\to\infty}||x_n-p||$ exists. Let $\lim||x_n-p|| = \delta \geq 0$. We prove that $\delta = 0$. Assume that $\delta > 0$. Then there exists a positive integer N_0 such that $||x_n - p|| \geq \frac{\delta}{2}$ for all $n \geq N_0$. Since

$$\frac{\phi(||x_{n+1} - p||)}{1 + \phi(||x_{n+1} - p||) + ||x_{n+1} - p||^2} ||x_n - p|| \geq \frac{\phi(\frac{\delta}{2})\delta}{2(1 + \phi(D) + D^2)} \quad \text{for all } n \geq N_0,$$

it follows from (6) that

$$||x_{n+1} - p|| \leq ||x_n - p|| - \frac{\phi(\frac{\delta}{2})\delta}{2(1 + \phi(D) + D^2)}\alpha_n + \lambda_n \quad \text{for all } n \geq N_0.$$

Hence

$$\frac{\phi(\frac{\delta}{2})\delta}{2(1 + \phi(D) + D^2)}\alpha_n \leq ||x_n - p|| - ||x_{n+1} - p|| + \lambda_n \quad \text{for all } n \geq N_0.$$

This implies that

$$\frac{\phi(\frac{\delta}{2})\delta}{2(1 + \phi(D) + D^2)} \sum_{j=N_0}^{n} \alpha_j \leq ||x_{N_0} - p|| + \sum_{j=N_0}^{n} \lambda_j \leq ||x_{N_0} - p|| + \sum_{j=0}^{\infty} \lambda_j,$$

so that $\sum_{n=0}^{\infty} \alpha_n < \infty$, contradicting condition (ii). Hence $\lim ||x_n - p|| = 0$, completing the proof of Lemma 1. □

REMARK 1. If T has bounded range, then the condition $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ can be dropped, and the condition $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ can be replaced by the condition $\lim_{n\to\infty} \alpha_n = 0$. To see this set $M := \sup\{||T^n x_n - p|| : n \geq 1\} + ||x_1 - p||$. Then it follows by a simple induction that

$$||x_n - p|| \leq \prod_{j=1}^{n-1}(1 + \gamma_j)\Big[M + \sum_{j=1}^{n-1} \rho_j\Big], \ n \geq 2,$$

so that $||x_n - p|| \leq D$ for all $n \geq 1$ and for some $D > 0$. It now follows from (6) that

$$\begin{aligned} ||x_{n+1} - p|| &\leq ||x_n - p|| - \frac{\phi(||x_{n+1} - p||)\alpha_n}{1 + \phi(||x_{n+1} - p||) + ||x_{n+1} - p||^2} ||x_n - p|| \\ &\quad + D\alpha_n[(k_n - 1) + \alpha_n(1 + 2(1 + L) + L(1 + L))] \\ &\quad + (L\alpha_n + 3)D\gamma_n + (3 + L)\rho_n \\ &= ||x_n - p|| - \frac{\phi(||x_{n+1} - p||)\alpha_n}{1 + \phi(||x_{n+1} - p||) + ||x_{n+1} - p||^2} ||x_n - p|| \\ &\quad + \alpha_n\lambda_n + \sigma_n, \end{aligned}$$

where $\lambda_n = D[(k_n - 1) + \alpha_n(1 + 2(1 + L) + L(1 + L))]$ and $\sigma_n = (L\alpha_n + 3)D\gamma_n + (3 + L)\rho_n$. Since $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$, it now follows from a standard argument that $\lim_{n\to\infty} ||x_n - p|| = 0$. It is clear that if $||X_n - p||$ is assumed bounded, then the condition $\gamma_n = 0(\alpha_n)$ could also be used inplace of $\sum_{n=1}^{\infty} \gamma_n < \infty$.

2.1. Convergence of the modified Mann and Ishikawa iteration methods with errors in the sense of Xu [13]

THEOREM 1. *Let E be a real Banach space and K a nonempty closed convex subset of E. Let $T : K \to K$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with $F(T) \neq \emptyset$, and a sequence $\{k_n\} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ be sequences in $[0, 1]$ such that*

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$,
(ii) $\sum_{n=1}^{\infty} b_n = \infty$,
(iii) $\sum_{n=1}^{\infty} b_n^2 < \infty$, $\sum_{n=1}^{\infty} b'_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$.

Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + c_n u_n, \; n \geq 1, \\ y_n &= a'_n x_n + b'_n T^n x_n + c'_n v_n, \; n \geq 1, \end{aligned}$$

where $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K. If there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x_{n+1} - p, j(x_{n+1} - p)\rangle \leq k_n ||x_{n+1} - p||^2 - \phi(||x_{n+1} - p||$$

for some $p \in F(T)$, then $\{x_n\}$ converges strongly to p.

Proof. Set $\alpha_n = b_n + c_n$, $\beta_n = b'_n + c'_n$. Then

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T^n y_n + c_n(u_n - T^n y_n), \; n \geq 1, \\ y_n &= (1 - \beta_n)x_n + \beta_n T^n x_n + c'_n(v_n - T^n x_n), \; n \geq 1. \end{aligned}$$

Observe that

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n + u'_n,$$

where $u'_n = \alpha_n(T^n y_n - T^n x_n) + c_n(u_n - T^n y_n)$. Furthermore,

$$||y_n - p|| \leq (1 - \beta_n)||x_n - p|| + \beta_n L||x_n - p|| + c'_n||v_n - p|| + c'_n L||x_n - p||,$$

so that

$$\begin{aligned}\|u_n'\| &\le [\alpha_n\beta_n L(1+L) + L^2 c_n'\alpha_n]\|x_n - p\| + Lc_n'\alpha_n\|v_n - p\| \\ &\quad + c_n\|u_n - p\| + c_n L\|y_n - p\| \\ &\le [\alpha_n\beta_n L(1+L) + L(1+L)c_n + L^2 c_n'\alpha_n + c_n c_n' L^2]\|x_n - p\| \\ &\quad + c_n\|u_n - p\| + [c_n c_n' L + Lc_n'\alpha_n]\|v_n - p\| \\ &= \gamma_n\|x_n - p\| + \rho_n,\end{aligned}$$

where $\gamma_n = \alpha_n\beta_n L(1+L) + L(1+L)c_n + L^2 c_n'\alpha_n + c_n c_n' L^2$ and $\rho_n = c_n||u_n - p|| + [Lc_n c_n' + L\alpha_n c_n']||v_n - p||$. Since $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\sum_{n=1}^{\infty}\rho_n < \infty$, it follows from our Lemma that $\{x_n\}$ converges strongly to p. □

COROLLARY 1. *Let E be a real Banach space and K a nonempty closed convex subset of E. Let $T : K \to K$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with $F(T) \neq \emptyset$, $T(K)$ bound -ed and with a sequence $\{k_n\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n'\}$, $\{b_n'\}$, $\{c_n'\}$ be sequences in $[0, 1]$ such that*

(i) $a_n + b_n + c_n = a_n' + b_n' + c_n' = 1$,
(ii) $\sum_{n=1}^{\infty} b_n = \infty$,
(iii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_n' = \lim_{n\to\infty} c_n' = 0$,
(iv) $\sum_{n=1}^{\infty} c_n < \infty$.

Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by

$$\begin{aligned}x_{n+1} &= a_n x_n + b_n T^n y_n + c_n u_n,\ n \ge 1, \\ y_n &= a_n' x_n + b_n' T^n x_n + c_n' v_n,\ n \ge 1,\end{aligned}$$

where $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K. If there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x_{n+1} - p, j(x_{n+1} - p)\rangle \le k_n||x_{n+1} - p||^2 - \phi(||x_{n+1} - p||)$$

for some $p \in F(T)$, then $\{x_n\}$ converges strongly to p.

Proof. Using Remark 1, the proof follows as in the proof of Theorem 1. □

REMARK 2. Theorem 2.1 of [1] is a special case of Corollary 1 for which $b_n = \alpha_n, a_n = 1 - \alpha_n, c_n = 0, b_n' = \beta_n, a_n' = 1 - \beta_n$.

2.2. Convergence of the modified Mann and Ishikawa iteration methods with errors in the sense of Liu [4]

THEOREM 2. *Let E be a real Banach space and let $T : E \to E$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with $F(T) \neq \emptyset$ and a sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences satisfying the conditions*

(i) $0 < \alpha_n, \beta_n < 1, \ \ n \geq 1,$
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty,$
(iii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty,$
(iv) $\sum_{n=1}^{\infty} \alpha_n\beta_n < \infty.$

Let $\{u_n\}$ and $\{v_n\}$ be sequences in E satisfying the conditions $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\sum_{n=1}^{\infty} \|v_n\| < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + u_n, \ n \geq 1, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n + v_n \ n \geq 1. \end{aligned}$$

If there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x_{n+1} - p, j(x_{n+1} - p)\rangle \leq k_n||x_{n+1} - p||^2 - \phi(||x_{n+1} - p||)$$

for some $p \in F(T)$, then $\{x_n\}$ converges strongly to p.

Proof. Observe that

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n + u_n',$$

where $u_n' = \alpha_n(T^n y_n - T^n x_n) + v_n$. Furthermore, $||u_n'|| \leq \alpha_n\beta_n L(1 + L)||x_n - p|| + \alpha_n L||u_n|| + ||v_n|| = \gamma_n||x_n - p|| + \rho_n$, where $\gamma_n = \alpha_n\beta_n L(1+L)$ and $\rho_n = \alpha_n L||u_n|| + ||v_n||$. Since $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \rho_n < \infty$, it follows from our Lemma that $\{x_n\}$ converges strongly to p. □

COROLLARY 2. *Let E be a real Banach space and let $T : E \to E$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with $F(T) \neq \emptyset$, a sequence $\{k_n\} \subseteq [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ and a bounded range. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences satisfying the conditions*

(i) $0 < \alpha_n, \beta_n < 1, \ \ n \geq 1,$
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty,$
(iii) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0.$

Let $\{u_n\}$ and $\{v_n\}$ be sequences in E satisfying the conditions $\lim_{n\to\infty} ||u_n|| = 0$ and $\sum_{n=1}^{\infty} ||v_n|| < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + u_n,\ n \geq 1, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n + v_n\ n \geq 1. \end{aligned}$$

If there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x_{n+1} - p, j(x_{n+1} - p)\rangle \leq k_n ||x_{n+1} - p||^2 - \phi(||x_{n+1} - p||)$$

for some $p \in F(T)$, then $\{x_n\}$ converges strongly to p.

Proof. Using Remark 1, the proof follows as in the proof of Theorem 2. □

REMARK 3. Theorem 2.1 of [1] is a special case of Corollary 2 for which $u_n = v_n = 0$ for all $n \geq 1$.

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