

GENERALIZED JENSEN'S FUNCTIONAL EQUATIONS AND APPROXIMATE ALGEBRA HOMOMORPHISMS

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ABSTRACT. We prove the generalized Hyers-Ulam-Rassias stability of generalized Jensen's functional equations in Banach modules over a unital C^* -algebra. It is applied to show the stability of algebra homomorphisms between Banach algebras associated with generalized Jensen's functional equations in Banach algebras.

1. Generalized Jensen's functional equations

Let E_1 and E_2 be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f : E_1 \rightarrow E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$. Th. M. Rassias [6] showed that there exists a unique \mathbb{R} -linear mapping $T : E_1 \rightarrow E_2$ such that $\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p} \|x\|^p$ for all $x \in E_1$.

According to Theorem 6 in [5], a mapping $f : E_1 \rightarrow E_2$ satisfying $f(0) = 0$ is a solution of the Jensen's functional equation $2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$ if and only if it satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$.

Throughout this paper, let n be a positive integer.

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LEMMA A. Let V and W be vector spaces. A mapping $f : V \rightarrow W$ with $f(0) = 0$ satisfies the functional equation

$$(A) \quad 2^n f\left(\frac{x_1 + \cdots + x_{2^n}}{2^n}\right) = \sum_{i=1}^{2^n} f(x_i)$$

for all $x_1, \dots, x_{2^n} \in V$ if and only if the mapping $f : V \rightarrow W$ satisfies the additive Cauchy equation $f(x + y) = f(x) + f(y)$ for all $x, y \in V$.

Proof. Assume that a mapping $f : V \rightarrow W$ satisfies (A). Put $x_1 = x_2 = \cdots = x_{2^{n-1}} = x$ and $x_{2^{n-1}+1} = x_{2^{n-1}+2} = \cdots = x_{2^n} = y$ in (A), then

$$2^n f\left(\frac{x+y}{2}\right) = 2^{n-1} f(x) + 2^{n-1} f(y)$$

for all $x, y \in V$. So $2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$ for all $x, y \in V$. Hence f is additive. The converse is obvious. \square

Throughout this paper, let A be a unital C^* -algebra with norm $|\cdot|$, $\mathcal{U}(A)$ the unitary group of A , $A_1 = \{a \in A \mid |a| = 1\}$, and A_1^+ the set of positive elements in A_1 . Let ${}_A\mathcal{B}$ and ${}_A\mathcal{C}$ be left Banach A -modules with norms $\|\cdot\|$ and $\|\cdot\|$, respectively.

The following is useful to prove the stability of the functional equation (A).

LEMMA B [4, Theorem 1]. Let $a \in A$ and $|a| < 1 - \frac{2}{m}$ for some integer m greater than 2. Then there are m elements $u_1, \dots, u_m \in \mathcal{U}(A)$ such that $ma = u_1 + \cdots + u_m$.

The main purpose of this paper is to prove the generalized Hyers-Ulam-Rassias stability of the functional equation (A) in Banach modules over a unital C^* -algebra, and to prove the Hyers-Ulam-Rassias stability of algebra homomorphisms between Banach algebras associated with the functional equation (A).

2. Stability of generalized Jensen's functional equations in Banach modules

We are going to prove the generalized Hyers-Ulam-Rassias stability of the functional equation (A) in Banach modules over a unital C^* -algebra

associated with its unitary group. For a given mapping $f : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ and a given $a \in A$, we set

$$(B) \quad D_a f(x_1, \dots, x_{2^n}) := 2^n a f\left(\frac{x_1 + \dots + x_{2^n}}{2^n}\right) - \sum_{i=1}^{2^n} f(ax_i)$$

for all $x_1, \dots, x_{2^n} \in {}_A\mathcal{B}$.

THEOREM 2.1. *Let $f : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_A\mathcal{B}^{2^n} \rightarrow [0, \infty)$ such that*

$$(2.i) \quad \tilde{\varphi}(x_1, \dots, x_{2^n}) := \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{1}{2^j}x_1, \frac{1}{2^j}x_2, \dots, \frac{1}{2^j}x_{2^n}\right) < \infty$$

$$(2.ii) \quad \|D_u f(x_1, \dots, x_{2^n})\| \leq \varphi(x_1, \dots, x_{2^n})$$

for all $u \in \mathcal{U}(A)$ and all $x_1, \dots, x_{2^n} \in {}_A\mathcal{B}$. Then there exists a unique A -linear mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ such that

$$(2.iii) \quad \|f(x) - T(x)\| \leq \frac{1}{2^{n-1}} \tilde{\varphi}(x, 0, \dots, x, 0)$$

for all $x \in {}_A\mathcal{B}$.

Proof. Put $u = 1 \in \mathcal{U}(A)$. Let $x_1 = x_3 = \dots = x_{2^{n-1}} = x$ and $x_2 = x_4 = \dots = x_{2^n} = 0$ in (2.ii). Then we get $\|f(x) - 2f\left(\frac{x}{2}\right)\| \leq \frac{1}{2^{n-1}} \varphi(x, 0, \dots, x, 0)$ for all $x \in {}_A\mathcal{B}$. We prove by induction on j that

$$(2.1) \quad \left\| 2^j f\left(\frac{1}{2^j}x\right) - 2^{j+1} f\left(\frac{1}{2^{j+1}}x\right) \right\| \leq \frac{2^j}{2^{n-1}} \varphi\left(\frac{1}{2^j}x, 0, \dots, \frac{1}{2^j}x, 0\right)$$

for all $x \in {}_A\mathcal{B}$. So we get

$$(2.2) \quad \left\| f(x) - 2^j f\left(\frac{1}{2^j}x\right) \right\| \leq \frac{1}{2^{n-1}} \sum_{m=0}^{j-1} 2^m \varphi\left(\frac{1}{2^m}x, 0, \dots, \frac{1}{2^m}x, 0\right)$$

for all $x \in {}_A\mathcal{B}$. Let x be an element in ${}_A\mathcal{B}$. By (2.1), for positive integers l and m with $l > m$,

$$\left\| 2^l f\left(\frac{1}{2^l}x\right) - 2^m f\left(\frac{1}{2^m}x\right) \right\| \leq \frac{1}{2^{n-1}} \sum_{j=m}^{l-1} 2^j \varphi\left(\frac{1}{2^j}x, 0, \dots, \frac{1}{2^j}x, 0\right),$$

which tends to zero as $m \rightarrow \infty$ by (2.i). So $\{2^j f(\frac{1}{2^j}x)\}$ is a Cauchy sequence for all $x \in {}_A\mathcal{B}$. Since ${}_A\mathcal{C}$ is complete, the sequence $\{2^j f(\frac{1}{2^j}x)\}$ converges for all $x \in {}_A\mathcal{B}$. We can define a mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ by

$$(2.3) \quad T(x) = \lim_{j \rightarrow \infty} 2^j f\left(\frac{1}{2^j}x\right)$$

for all $x \in {}_A\mathcal{B}$. By (2.ii) and (2.3), we get

$$\begin{aligned} \|D_1T(x_1, \dots, x_{2^n})\| &= \lim_{j \rightarrow \infty} 2^j \left\| D_1f\left(\frac{1}{2^j}x_1, \dots, \frac{1}{2^j}x_{2^n}\right) \right\| \\ &\leq \lim_{j \rightarrow \infty} 2^j \varphi\left(\frac{1}{2^j}x_1, \dots, \frac{1}{2^j}x_{2^n}\right) = 0 \end{aligned}$$

for all $x_1, \dots, x_{2^n} \in {}_A\mathcal{B}$. Hence $D_1T(x_1, \dots, x_{2^n}) = 0$ for all $x_1, \dots, x_{2^n} \in {}_A\mathcal{B}$. By Lemma A, T is additive. Moreover, by passing to the limit in (2.2) as $j \rightarrow \infty$, we get the inequality (2.iii). Now let $L : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ be another additive mapping satisfying $\|f(x) - L(x)\| \leq \frac{1}{2^{n-1}}\tilde{\varphi}(x, 0, \dots, x, 0)$ for all $x \in {}_A\mathcal{B}$.

$$\begin{aligned} \|T(x) - L(x)\| &= 2^j \left\| T\left(\frac{1}{2^j}x\right) - L\left(\frac{1}{2^j}x\right) \right\| \\ &\leq 2^j \left\| T\left(\frac{1}{2^j}x\right) - f\left(\frac{1}{2^j}x\right) \right\| + 2^j \left\| f\left(\frac{1}{2^j}x\right) - L\left(\frac{1}{2^j}x\right) \right\| \\ &\leq \frac{2}{2^{n-1}} 2^j \tilde{\varphi}\left(\frac{1}{2^j}x, 0, \dots, \frac{1}{2^j}x, 0\right), \end{aligned}$$

which tends to zero as $j \rightarrow \infty$ by (2.i). Thus $T(x) = L(x)$ for all $x \in {}_A\mathcal{B}$. This proves the uniqueness of T . By the assumption, for each $u \in \mathcal{U}(A)$,

$$2^j \left\| D_u f\left(\frac{1}{2^j}x, \dots, \frac{1}{2^j}x\right) \right\| \leq 2^j \varphi\left(\frac{1}{2^j}x, \frac{1}{2^j}x, \dots, \frac{1}{2^j}x\right)$$

for all $x \in {}_A\mathcal{B}$, and $2^j \left\| D_u f\left(\frac{1}{2^j}x, \dots, \frac{1}{2^j}x\right) \right\| \rightarrow 0$ as $j \rightarrow \infty$ for all $x \in {}_A\mathcal{B}$. So

$$D_u T(x, \dots, x) = \lim_{j \rightarrow \infty} 2^j D_u f\left(\frac{1}{2^j}x, \dots, \frac{1}{2^j}x\right) = 0$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Hence

$$D_u T(x, \dots, x) = 2^n uT(x) - 2^n T(ux) = 0$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. So $uT(x) = T(ux)$ for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Now let $a \in A$ ($a \neq 0$) and M an integer greater than $4|a|$. Then

$$\left| \frac{a}{M} \right| = \frac{1}{M}|a| < \frac{|a|}{4|a|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By Lemma B, there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(A)$ such that $3\frac{a}{M} = u_1 + u_2 + u_3$. And $T(x) = T(3 \cdot \frac{1}{3}x) = 3T(\frac{1}{3}x)$ for all $x \in {}_A\mathcal{B}$. So $T(\frac{1}{3}x) = \frac{1}{3}T(x)$ for all $x \in {}_A\mathcal{B}$. Thus

$$\begin{aligned} T(ax) &= T\left(\frac{M}{3} \cdot 3\frac{a}{M}x\right) = M \cdot T\left(\frac{1}{3} \cdot 3\frac{a}{M}x\right) = \frac{M}{3}T\left(3\frac{a}{M}x\right) \\ &= \frac{M}{3}T(u_1x + u_2x + u_3x) = \frac{M}{3}(T(u_1x) + T(u_2x) + T(u_3x)) \\ &= \frac{M}{3}(u_1 + u_2 + u_3)T(x) = \frac{M}{3} \cdot 3\frac{a}{M}T(x) = aT(x) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$. Obviously, $T(0x) = 0T(x)$ for all $x \in {}_A\mathcal{B}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{B}$. So the unique additive mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ is an A -linear mapping, as desired. \square

Applying the unital C^* -algebra \mathbb{C} to Theorem 2.1, one can obtain the following.

COROLLARY 2.2. *Let E_1 and E_2 be complex Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Let $f : E_1 \rightarrow E_2$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : E_1^{2^n} \rightarrow [0, \infty)$ such that (2.i) and $\|D_\lambda f(x_1, \dots, x_{2^n})\| \leq \varphi(x_1, \dots, x_{2^n})$ for all $\lambda \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and all $x_1, \dots, x_{2^n} \in E_1$. Then there exists a unique \mathbb{C} -linear mapping $T : E_1 \rightarrow E_2$ such that*

$$\|f(x) - T(x)\| \leq \frac{1}{2^{n-1}}\tilde{\varphi}(x, 0, \dots, x, 0)$$

for all $x \in E_1$.

From now on, assume that $\varphi : {}_A\mathcal{B}^{2^n} \rightarrow [0, \infty)$ is a function satisfying (2.i).

THEOREM 2.3. *Let $f : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ be a continuous mapping with $f(0) = 0$ such that (2.ii) for all $u \in \mathcal{U}(A)$ and all $x_1, \dots, x_{2^n} \in {}_A\mathcal{B}$. If the sequence $\{2^j f(\frac{1}{2^j}x)\}$ converges uniformly on ${}_A\mathcal{B}$, then there exists a unique continuous A -linear mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ satisfying (2.iii).*

Proof. Put $u = 1 \in \mathcal{U}(A)$. By Theorem 2.1, there exists a unique A -linear mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ satisfying (2.iii). By the continuity of f , the uniform convergence and the definition of T , the A -linear mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ is continuous, as desired. \square

THEOREM 2.4. *Let $f : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that $\|D_1 f(x_1, \dots, x_{2^n})\| \leq \varphi(x_1, \dots, x_{2^n})$ for all $x_1, \dots, x_{2^n} \in {}_A\mathcal{B}$. Then there exists a unique additive mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ satisfying (2.iii). Further, if $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$, then the additive mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ is \mathbb{R} -linear.*

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique additive mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ satisfying (2.iii).

Assume that $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$. The additive mapping T given above is similar to the additive mapping T given in the proof of [6, Theorem]. By the same reasoning as the proof of [6, Theorem], the additive mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ is \mathbb{R} -linear. \square

THEOREM 2.5. *Let $f : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ be a continuous mapping with $f(0) = 0$ such that $\|D_a f(x_1, \dots, x_{2^n})\| \leq \varphi(x_1, \dots, x_{2^n})$ for all $a \in A_1^+ \cup \{i\}$ and all $x_1, \dots, x_{2^n} \in {}_A\mathcal{B}$. If the sequence $\{2^j f(\frac{1}{2^j}x)\}$ converges uniformly on ${}_A\mathcal{B}$, then there exists a unique continuous A -linear mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ satisfying (2.iii).*

Proof. Put $a = 1 \in A_1^+$. By Theorem 2.4, there exists a unique \mathbb{R} -linear mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ satisfying (2.iii). By the continuity of f and the uniform convergence, the \mathbb{R} -linear mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ is continuous.

By the same reasoning as the proof of Theorem 2.1, $T(ax) = aT(x)$ for all $a \in A_1^+ \cup \{i\}$.

For any element $a \in A$, $a = a_1 + ia_2$, where $a_1 := \frac{a+a^*}{2}$ and $a_2 := \frac{a-a^*}{2i}$ are self-adjoint elements, furthermore, $a = a_1^+ - a_1^- + ia_2^+ - ia_2^-$, where a_1^+ , a_1^- , a_2^+ , and a_2^- are positive elements (see [2, Lemma 38.8]). So

$$\begin{aligned} T(ax) &= T(a_1^+ x - a_1^- x + ia_2^+ x - ia_2^- x) \\ &= (a_1^+ - a_1^- + ia_2^+ - ia_2^-)T(x) = aT(x) \end{aligned}$$

for all $a \in A$ and all $x \in {}_A\mathcal{B}$. Hence T is A -linear, as desired. \square

THEOREM 2.6. *Let $f : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that $\|D_a f(x_1, \dots, x_{2^n})\| \leq \varphi(x_1, \dots, x_{2^n})$ for all $a \in A_1^+ \cup \{i\}$ and all $x_1, \dots, x_{2^n} \in {}_A\mathcal{B}$. If $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$, then there exists a unique A -linear mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ satisfying (2.iii).*

Proof. Put $a = 1 \in A_1^+$. By Theorem 2.4, there exists a unique \mathbb{R} -linear mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ satisfying (2.iii). The rest of the proof is similar to the proof of Theorem 2.5. \square

3. Stability of generalized Jensen's functional equations in Banach algebras and approximate algebra homomorphisms

In this section, let \mathcal{A} and \mathcal{B} be Banach algebras with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. D.G. Bourgin [3] proved the stability of ring homomorphisms between Banach algebras. In [1], R. Badora generalized the Bourgin's result.

We prove the generalized Hyers-Ulam-Rassias stability of algebra homomorphisms between Banach algebras associated with the functional equation (A).

THEOREM 3.1. *Let \mathcal{A} and \mathcal{B} be real Banach algebras, and $f : \mathcal{A} \rightarrow \mathcal{B}$ a mapping with $f(0) = 0$ for which there exist functions $\varphi : \mathcal{A}^{2^n} \rightarrow [0, \infty)$ and $\psi : \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ such that (2.i),*

$$(3.i) \quad \|D_1 f(x_1, \dots, x_{2^n})\| \leq \varphi(x_1, \dots, x_{2^n}),$$

$$(3.ii) \quad \tilde{\psi}(x, y) := \sum_{j=0}^{\infty} 2^j \psi\left(\frac{1}{2^j}x, y\right) < \infty,$$

$$(3.iii) \quad \|f(x \cdot y) - f(x)f(y)\| \leq \psi(x, y)$$

for all $x, y, x_1, \dots, x_{2^n} \in \mathcal{A}$, where D_1 is in (B). If $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then there exists a unique algebra homomorphism $T : \mathcal{A} \rightarrow \mathcal{B}$ satisfying (2.iii). Further, if \mathcal{A} and \mathcal{B} are unital, then f itself is an algebra homomorphism.

Proof. By the same method as the proof of Theorem 2.4, one can show that there exists a unique \mathbb{R} -linear mapping $T : \mathcal{A} \rightarrow \mathcal{B}$ satisfying (2.iii). The \mathbb{R} -linear mapping $T : \mathcal{A} \rightarrow \mathcal{B}$ was given by $T(x) = \lim_{j \rightarrow \infty} 2^j f\left(\frac{1}{2^j}x\right)$ for all $x \in \mathcal{A}$. Let $R(x, y) = f(xy) - f(x)f(y)$ for all $x, y \in \mathcal{A}$. By (3.ii), we get $\lim_{j \rightarrow \infty} 2^j R\left(\frac{1}{2^j}x, y\right) = 0$ for all $x, y \in \mathcal{A}$. So

$$\begin{aligned} T(xy) &= \lim_{j \rightarrow \infty} 2^j f\left(\frac{1}{2^j}xy\right) = \lim_{j \rightarrow \infty} 2^j f\left[\left(\frac{1}{2^j}x\right)y\right] \\ (3.1) \quad &= \lim_{j \rightarrow \infty} 2^j \left[f\left(\frac{1}{2^j}x\right)f(y) + R\left(\frac{1}{2^j}x, y\right) \right] = T(x)f(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$. Thus

$$\begin{aligned} T(x)f\left(\frac{1}{2^j}y\right) &= T\left[x\left(\frac{1}{2^j}y\right)\right] = T\left[\left(\frac{1}{2^j}x\right)y\right] = T\left(\frac{1}{2^j}x\right)f(y) \\ &= \frac{1}{2^j}T(x)f(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$. Hence

$$(3.2) \quad T(x)2^j f\left(\frac{1}{2^j}y\right) = T(x)f(y)$$

for all $x, y \in \mathcal{A}$. Taking the limit in (3.2) as $j \rightarrow \infty$, we obtain $T(x)T(y) = T(x)f(y)$ for all $x, y \in \mathcal{A}$. Therefore, $T(xy) = T(x)T(y)$ for all $x, y \in \mathcal{A}$. So $T : \mathcal{A} \rightarrow \mathcal{B}$ is an algebra homomorphism. Now assume that \mathcal{A} and \mathcal{B} are unital. By (3.1),

$$T(y) = T(1y) = T(1)f(y) = f(y)$$

for all $y \in \mathcal{A}$. So $f : \mathcal{A} \rightarrow \mathcal{B}$ is an algebra homomorphism, as desired. \square

THEOREM 3.2. *Let \mathcal{A} and \mathcal{B} be complex Banach algebras. Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a mapping with $f(0) = 0$ for which there exist functions $\varphi : \mathcal{A}^{2^n} \rightarrow [0, \infty)$ and $\psi : \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ such that (2.i), (3.ii), (3.iii) and*

$$(3.iv) \quad \|D_\lambda f(x_1, \dots, x_{2^n})\| \leq \varphi(x_1, \dots, x_{2^n})$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y, x_1, \dots, x_{2^n} \in \mathcal{A}$, where D_λ is in (B). Then there exists a unique algebra homomorphism $T : \mathcal{A} \rightarrow \mathcal{B}$ satisfying (2.iii). Further, if \mathcal{A} and \mathcal{B} are unital, then f itself is an algebra homomorphism.

Proof. Under the assumption (2.i) and (3.iv), in Corollary 2.2, we showed that there exists a unique \mathbb{C} -linear mapping $T : \mathcal{A} \rightarrow \mathcal{B}$ satisfying (2.iii).

The rest of the proof is the same as the proof of Theorem 3.1. \square

THEOREM 3.3. *Let \mathcal{A} and \mathcal{B} be complex Banach $*$ -algebras. Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a mapping with $f(0) = 0$ for which there exist functions $\varphi : \mathcal{A}^{2^n} \rightarrow [0, \infty)$ and $\psi : \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ such that (2.i), (3.ii), (3.iii), (3.iv) and $\|f(x^*) - f(x)^*\| \leq \varphi(x, \dots, x)$ for all $\lambda \in \mathbb{T}^1$ and all $x, y, x_1, \dots, x_{2^n} \in \mathcal{A}$. Then there exists a unique $*$ -algebra homomorphism $T : \mathcal{A} \rightarrow \mathcal{B}$ satisfying (2.iii). Further, if \mathcal{A} and \mathcal{B} are unital, then f itself is a $*$ -algebra homomorphism.*

Proof. By the same reasoning as the proof of Theorem 3.2, there exists a unique \mathbb{C} -linear mapping $T : \mathcal{A} \rightarrow \mathcal{B}$ satisfying (2.iii).

Now

$$2^j \left\| f\left(\frac{1}{2^j}x^*\right) - f\left(\frac{1}{2^j}x\right)^* \right\| \leq 2^j \varphi\left(\frac{1}{2^j}x, \dots, \frac{1}{2^j}x\right)$$

for all $x \in \mathcal{A}$. Thus $2^j \left\| f\left(\frac{1}{2^j}x^*\right) - f\left(\frac{1}{2^j}x\right)^* \right\| \rightarrow 0$ as $j \rightarrow \infty$ for all $x \in \mathcal{A}$. Hence

$$T(x^*) = \lim_{j \rightarrow \infty} 2^j f\left(\frac{1}{2^j}x^*\right) = \lim_{j \rightarrow \infty} 2^j f\left(\frac{1}{2^j}x\right)^* = T(x)^*$$

for all $x \in \mathcal{A}$. The rest of the proof is the same as the proof of Theorem 3.1. \square

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