SELF-ADJOINT INTERPOLATION FOR OPERATORS IN TRIDIAGONAL ALGEBRAS

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ABSTRACT. Given operators $X$ and $Y$ acting on a Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator $A$ such that $AX = Y$. An interpolating operator for $n$-operators satisfies the equation $AX_i = Y_i$ for $i = 1, 2, \ldots, n$. In this article, we obtained the following: Let $X = (x_{ij})$ and $Y = (y_{ij})$ be operators in $\mathcal{B}(\mathcal{H})$ such that $x_{i\sigma(i)} \neq 0$ for all $i$. Then the following statements are equivalent.

1. There exists an operator $A$ in $\text{Alg} \mathcal{L}$ such that $AX = Y$, every $E$ in $\mathcal{L}$ reduces $A$ and $A$ is a self-adjoint operator.

2. $\sup \left\{ \frac{\sum_{i=1}^{n} E_iX_i f_i}{\sum_{i=1}^{n} E_i X_i f_i} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H} \right\} < \infty$

and $x_{i\sigma(i)} y_{i\sigma(i)}$ is real for all $i = 1, 2, \ldots$.

1. Introduction

Let $\mathcal{C}$ be a collection of operators acting on a Hilbert space $\mathcal{H}$ and let $X$ and $Y$ be operators acting on $\mathcal{H}$. An interpolation question for $\mathcal{C}$ asks for which $X$ and $Y$ is there a bounded operator $T$ in $\mathcal{C}$ such that $TX = Y$. A variation, the `$n$-operator interpolation problem', asks for an operator $T$ such that $TX_i = Y_i$ for fixed finite collections $\{X_1, X_2, \ldots, X_n\}$ and $\{Y_1, Y_2, \ldots, Y_n\}$.

In this article, we investigate self-adjoint interpolation problems in tridiagonal algebras: Given operators $X$ and $Y$ acting on a Hilbert space $\mathcal{H}$, when does there exists a self-adjoint operator $A$ in a tridiagonal algebra such that $AX = Y$?


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First, we establish some notations and conventions. A commutative subspace lattice $L$, or CSL $\mathcal{L}$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections $0$ and $I$ lie in $L$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $L$ is CSL, Alg$L$ is called a CSL-algebra. The symbol Alg$L$ is the algebra of all bounded linear operators on $\mathcal{H}$ that leave invariant all the projections in $L$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers. Let $z \in \mathbb{C}$. Then $\overline{z}$ means the complex conjugate of $z$.

2. Results

Let $\mathcal{H}$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \cdots\}$. Let $x_1, x_2, \cdots, x_n$ be vectors in $\mathcal{H}$. Then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2, \cdots, x_n$. Let $M$ be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of $M$ and $\overline{M}^\perp$ the orthogonal complement of $M$. Let $\mathcal{L}$ be a subspace lattice of orthogonal projections generated by the subspaces $[e_{2k-1}, e_{2k-1}, e_{2k}, e_{2k+1}]$ ($k = 1, 2, \cdots$). Then the algebra Alg$L$ is called a tridiagonal algebra which was introduced by F. Gilfeather and D. Larson [3]. These algebras have been found to be useful counterexamples to a number of plausible conjectures. Recently, such algebras have been found to be use in physics, in electrical engineering and in general system theory.

Let $\mathcal{A}$ be the algebra consisting of all bounded operators acting on $\mathcal{H}$ of the form

\[
\begin{pmatrix}
* & * \\
* & * \\
& * \\
& & * \\
& & & & \ddots
\end{pmatrix}
\]

with respect to the orthonormal basis $\{e_1, e_2, \cdots\}$, where all non-starred entries are zero. It is easy to see that Alg$L$ $= \mathcal{A}$. Let $D = \{A : A$ is diagonal in $B(\mathcal{H}) \}$. Then $D$ is a masa of Alg$L$ and $D = (\text{Alg$L$}) \cap (\text{Alg$L$})^*$, where $(\text{Alg$L$})^* = \{A^* : A \in \text{Alg$L$}\}$.

In this paper, we use the convention $\frac{0}{0} = 0$, when necessary.

From now, let $\sigma : \mathbb{N} \to \mathbb{N}$ be a mapping in this paper.
Self-adjoint interpolation for operators in tridiagonal algebras

THEOREM 1. Let $X = (x_{ij})$ and $Y = (y_{ij})$ be operators in $B(H)$ such that $x_{i\sigma(i)} \neq 0$ for all $i$. Then the following statements are equivalent.

(1) There exists an operator $A$ in AlgC such that $AX = Y$, every $E$ in $L$ reduces $A$ and $A$ is a self-adjoint operator.

(2) $\sup \left\{ \left\| \frac{\sum_{i=1}^{n} E_i Y f_i}{\sum_{i=1}^{n} E_i X f_i} \right\| : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{M} \right\} < \infty$ and $\bar{x}_{i,\sigma(i)} y_{i,\sigma(i)}$ is real for all $i = 1, 2, \ldots$.

Proof. (1) $\Rightarrow$ (2): Since $E$ reduces $A$ and $AX = Y$, $AE = EY$ for every $E$ in $L$. So $A(\sum_{i=1}^{n} E_i X f_i) = \sum_{i=1}^{n} E_i Y f_i$ and hence $\left\| \sum_{i=1}^{n} E_i Y f_i \right\| \leq \|A\| \left\| \sum_{i=1}^{n} E_i X f_i \right\|$, $n \in \mathbb{N}$, $E_i \in \mathcal{L}$ and $f_i \in \mathcal{M}$. If $\left\| \sum_{i=1}^{n} E_i X f_i \right\| \neq 0$, then $\left\| \frac{\sum_{i=1}^{n} E_i Y f_i}{\sum_{i=1}^{n} E_i X f_i} \right\| \leq \|A\|$. Hence $\sup \left\{ \left\| \frac{\sum_{i=1}^{n} E_i Y f_i}{\sum_{i=1}^{n} E_i X f_i} \right\| : n \in \mathbb{N}, E_i \in \mathcal{L}, \text{ and } f_i \in \mathcal{M} \right\} < \infty$. Since every $E$ in $L$ reduces $A$, $A$ is a diagonal operator. Let $A = (a_{ii})$. Since $AX = Y$, $y_{ij} = a_{ii} x_{ij}$ for all $i$ and all $j$. Since $A$ is a self-adjoint operator, $\bar{x}_{i,\sigma(i)} y_{i,\sigma(i)}$ is real for all $i = 1, 2, \ldots$.

(2) $\Rightarrow$ (1): If $\sup \left\{ \left\| \frac{\sum_{i=1}^{n} E_i Y f_i}{\sum_{i=1}^{n} E_i X f_i} \right\| : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{M} \right\} < \infty$, then without loss of generality, we may assume that $\sup \left\{ \left\| \frac{\sum_{i=1}^{n} E_i Y f_i}{\sum_{i=1}^{n} E_i X f_i} \right\| : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{M} \right\} = 1$. So, $\left\| \sum_{i=1}^{n} E_i Y f_i \right\| \leq \left\| \sum_{i=1}^{n} E_i X f_i \right\|$, $n \in \mathbb{N}$, $E_i \in \mathcal{L}$ and $f_i \in \mathcal{M}$.

Let $\mathcal{M} = \left\{ \sum_{i=1}^{n} E_i X f_i : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{M} \right\}$. Then $\mathcal{M}$ is a linear manifold.

Define $A : \mathcal{M} \rightarrow \mathcal{M}$ by $A(\sum_{i=1}^{n} E_i X f_i) = \sum_{i=1}^{n} E_i Y f_i$. Then $A$ is well-defined by $(\ast)$. Extend $A$ to $\overline{\mathcal{M}}$ by continuity. Define $A|_{\overline{\mathcal{M}}^{-}} = 0$. Then $\|A\| \leq 1$ and $AX = Y$. $AE(\sum_{i=1}^{n} E_i X f_i) = A(\sum_{i=1}^{n} EE_i X f_i) = \sum_{i=1}^{n} EE_i E_Y f_i$ and $EA(\sum_{i=1}^{n} E_i X f_i) = E(\sum_{i=1}^{n} E_i Y f_i) = \sum_{i=1}^{n} EE_i E_Y f_i$. And $EA(g) = E(0) = 0$ and $AE(g) = 0$ for $g$ in $\overline{\mathcal{M}}^{-}$ since $\langle Eg, \sum_{i=1}^{n} E_i X f_i \rangle = \langle g, \sum_{i=1}^{n} EE_i X f_i \rangle = 0$. Hence every $E$ in $L$ reduces $A$. Therefore, $A$ is a diagonal operator. Let $A = (a_{ii})$. Since $AX = Y$, $y_{ij} = a_{ii} x_{ij}$ for all $i$ and all $j$. Since $\bar{x}_{i,\sigma(i)} y_{i,\sigma(i)}$ is real for all $i = 1, 2, \ldots$, $A$ is a self-adjoint operator. \qed
THEOREM 2. Let $X_p = (x_{ij}^{(p)})$ and $Y_p = (y_{ij}^{(p)})$ be operators in $B(ℋ)$ $(p = 1, 2, \ldots, n)$ such that $x_{ij}^{(q)} \neq 0$ for some $q$. Then the following statements are equivalent.

(1) There exists an operator $A$ in $\text{Alg} ℋ$ such that $AX_p = Y_p$ $(p = 1, 2, \ldots, n)$, every $E$ in $ℒ$ reduces $A$ and $A$ is a self-adjoint operator.

(2) $\sup \left\{ \frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i} \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \|} : \ m_i \in ℤ, l \leq n, E_{k,i} \in ℒ \text{ and } f_{k,i} \in ℋ \right\} < \infty$ and $x_{i,\sigma(i)}^{(q)}, y_{i,\sigma(i)}^{(q)}$ is real for all $i = 1, 2, \ldots$.

Proof. We assume the condition (2) holds. Then, without loss of generality, we may assume that $\sup \left\{ \frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i} \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \|} : l \leq n, m_i \in ℤ, E_{k,i} \in ℒ \text{ and } f_{k,i} \in ℋ \right\} = 1$.

Then $\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i} \| \leq \| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \| \cdots \cdots (\ast)$. Let $ℳ = \left\{ \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} : l \leq n, m_i \in ℤ, E_{k,i} \in ℒ \text{ and } f_{k,i} \in ℋ \right\}$.

Then $ℳ$ is a linear manifold. Define $A : ℳ \rightarrow ℋ$ by $A(\sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i}) = \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i}$. Then $A$ is well-defined by (\ast). Extend $A$ to $ℳ$ by continuity. Define $A|_{ℳ^{-1}} = 0$. Clearly $AX_p = Y_p$ and we know that $\| A|ℳ \| \leq 1$ $(p = 1, 2, \ldots, n)$. For $m_i \in ℤ, l \leq n, E_{k,i} \in ℒ$ and $f_{k,i} \in ℋ$,

$$AE\left( \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \right) = A\left( \sum_{i=1}^{l} \sum_{k=1}^{m_i} E E_{k,i} X_i f_{k,i} \right)$$

$$= \sum_{i=1}^{l} \sum_{k=1}^{m_i} E E_{k,i} Y_i f_{k,i},$$
\[ EA \left( \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \right) = E \left( \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i} \right) = \sum_{i=1}^{l} \sum_{k=1}^{m_i} E E_{k,i} Y_i f_{k,i}. \]

For every \( g \) in \( \mathfrak{M}^l \), \( EAg = E0 = 0 \) and \( AEg = 0 \) since \( \langle Eg, \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \rangle = \langle g, \sum_{i=1}^{l} \sum_{k=1}^{m_i} E E_{k,i} X_i f_{k,i} \rangle = 0 \). So every \( E \) in \( \mathcal{L} \) reduces \( A \). Therefore, \( A \) is diagonal. Let \( A = (a_{ij}) \). Since \( AX_p = Y_p, y_{ij}^{(p)} = a_{ij} x_{ij}^{(p)} \) for all \( i, j \) and \( p = 1, 2, \ldots, n \). Since \( x_{i,\sigma(i)}^{(q)} \) is real for all \( i = 1, 2, \ldots, k \), \( A \) is a self-adjoint operator.

Conversely, if the condition (1) holds, then \( EAX_i = AX_i = EY_i \) for every \( E \) in \( \mathcal{L} (i = 1, 2, \ldots, n) \). So \( AX_i f = EY_i f \) for every \( E \) in \( \mathcal{L} \) and every \( f \) in \( \mathcal{H} (i = 1, 2, \ldots, n) \). Thus \( A(\sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i}) = \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i} \), for \( m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \) and \( f_{k,i} \in \mathcal{H} \). So

\[
\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i} \| \leq \| A(\sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i}) \|
\leq \| A \| \| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \|.
\]

If \( \| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \| \neq 0 \), then \( \| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i} \| \leq \| A \| \| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \|. \)

Hence, sup \( \left\{ \frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i} \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \|} : l \leq n, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \) and \( f_{k,i} \in \mathcal{H} \} \leq \| A \|. \)

Since every \( E \) in \( \mathcal{L} \) reduces \( A \), \( A \) is diagonal. Let \( A = (a_{ii}) \). Since \( AX_p = Y_p, y_{ij}^{(p)} = a_{ij} x_{ij}^{(p)} \) for all \( i, j \) and \( p \). Since \( A \) is a self-adjoint operator, \( x_{i,\sigma(i)}^{(q)} \) is real for all \( i = 1, 2, \ldots \).

**Theorem 3.** Let \( X_p = (x_{ij}^{(p)}) \) and \( Y_p = (y_{ij}^{(p)}) \) be in \( \mathcal{B}(\mathcal{H}) (p = 1, 2, \ldots) \) such that \( x_{i,\sigma(i)}^{(q)} \neq 0 \) for some fixed \( q \) and for all \( i \). Then the following statements are equivalent.
(1) There exists an operator $A$ in $\text{AlgCl}$ such that $AX_p = Y_p$ ($p = 1, 2, \cdots$), every $E$ in $\mathcal{L}$ reduces $A$ and $A$ is a self-adjoint operator.

$$
\sup \left\{ \frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i} \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \|} : m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H} \right\}
$$

$< \infty$ and $x^{(q)}_{i,a(i)} y^{(q)}_{i,a(i)}$ is real for all $i = 1, 2, \cdots$.

**Proof.** If $\sup \left\{ \frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i} \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \|} : l, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H} \right\} < \infty$, then without loss of generality, we may assume that

$$
\sup \left\{ \frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i} \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \|} : l, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H} \right\} = 1.
$$

So $\sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i} \leq \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \cdots \cdots (\ast)$. Let $\mathcal{M} = \left\{ \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} : l, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H} \right\}$.

Then $\mathcal{M}$ is a linear manifold. Define $A : \mathcal{M} \to \mathcal{H}$ by $A(\sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i}) = \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i}$. Then $A$ is well-defined by (\ast). Extend $A$ to $\overline{\mathcal{M}}$ by continuity. Define $A|_{\mathcal{M}} = 0$. Clearly $AX_p = Y_p$ and we know that $\|A|\mathcal{M}\| \leq 1$ ($p = 1, 2, \cdots$).

$$
AE\left( \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \right) = A\left( \sum_{i=1}^{l} \sum_{k=1}^{m_i} E E_{k,i} X_i f_{k,i} \right) = \sum_{i=1}^{l} \sum_{k=1}^{m_i} E E_{k,i} Y_i f_{k,i}
$$

and

$$
EA\left( \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \right) = E\left( \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i} \right) = \sum_{i=1}^{l} \sum_{k=1}^{m_i} E E_{k,i} Y_i f_{k,i}.
$$
For every $g$ in $\mathcal{M}^1$, $EA(g) = E(0) = 0$ and $AE(g) = 0$ since

$$\langle Eg, \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \rangle = \langle g, \sum_{i=1}^{l} \sum_{k=1}^{m_i} EE_{k,i} X_i f_{k,i} \rangle = 0.$$ 

So every $E$ in $\mathcal{L}$ reduces $A$. Therefore, $A$ is diagonal. Let $A = (a_{ii})$. Since $AX_p = Y_p$, $y_{ij}^{(p)} = a_{ii} x_{ij}^{(p)}$ for all $i, j$, and $p = 1, 2, \cdots$. Since $x_{ij}^{(q)} y_{ij}^{(q)}$ is real for all $i = 1, 2, \cdots$, $A$ is a self-adjoint operator.

Conversely, if $AX_i = Y_i$, then $EA X_i = EA X_i = EY_i$ for every $E$ in $\mathcal{L}$ ($i = 1, 2, \cdots$). So $AEX_i f = EY_i f$ for every $E$ in $\mathcal{L}$ and every $f$ in $\mathcal{H}$ ($i = 1, 2, \cdots$). Thus $A(\sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i}) = \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} x_{ij}^{(p)} Y_i f_{k,i}$, $m_i, l \in \mathbb{N}$, $E_{k,i} \in \mathcal{L}$ and $f_{k,i} \in \mathcal{H}$. So

$$\left\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \right\| \leq \left\| A(\sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i}) \right\|$$

$$\leq \left\| A \right\| \left\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \right\|.$$ 

If $\sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \neq 0$, then

$$\frac{\left\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \right\|}{\left\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \right\|} \leq \left\| A \right\|.$$ 

Hence

$$\left\{ \frac{\left\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_i f_{k,i} \right\|}{\left\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_i f_{k,i} \right\|} : l, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H} \right\} \leq \left\| A \right\|.$$

Since every $E$ in $\mathcal{L}$ reduces $A$, $A$ is diagonal. Let $A = (a_{ii})$. Since $AX_p = Y_p$, $y_{ij}^{(p)} = a_{ii} x_{ij}^{(p)}$ for all $p, i$, and $j$. Since $A$ is a self-adjoint operator, $x_{ij}^{(q)} y_{ij}^{(q)}$ is real for all $i = 1, 2, \cdots$. \qed

References


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