Bull. Korean Math. Soc. **39** (2002), No. 3, pp. 423–430

SELF-ADJOINT INTERPOLATION FOR OPERATORS IN TRIDIAGONAL ALGEBRAS

Joo Ho Kang and Young Soo Jo

Abstract. Given operators X and Y acting on a Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator A such that $AX = Y$. An interpolating operator for n-operators satisfies the equation $AX_i = Y_i$ for $i = 1, 2, \cdots, n$. In this article, we obtained the following : Let $X = (x_{ij})$ and $Y = (y_{ij})$ be operators in $\mathcal{B}(\mathcal{H})$ such that $x_{i\sigma(i)} \neq 0$ for all i. Then the following statements are equivalent.

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$, every E in $\mathcal{L}$ reduces A and A is a self-adjoint operator.

(2) $\sup\left\{\frac{\|\sum_{i=1}^n E_iYf_i\|}{\|\sum_{i=1}^n E_iXf_i\|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\} < \infty$ and $\overline{x_{i,\sigma(i)}}y_{i,\sigma(i)}$ is real for all $i = 1, 2, \cdots$.

1. Introduction

Let $\mathcal{C}$ be a collection of operators acting on a Hilbert space $\mathcal{H}$ and let X and Y be operators acting on $\mathcal{H}$. An *interpolation question* for $\mathcal{C}$ asks for which X and Y is there a bounded operator T in $\mathcal{C}$ such that $TX = Y$. A variation, the 'n-operator interpolation problem', asks for an operator T such that $TX_i = Y_i$ for fixed finite collections $\{X_1, X_2, \cdots, X_n\}$ and $\{Y_1, Y_2, \cdots, Y_n\}$.

In this article, we investigate self-adjoint interpolation problems in tridiagonal algebras: Given operators X and Y acting on a Hilbert space $\mathcal{H}$, when does there exists a self-adjoint operator A in a tridiagonal algebra such that $AX = Y$?

Received July 10, 2001. Revised February 26, 2002.
2000 Mathematics Subject Classification: 47L35.
Key words and phrases: interpolation problem, self-adjoint interpolation, tridiagonal algebra, Alg $\mathcal{L}$, CSL-algebra.

First, we establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $\mathcal{L}$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is CSL, Alg$\mathcal{L}$ is called a CSL-algebra. The symbol Alg$\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers. Let $z \in \mathbb{C}$. Then $\overline{z}$ means the complex conjugate of z.

2. Results

Let $\mathcal{H}$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \cdots\}$. Let $x_1, x_2, \cdots, x_n$ be vectors in $\mathcal{H}$. Then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2, \cdots, x_n$. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ the orthogonal complement of M. Let $\mathcal{L}$ be a subspace lattice of orthogonal projections generated by the subspaces $[e_{2k-1}]$, $[e_{2k-1}, e_{2k}, e_{2k+1}]$ $(k = 1, 2, \cdots)$. Then the algebra Alg$\mathcal{L}$ is called a tridiagonal algebra which was introduced by F. Gilfeather and D. Larson [3]. These algebras have been found to be useful counterexamples to a number of plausible conjectures. Recently, such algebras have been found to be use in physics, in electrical engineering and in general system theory.

Let $\mathcal{A}$ be the algebra consisting of all bounded operators acting on $\mathcal{H}$ of the form

$$\begin{pmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \ddots \end{pmatrix}$$

with respect to the orthonormal basis $\{e_1, e_2, \cdots\}$, where all non-starred entries are zero. It is easy to see that Alg$\mathcal{L}$=$\mathcal{A}$. Let D=$\{A : A$ is diagonal in $\mathcal{B}(\mathcal{H})\}$. Then D is a masa of Alg$\mathcal{L}$ and D=(Alg$\mathcal{L}$)$\cap$ (Alg$\mathcal{L}$)*, where (Alg$\mathcal{L}$)$^* = \{A^* : A \in$Alg$\mathcal{L}\}$.

In this paper, we use the convention $\frac{0}{0} = 0$, when necessary.

From now, let $\sigma : \mathbb{N} \to \mathbb{N}$ be a mapping in this paper.

THEOREM 1. *Let $X = (x_{ij})$ and $Y = (y_{ij})$ be operators in $\mathcal{B}(\mathcal{H})$ such that $x_{i\sigma(i)} \neq 0$ for all i. Then the following statements are equivalent.*

(1) *There exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$, every E in $\mathcal{L}$ reduces A and A is a self-adjoint operator.*

(2) $\sup\left\{\frac{\|\sum_{i=1}^n E_iYf_i\|}{\|\sum_{i=1}^n E_iXf_i\|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\} < \infty$ *and* $\overline{x_{i,\sigma(i)}}y_{i,\sigma(i)}$ *is real for all* $i = 1, 2, \cdots$.

Proof. (1) $\Rightarrow$ (2) : Since E reduces A and $AX = Y$, $AEX = EY$ for every E in $\mathcal{L}$. So $A(\sum_{i=1}^n E_iXf_i) = \sum_{i=1}^n E_iYf_i$ and hence $\|\sum_{i=1}^n E_iYf_i\| \leq \|A\|\|\sum_{i=1}^n E_iXf_i\|$, $n \in \mathbb{N}$, $E_i \in \mathcal{L}$ and $f_i \in \mathcal{H}$. If $\|\sum_{i=1}^n E_iXf_i\| \neq 0$, then $\frac{\|\sum_{i=1}^n E_iYf_i\|}{\|\sum_{i=1}^n E_iXf_i\|} \leq \|A\|$.

Hence $\sup\left\{\frac{\|\sum_{i=1}^n E_iYf_i\|}{\|\sum_{i=1}^n E_iXf_i\|} : n \in \mathbb{N},\ E_i \in \mathcal{L}, \text{ and } f_i \in \mathcal{H}\right\} < \infty$. Since every E in $\mathcal{L}$ reduces A, A is a diagonal operator. Let $A = (a_{ii})$. Since $AX = Y$, $y_{ij} = a_{ii}x_{ij}$ for all i and all j. Since A is a self-adjoint operator, $\overline{x_{i,\sigma(i)}}y_{i,\sigma(i)}$ is real for all $i = 1, 2, \cdots$.

(2) $\Rightarrow$ (1) : If $\sup\left\{\frac{\|\sum_{i=1}^n E_iYf_i\|}{\|\sum_{i=1}^n E_iXf_i\|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\} < \infty$, then without loss of generality, we may assume that

$\sup\left\{\frac{\|\sum_{i=1}^n E_iYf_i\|}{\|\sum_{i=1}^n E_iXf_i\|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\} = 1$. So,

$\|\sum_{i=1}^n E_iYf_i\| \leq \|\sum_{i=1}^n E_iXf_i\|$, $n \in \mathbb{N}$, $E_i \in \mathcal{L}$ and $f_i \in \mathcal{H} \cdots (*)$.

Let $\mathcal{M} = \left\{\sum_{i=1}^n E_iXf_i : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\}$. Then $\mathcal{M}$ is a linear manifold.

Define $A : \mathcal{M} \longrightarrow \mathcal{H}$ by $A(\sum_{i=1}^n E_iXf_i) = \sum_{i=1}^n E_iYf_i$. Then A is well-defined by $(*)$. Extend A to $\overline{\mathcal{M}}$ by continuity. Define $A|_{\overline{\mathcal{M}}^\perp} = 0$. Then $\|A\| \leq 1$ and $AX = Y$. $AE(\sum_{i=1}^n E_iXf_i) = A(\sum_{i=1}^n EE_iXf_i) = \sum_{i=1}^n EE_iYf_i$ and $EA(\sum_{i=1}^n E_iXf_i) = E(\sum_{i=1}^n E_iYf_i) = \sum_{i=1}^n EE_iYf_i$. And $EA(g) = E(0) = 0$ and $AE(g) = 0$ for g in $\overline{\mathcal{M}}^\perp$ since $\langle Eg, \sum_{i=1}^n E_iXf_i\rangle = \langle g, \sum_{i=1}^n EE_iXf_i\rangle = 0$. Hence every E in $\mathcal{L}$ reduces A. Therefore, A is a diagonal operator. Let $A = (a_{ii})$. Since $AX = Y$, $y_{ij} = a_{ii}x_{ij}$ for all i and all j. Since $\overline{x_{i,\sigma(i)}}y_{i,\sigma(i)}$ is real for all $i = 1, 2, \cdots$, A is a self-adjoint operator. $\square$

THEOREM 2. *Let $X_p = (x_{ij}^{(p)})$ and $Y_p = (y_{ij}^{(p)})$ be operators in $\mathcal{B}(\mathcal{H})$ $(p = 1, 2, \cdots, n)$ such that $x_{i\sigma(i)}^{(q)} \neq 0$ for some q. Then the following statements are equivalent.*

(1) *There exists an operator A in Alg$\mathcal{L}$ such that $AX_p = Y_p$ $(p = 1, 2, \cdots, n)$, every E in $\mathcal{L}$ reduces A and A is a self-adjoint operator.*

(2) $\sup\left\{\frac{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i}\|} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty$ *and* $\overline{x_{i,\sigma(i)}^{(q)}}\, y_{i,\sigma(i)}^{(q)}$ *is real for all $i = 1, 2, \cdots$.*

Proof. We assume the condition (2) holds. Then, without loss of generality, we may

assume that $\sup\left\{\frac{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i}\|} : l \leq n, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} = 1.$

Then $\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}Y_i f_{k,i}\| \leq \|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i}\| \cdots\cdots (*)$.

Let $\mathcal{M} = \left\{\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i} : l \leq n, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\}$.

Then $\mathcal{M}$ is a linear manifold. Define $A : \mathcal{M} \to \mathcal{H}$ by $A(\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i}) = \sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}Y_i f_{k,i}$. Then A is well-defined by $(*)$. Extend A to $\overline{\mathcal{M}}$ by continuity. Define $A|_{\overline{\mathcal{M}}^{\perp}} = 0$. Clearly $AX_p = Y_p$ and we know that $\|A|_{\mathcal{M}}\| \leq 1$ $(p = 1, 2, \cdots, n)$. For $m_i \in \mathbb{N}$, $l \leq n$, $E_{k,i} \in \mathcal{L}$ and $f_{k,i} \in \mathcal{H}$,

$$\begin{aligned} AE\Big(\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i}\Big) &= A\Big(\sum_{i=1}^{l}\sum_{k=1}^{m_i} EE_{k,i}X_i f_{k,i}\Big) \\ &= \sum_{i=1}^{l}\sum_{k=1}^{m_i} EE_{k,i}Y_i f_{k,i}, \end{aligned}$$

$$EA\Big(\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}X_if_{k,i}\Big) = E\Big(\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}Y_if_{k,i}\Big) = \sum_{i=1}^{l}\sum_{k=1}^{m_i}EE_{k,i}Y_if_{k,i}.$$

For every g in $\overline{\mathcal{M}}^{\perp}$, $EAg = E0 = 0$ and $AEg = 0$ since $\langle Eg, \sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_if_{k,i}\rangle = \langle g, \sum_{i=1}^{l}\sum_{k=1}^{m_i} EE_{k,i}X_if_{k,i}\rangle = 0$. So every E in $\mathcal{L}$ reduces A. Therefore, A is diagonal. Let $A = (a_{ii})$. Since $AX_p = Y_p$, $y_{ij}^{(p)} = a_{ii}x_{ij}^{(p)}$ for all i, j, and $p = 1, 2, \cdots, n$. Since $\overline{x_{i,\sigma(i)}^{(q)}}y_{i,\sigma(i)}^{(q)}$ is real for all $i = 1, 2, \cdots$, A is a self-adjoint operator.

Conversely, if the condition (1) holds, then $EAX_i = AEX_i = EY_i$ for every E in $\mathcal{L}$ $(i = 1, 2, \cdots, n)$. So $AEX_if = EY_if$ for every E in $\mathcal{L}$ and every f in $\mathcal{H}$ $(i = 1, 2, \cdots, n)$. Thus $A(\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_if_{k,i}) = \sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}Y_if_{k,i}$, for $m_i \in \mathbb{N}$, $l \le n$, $E_{k,i} \in \mathcal{L}$ and $f_{k,i} \in \mathcal{H}$. So

$$\|\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}Y_if_{k,i}\| \le \|A(\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}X_if_{k,i})\| \le \|A\|\|\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}X_if_{k,i}\|.$$

If $\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_if_{k,i}\| \ne 0$, then $\dfrac{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}Y_if_{ki}\|}{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_if_{ki}\|} \le \|A\|$.

Hence, $\sup\left\{\dfrac{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}Y_if_{k,i}\|}{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_if_{k,i}\|} : l \le n,\ m_i \in \mathbb{N},\ E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} \le \|A\|$.

Since every E in $\mathcal{L}$ reduces A, A is diagonal. Let $A = (a_{ii})$. Since $AX_p = Y_p$, $y_{ij}^{(p)} = a_{ii}x_{ij}^{(p)}$ for all i, j, and p. Since A is a self-adjoint operator, $\overline{x_{i,\sigma(i)}^{(q)}}y_{i,\sigma(i)}^{(q)}$ is real for all $i = 1, 2, \cdots$. □

THEOREM 3. *Let $X_p = (x_{ij}^{(p)})$ and $Y_p = (y_{ij}^{(p)})$ be in $\mathcal{B}(\mathcal{H})$ $(p = 1, 2, \cdots)$ such that $x_{i\sigma(i)}^{(q)} \ne 0$ for some fixed q and for all i. Then the following statements are equivalent.*

(1) *There exists an operator A in $Alg\mathcal{L}$ such that $AX_p = Y_p$ ($p = 1, 2, \cdots$), every E in $\mathcal{L}$ reduces A and A is a self-adjoint operator.*

(2) $\sup\left\{\dfrac{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}Y_i f_{k,i}\|}{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}X_i f_{k,i}\|}: m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\}$ $< \infty$ *and* $\overline{x_{i,\sigma(i)}^{(q)}}y_{i,\sigma(i)}^{(q)}$ *is real for all* $i = 1, 2, \cdots$.

Proof. If $\sup\left\{\dfrac{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}Y_i f_{k,i}\|}{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}X_i f_{k,i}\|} : l, m_i \in \mathbb{N},\ E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty$, then without loss of generality, we may assume that

$$\sup\left\{\frac{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}Y_i f_{k,i}\|}{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}X_i f_{k,i}\|} : l, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} = 1.$$

So $\|\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}Y_i f_{k,i}\| \leq \|\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}X_i f_{k,i}\| \cdots\cdots (*)$.

Let $\mathcal{M} = \left\{\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}X_i f_{k,i} : l, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\}$.

Then $\mathcal{M}$ is a linear manifold. Define $A : \mathcal{M} \to \mathcal{H}$ by $A(\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i} X_i f_{k,i}) = \sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}Y_i f_{k,i}$. Then A is well-defined by $(*)$. Extend A to $\overline{\mathcal{M}}$ by continuity. Define $A|_{\overline{\mathcal{M}}^{\perp}} = 0$. Clearly $AX_p = Y_p$ and we know that $\|A|_{\mathcal{M}}\| \leq 1$ ($p = 1, 2, \cdots$).

$$\begin{aligned} AE\Big(\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}X_i f_{k,i}\Big) &= A\Big(\sum_{i=1}^{l}\sum_{k=1}^{m_i}EE_{k,i}X_i f_{k,i}\Big) \\ &= \sum_{i=1}^{l}\sum_{k=1}^{m_i}EE_{k,i}Y_i f_{k,i} \end{aligned}$$

and

$$\begin{aligned} EA\Big(\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}X_i f_{k,i}\Big) &= E\Big(\sum_{i=1}^{l}\sum_{k=1}^{m_i}E_{k,i}Y_i f_{k,i}\Big) \\ &= \sum_{i=1}^{l}\sum_{k=1}^{m_i}EE_{k,i}Y_i f_{k,i}. \end{aligned}$$

For every g in $\overline{\mathcal{M}}^{\perp}$, $EA(g) = E(0) = 0$ and $AE(g) = 0$ since

$$\langle Eg, \sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i}\rangle = \langle g, \sum_{i=1}^{l}\sum_{k=1}^{m_i} EE_{k,i}X_i f_{k,i}\rangle = 0.$$

So every E in $\mathcal{L}$ reduces A. Therefore, A is diagonal. Let $A = (a_{ii})$. Since $AX_p = Y_p$, $y_{ij}^{(p)} = a_{ii}x_{ij}^{(p)}$ for all i, j, and $p = 1, 2, \cdots$. Since $\overline{x_{i,\sigma(i)}^{(q)}}y_{i,\sigma(i)}^{(q)}$ is real for all $i = 1, 2, \cdots$, A is a self-adjoint operator.

Conversely, if $AX_i = Y_i$, then $EAX_i = AEX_i = EY_i$ for every E in $\mathcal{L}$ $(i = 1, 2, \cdots)$. So $AEX_i f = EY_i f$ for every E in $\mathcal{L}$ and every f in $\mathcal{H}$ $(i = 1, 2, \cdots)$. Thus $A(\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i}) = \sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}xY_i f_{k,i}$, $m_i, l \in \mathbb{N}$, $E_{k,i} \in \mathcal{L}$ and $f_{k,i} \in \mathcal{H}$. So

$$\begin{aligned}\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}Y_i f_{k,i}\| &\le \|A(\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i})\| \\ &\le \|A\|\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i}\|.\end{aligned}$$

If $\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i}\| \neq 0$, then $\dfrac{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}Y_i f_{ki}\|}{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{ki}\|} \le \|A\|$.

Hence $\sup\left\{\dfrac{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{i=1}^{l}\sum_{k=1}^{m_i} E_{k,i}X_i f_{k,i}\|} : l, m_i \in \mathbb{N},\ E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} \le \|A\|$.

Since every E in $\mathcal{L}$ reduces A, A is diagonal. Let $A = (a_{ii})$. Since $AX_p = Y_p$, $y_{ij}^{(p)} = a_{ii}x_{ij}^{(p)}$ for all p, i, and j. Since A is a self-adjoint operator, $\overline{x_{i,\sigma(i)}^{(q)}}y_{i,\sigma(i)}^{(q)}$ is real for all $i = 1, 2, \cdots$. □

References

[1] W. B. Arveson, *Interpolation problems in nest algebras*, J. Functional Analysis **3** (1975), 208–233.

[2] R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415.

[3] F. Gilfeather and D. Larson, *Commutants modulo the compact operators of certain CSL algebras*, Operator Theory: Adv. Appl. **2** (Birkhauser, Basel, 1981), 105–120.

[4] A. Hopenwasser, *The equation $Tx = y$ in a reflexive operator algebra*, Indiana University Math. J. **29** (1980), 121–126.

[5] ———, *Hilbert-Schmidt interpolation in CSL algebras*, Illinois J. Math. **33** (1989), 657–672.

[6] Y. S. Jo, *Isometries of Tridiagonal Algebras*, Pacific Journal of Mathematics **140** (1989), no.1, 97–115.

[7] Y. S. Jo and T. Y. Choi, *Isomorphisms of $Alg\mathcal{L}_n$ and $Alg\mathcal{L}_\infty$*, Michigan Math. J. **37** (1990), 305–314.

[8] Y. S. Jo and J. H. Kang, *Interpolation problems in CSL-Algebra $Alg\mathcal{L}$*, to appear in Rocky Mountain Journal of Math.

[9] R. Kadison, *Irreducible Operator Algebras*, Proc. Nat. Acad. Sci. U. S. A. (1957), 273–276.

[10] E. G. Katsoulis; R. L. Moore; T. T. Trent, *Interpolation in nest algebras and applications to operator corona theorems*, J. Operator Theory **29** (1993), 115–123.

[11] E. C. Lance, *Some properties of nest algebras*, Proc. London Math. Soc. **19** (1969), 45–68.

[12] N. Munch, *Compact causal data interpolation*, Aarhus University reprint series no. 11, 1986–1987.

Joo Ho Kang, Department of Mathematics, Taegu University, Taegu 713-714, Korea
E-mail: jhkang@biho.taegu.ac.kr

Young Soo Jo, Department of Mathematics, Keimyung University, Taegu 704-701, Korea
E-mail: ysjo@kmucc.keimyung.ac.kr