Bull. Korean Math. Soc. **39** (2002), No. 3, pp. 431–440

REMARKS ON HIGHER TYPE ADJUNCTION INEQUALITIES OF 4-MANIFOLDS OF NON-SIMPLE TYPE

JIN-HONG KIM

ABSTRACT. Recently P. Ozsváth and Z. Szabó proved higher type adjunction inequalities for embedded surfaces in 4-manifolds of non-simple type. The aim of this short paper is to give a simple and direct proof of such higher type adjunction inequalities for smoothly embedded surfaces with *negative* self-intersection number in smooth 4-manifolds of non-simple type. This will be achieved through a relation between the Seiberg-Witten invariants used to get adjunction inequalities of 4-manifolds of simple type and a blow-up formula.

1. Introduction

N. Seiberg and E. Witten introduced the Seiberg-Witten invariants for 4-manifolds. They are differential-topological invariants for 4-manifolds. Kronheimer and Mrowka [2] and Morgan, Szabó, and Taubes [3] proved the Thom conjecture for curves with non-negative self-intersection number in any symplectic 4-manifold. Recently, Ozsváth and Szabó completed the Thom conjecture in its full generality by proving the conjecture for curves with negative self-intersection number [4]. Their proof is based on a new relation among the Seiberg-Witten invariants. As a consequence, they proved a generalized adjunction inequality for negative self-intersection number on a smooth 4-manifold of simple type. Moreover, using the refinement of their relation between the Seiberg-Witten invariants they proved higher type adjunction inequalities for smoothly embedded surfaces in 4-manifolds of non-simple type [5]. It is a well known conjecture [6] that any smooth 4-manifold with $b_2^+(X) > 1$

Received September 28, 2001.
2000 Mathematics Subject Classification: Primary 57R57, 57M40.
Key words and phrases: 4-manifolds, non-simple type, higher type adjunction inequalities, Seiberg-Witten invariants.
This research was supported by the Grant-in-Aid of KAIST.

and $b_1(X) = 0$ cannot admit basic classes of non-zero dimension. However, in this paper we do not require that $b_1(X)$ be zero, and there exist some examples of 4-manifolds with $b_2^+(X) > 1$ and $b_1(X) > 0$ which admit basic classes of non-zero dimension (see [5] or Section 3).

The aim of this short paper is to give a simple and direct proof of such higher type adjunction inequalities for smoothly embedded surfaces with *negative* self-intersection number in smooth 4-manifolds of non-simple type. These adjunction inequalities can also be obtained from Theorem 1.6 in [5] that is a consequence of more general and difficult results. The main idea of this paper is to use just the relation of P. Ozsváth and Z. Szabó in [4] between the Seiberg-Witten invariants and the blow-up formula in [1]. In view of our results of this paper we speculate that we can deduce any higher type adjunction inequalities only from some relations used to get adjunction inequalities of 4-manifold of simple type and a blow-up formula, contrary to the approach of [5]. Similar adjunction inequalities for immersed 2-spheres were already proved by R. Fintushel and R. Stern in [1].

To state our main results, we need some notations (see Section 2 for more details). Let $M_X(L)$ denote the Seiberg-Witten moduli space for a spin^c structure L, and $d(L)$ denote its virtual dimension. A *Seiberg-Witten basic class* is a spin^c structure L for which the Seiberg-Witten invariant does not vanish identically, and a smooth 4-manifold is of *Seiberg-Witten type* $2m$ if the Seiberg-Witten invariants vanish for all spin^c structures L with $d(L) > 2m$. In particular, if a smooth 4-manifold X is of type 0, then X is called of *simple type*. Moreover, if $b_2^+(X) = 1$, then we say that X is of type $2m$ for a common chamber for the set $\mathcal{C}_+$ of all spin^c structure L with $d(L) \geq 0$.

Now, we are in a position to state our main result. Our main result is Theorem 3.3, and one special case that is particularly interesting is the following

THEOREM 1.1. *Let X be a closed, connected, oriented, smooth 4-manifold of the Seiberg-Witten $2d$ type with $b_2^+(X) > 1$ and $\Sigma \subset X$ be a smoothly embedded, oriented, closed surface with genus $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma < 0$. Let L be a Seiberg-Witten basic class with the Seiberg-Witten invariant $SW_{X,L}(x^{\frac{\dim M_X(L)}{2}}) \neq 0$, where x is a 2-dimensional generator. If either $g(\Sigma) > d$ or $|\langle L, [\Sigma]\rangle| + [\Sigma] \cdot [\Sigma] > 0$, then we have*

$$|\langle L, [\Sigma]\rangle| + \Sigma \cdot \Sigma + 2d \leq 2g(\Sigma) - 2. \tag{1.1}$$

Moreover, if $b_2^+(X) = 1$, then the inequality still holds in the common chambers for $\mathcal{C}_+$ which are perpendicular to $[\Sigma]$.

We organize this paper as follows. In Section 2, we set up some basic notations. The main results will be stated and proved in Section 3. Finally, in Section 4 we give two examples which show that our main results are in fact sharp.

2. Preliminaries

In this section, we will set up some notations for the next section. Let X be a closed, oriented, smooth 4-manifold with $b_2^+(X) > 0$. We let x to be a two-dimensional generator. Let $\mathbb{A}(X)$ denote the graded algebra of the tensor product of the exterior algebra on $H_1(X, \mathbb{Z})$ with the polynomial algebra $\mathbb{Z}[x]$ on a single two-dimensional generator x, i.e.,

$$\mathbb{A}(X) = \mathbb{Z}[x] \otimes \Lambda^* H_1(X).$$

Given an orientation for $H^1(X, \mathbb{R}) \oplus H_+^2(X, \mathbb{R})$ with a spin^c structure L, the Seiberg-Witten invariant form an integer-valued function

$$SW_{X,L} : \mathbb{A}(X) \to \mathbb{Z}.$$

Given a spin^c structure L over X, the Seiberg-Witten equations are for a connection A on $\det(W_+)$ and a section $\Phi \in \Gamma(W_+)$

$$\rho(F_A^+ + i\eta) = (\Phi \otimes \Phi^*)_0 \quad \text{and} \quad \not{D}_A \Phi = 0, \tag{2.1}$$

where ρ denotes the isomorphism from $i\Lambda^+$ to $isu(W_+)$, $\not{D}$ denotes the Dirac operator, and η denotes some fixed real self-dual 2-form.

Let $M_X(L)$ denote the moduli space of solutions to the equations (2.1) modulo gauge group. Then the virtual dimension of the moduli space is given by

$$d(L) = \frac{L^2 - (2\chi(X) + 3\sigma(X))}{4},$$

where $\chi(X)$ and $\sigma(X)$ denote the Euler characteristic and the signature of X, respectively. Let $\mathcal{L}$ denote the universal line bundle over $X \times M_X(L)$ and

$$\mu : \mathbb{A}(X) \to H^*(M_X(L), \mathbb{Z}), \quad \lambda \mapsto c_1(\mathcal{L})/\lambda$$

for any $\lambda \in H_*(X, \mathbb{Z})$.

Then, the Seiberg-Witten invariant is defined to be

$$SW_{X,L}(a) = \langle \mu(a), [M_X(L)] \rangle,$$

where $[M_X(L)]$ denotes the fundamental class for the moduli space induced from the homology orientation of $H^1(X, \mathbb{Z}) \oplus H_+^2(X, \mathbb{Z})$.

The Seiberg-Witten invariant is a smooth invariant of the 4-manifold X when $b_2^+(X) > 1$. When $b_2^+(X) = 1$, the invariant depends on the chamber structure. Here, we briefly review some basic terminologies (see [5] for details). Let

$$\Omega(X) = \{\alpha \in H^2(X, \mathbb{R}) \mid \alpha^2 = 1\}.$$

Given an orientation of $H_+^2(X, \mathbb{R})$, we can choose the positive component $\Omega^+(X)$ of two components of $\Omega(X)$. For a spin^c structure L on X, we define the *wall* determined by L, denoted W_L, to be the set of $(\omega, t) \in \Omega^+(X) \times \mathbb{R}$ such that $2\pi\omega \cdot L + t = 0$, and the *chamber* determined by L is a connected component of $\Omega^+(X) \times \mathbb{R} - W_L$. We can also define so-called a *period map* from the space of metrics and perturbations to the space $\Omega^+(X) \times \mathbb{R}$ given by $(g, \eta) \mapsto (\omega_g, \int_X \omega_g \wedge \eta)$, where ω_g is the unique harmonic, self-dual two-form in $\Omega^+(X)$. It is well known that the Seiberg-Witten invariant of L for g and η is well defined if the corresponding period point does not lie on a wall, and that it depends on g and η if the period point varies in the chamber.

To state our main results for $b_2^+(X) = 1$, we need to define a *common chamber* for a collection $\mathcal{C}$ of spin^c structures. This is defined to be a connected component of $\Omega^+(X) \times \mathbb{R} - \cup_{L \in \mathcal{C}} W_L$, and we say that a common chamber for $\mathcal{C}$ is *perpendicular* to $c \in H^2(X, \mathbb{R})$ of negative square if it contains a pair (ω, t), where ω is perpendicular to c.

Finally, we need one more notation. Let $\Sigma \subset X$ be a smoothly embedded, oriented, closed surface. We define the class $\xi(\Sigma) \in \mathbb{A}(X)$ by $\xi(\Sigma) = \prod_{i=1}^{g}(x - a_i \cdot b_i)$, where $\{a_i, b_i\}$ are the images in $H_1(X, \mathbb{Z})$ of a standard symplectic basis for $H_1(\Sigma, \mathbb{Z})$, and the product $a_i \cdot b_i$ is taken in the algebra $\mathbb{A}(X)$.

Now, we state two important theorems of P. Ozsváth and Z. Szabó and R. Fintushel and R. Stern. We denote by $[\Sigma]$ the Poincaré dual of a smoothly embedded, oriented, closed surface Σ.

THEOREM 2.1 ([4]). *Let X be a closed, smooth 4-manifold with $b_2^+(X) > 1$ and $\Sigma \subset X$ a smoothly embedded, oriented, closed surface of genus $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma = -n < 0$. Then, for each spin^c structure L with $d(L) \geq 0$ and $|\langle L, [\Sigma]\rangle| \geq 2g(\Sigma) + n$, we have for each $a \in \mathbb{A}(X)$*

$$SW_{X,L+2\varepsilon[\Sigma]}(\xi(\varepsilon\Sigma)x^m \cdot a) = SW_{X,L}(a),$$

where $\varepsilon = \pm 1$ is the sign of $\langle L, [\Sigma]\rangle$ and $2m = |\langle L, [\Sigma]\rangle| - 2g(\Sigma) - n$.

Moreover, if $b_2^+(X) = 1$, then the relation still holds in any common chamber for L and $L + 2\varepsilon[\Sigma]$ which is perpendicular to $[\Sigma]$.

Next, we state the blow-up formula of R. Fintushel and R. Stern.

THEOREM 2.2 ([1]). *Let X be a smooth, closed 4-manifold, and let $\tilde{X} = X \# \overline{\mathbb{C}P}^2$ denote its blow-up, with an exceptional class $E \in H^2(\tilde{X}, \mathbb{R})$. Then, for each spin^c structure $\tilde{L}$ on $\tilde{X}$ with $d(\tilde{L}) \geq 0$, and each $a \in \mathbb{A}(X) \cong \mathbb{A}(\tilde{X})$, we have $SW_{\tilde{X},\tilde{L}}(a) = SW_{X,L}(x^q a)$, where L is the spin^c structure on X obtained by restricting $\tilde{X}$ and $2q = d(L) - d(\tilde{L})$. Moreover, if $b_2^+(X) = 1$, then the relation still holds for any common chamber perpendicular to E.*

3. Main result

In this section, we will prove a generalized adjunction inequality for smoothly embedded surfaces with negative self-intersection number. We first start with the following relation between the Seiberg-Witten invariants. This is a refinement of Theorem 2.1.

LEMMA 3.1. *Let X be a closed, smooth 4-manifold with $b_2^+(X) > 1$ and $\Sigma \subset X$ a smoothly embedded, oriented, closed surface of genus $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma = -n < 0$. Let L be a spin^c structure with $\dim M_X(L) = \sum_{i=1}^r l_i(l_i+1)$ with each integer $l_i \geq 0$. Suppose that $|\langle L, [\Sigma]\rangle| \geq 2g(\Sigma) - \Sigma\cdot\Sigma - 2\sum_{i=1}^r l_i$. Then, the following relation holds:*

$$SW_{X,L+2\varepsilon[\Sigma]}(\xi(\varepsilon\Sigma)x^{m+p}) = SW_{X,L}(x^{\frac{\dim M_X(L)}{2}}),$$

where $\varepsilon = \pm 1$ is the sign of $\langle L, [\Sigma]\rangle$, $2p = \sum_{i=1}^r l_i(l_i - 1)$, and

$$2m = |\langle L, [\Sigma]\rangle| - 2g(\Sigma) + \Sigma\cdot\Sigma + 2\sum_{i=1}^r l_i.$$

Moreover, if $b_2^+(X) = 1$, then the relation still holds in any common chamber for L and $L + 2\varepsilon[\Sigma]$ which is perpendicular to $[\Sigma]$.

REMARK 3.2. P. Ozsváth and Z. Szabó proved a similar relation (Theorem 1.6 in [5]) with an extra condition $-\langle L, [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] \geq 0$ with $d(L)$ in place of $\sum_{i=1}^r l_i$.

Proof. Let l be a positive number greater than r. We first blow up our manifold X l times to get a new manifold $\tilde{X} = X \# l\overline{\mathbb{C}P^2}$. Assume without loss of generality that the sign of $\langle L, [\Sigma]\rangle$ is positive, i.e., $\varepsilon = 1$. Let $\tilde{\Sigma}$ be the proper transform of Σ so that we get an embedded submanifold of the same genus with $[\tilde{\Sigma}] = [\Sigma] - E_1 - \ldots - E_r$, where E_i $(1 \leq i \leq l)$ are the exceptional divisors. Now consider the spin^c

structure $\tilde{L}$ on $\tilde{X}$ which extends L on X such that $\tilde{L} = L + \sum_{i=1}^{r}(2l_i + 1)E_i + E_{r+1} + \ldots + E_l$. Then, we get

(3.1)

$$\begin{aligned} 4 \cdot d(\tilde{L}) &= \tilde{L}^2 - (2\chi(\tilde{X}) + 3\sigma(\tilde{X})) \\ &= (L + \sum_{i=1}^{r}(2l_i + 1)E_i + E_{r+1} + \ldots + E_l)^2 - (2\chi(X) + 3\sigma(X) - l) \\ &= (L^2 - (2\chi(X) + 3\sigma(X))) - \sum_{i=1}^{r}(2l_i + 1)^2 - (l - r) + l \\ &= 4 \cdot d(L) - 4\sum_{i=1}^{r} l_i(l_i + 1) \geq 0. \end{aligned}$$

Note also that

(3.2)

$$\begin{aligned} |\langle \tilde{L}, [\tilde{\Sigma}] \rangle| &= \Big| L + \sum_{i=1}^{r}(2l_i + 1)E_i + E_{r+1} + \ldots + E_l, \\ &\qquad ([\Sigma] - (E_1 + \ldots + E_r))\Big| \\ &= |\langle L, [\Sigma] \rangle| + \sum_{i=1}^{r}(2l_i + 1) \geq 2g(\Sigma) + n - 2\sum_{i=1}^{r} l_i + \sum_{i=1}^{r}(2l_i + 1) \\ &\geq 2g(\Sigma) + n + r = 2g(\tilde{\Sigma}) + (n + r). \end{aligned}$$

Finally, note that

$$\tilde{\Sigma} \cdot \tilde{\Sigma} = \left(\Sigma - \sum_{i=1}^{r} E_i\right) \cdot \left(\Sigma - \sum_{i=1}^{r} E_i\right) = \Sigma \cdot \Sigma - r = -(n + r). \tag{3.3}$$

From (3.1), (3.2), and (3.3), we can apply Theorem 2.1 to our 4-manifold $\tilde{X}$ with $\tilde{L}$ and $\tilde{\Sigma}$ to obtain a relation

$$SW_{\tilde{X}, \tilde{L}+2[\tilde{\Sigma}]}(\xi(\tilde{\Sigma})x^m \cdot a) = SW_{\tilde{X}, \tilde{L}}(a), \tag{3.4}$$

where $2m = \langle \tilde{L}, [\tilde{\Sigma}] \rangle - 2g(\tilde{\Sigma}) - (n + r)$. Here, we used $\langle \tilde{L}, [\tilde{\Sigma}] \rangle = \langle L, [\Sigma] \rangle + \sum_{i=1}^{r}(2l_i + 1)$.

On the other hand, since

$$\tilde{L} + 2[\tilde{\Sigma}]\Big|_X = L + 2[\Sigma] \quad \text{and} \quad \tilde{L}\Big|_X = L,$$

and

$$\begin{aligned}4d(L+2[\Sigma]) - 4d(\tilde{L}+2[\tilde{\Sigma}]) &= (L+2[\Sigma])^2 - (2\chi(X)+3\sigma(X)) \\ &- (\tilde{L}+2[\tilde{\Sigma}])^2 + (2\chi(X)+3\sigma(X)-l) \\ &= \sum_{i=1}^{r}(2l_i+1)^2 + (l-r) - 4\sum_{i=1}^{r}(2l_i+1) + 4r - l \\ &= 4\sum_{i=1}^{r} l_i(l_i-1) = 8p,\end{aligned}$$

it follows from the blow-up formula Theorem 2.2 and (3.1) that we have

$$SW_{\tilde{X},\tilde{L}+2[\tilde{\Sigma}]}(\xi(\tilde{\Sigma})x^m \cdot a) = SW_{X,L+2[\Sigma]}(\xi(\Sigma)x^{m+p} \cdot a)$$
$$SW_{\tilde{X},\tilde{L}}(a) = SW_{X,L}(x^{\frac{\dim M_X(L)}{2}} \cdot a). \tag{3.5}$$

Since the Seiberg-Witten invariants are zero on homogeneous elements whose degree is not $d(L)$, combining equations (3.4) and (3.5) together completes the proof. If $b_2^+(X) = 1$, then a common chamber for L and $L + 2[\Sigma]$ which is perpendicular to $[\Sigma]$ yields a common chamber for $\tilde{L}$ and $\tilde{L} + 2[\tilde{\Sigma}]$ which is perpendicular to $[\tilde{\Sigma}]$. Thus, the above arguments work to give the relation for such a common chamber. □

Now, we are in a position to state and prove a generalized adjunction inequality for 4-manifolds of non-simple type.

THEOREM 3.3. *Let X be a closed, connected, oriented, 4-manifold of the Seiberg-Witten $\sum_{i=1}^{r} l_i(l_i+1)$ type with $b_2^+(X) > 1$ and $\Sigma \subset X$ be a smoothly embedded, oriented, closed surface with genus $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma < 0$. Let L be a Seiberg-Witten basic class with the Seiberg-Witten invariant $SW_{X,L}(x^{\frac{\dim M_X(L)}{2}}) \neq 0$. If either $g(\Sigma) > \sum_{i=1}^{r} l_i$ or $|\langle L, [\Sigma]\rangle| + [\Sigma]\cdot[\Sigma] > 0$, then we have*

$$|\langle L, [\Sigma]\rangle| + \Sigma\cdot\Sigma + 2\sum_{i=1}^{r} l_i \leq 2g(\Sigma) - 2. \tag{3.6}$$

Moreover, if $b_2^+(X) = 1$, then the inequality still holds in the common chambers for C_+ which are perpendicular to $[\Sigma]$.

Proof. We prove this by contradiction. It suffices to prove the first case. Suppose that $g(\Sigma) > \sum_{i=1}^{r} l_i$ and that there is a basic class L with $SW_{X,L}(x^{\frac{\dim M_X(L)}{2}}) \neq 0$ for which the adjunction inequality does not hold. Then, Lemma 3.1 implies that $L + 2\varepsilon[\Sigma]$ is also a basic class. Moreover, we have

$$\begin{aligned} d(L+2\varepsilon[\Sigma]) &= d(L) + \varepsilon\langle L, [\Sigma]\rangle + \Sigma\cdot\Sigma \\ &\geq \sum_{i=1}^{r} l_i(l_i+1) + 2(g(\Sigma) - \sum_{i=1}^{r} l_i) > d(L). \end{aligned}$$

But, this implies that X is not of the Seiberg-Witten type $\sum_{i=1}^{r} l_i(l_i+1)$, which completes the proof of the first case. □

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. To get the inequality (1.1), it suffices to take $r = d$ and $l_1 = \ldots = l_d = 1$ in Theorem 3.3. □

Finally, we close this section with a simple algebraic fact which tells that the lower bound of the inequality (1.1) in Theorem 1.1 is the best possible.

LEMMA 3.4. *Let* $\mathcal{L}_r = \{(l_1, \ldots, l_r) \in \mathbb{N}^r \mid \sum_{i=1}^{r} l_i(l_i+1) = 2d\}$ *for a positive integer* d, *and let* $\mathcal{L} = \cup_{r=1}^{\infty}\mathcal{L}_r$. *Define a function* $\varphi : \mathcal{L} \to \mathbb{N}$, $(l_1, \ldots, l_r) \mapsto \sum_{i=1}^{r} l_i$. *Then, the function* φ *attains its maximum* d *at* $(l_1, \ldots, l_d) = (1, \ldots, 1)$.

Proof. Let k_i be a positive integer such that $2k_i = l_i(l_i+1)$ for each $i = 1, \ldots, r$. Note that $r \leq d$. It is also easy to see that $l_i \leq k_i$ for all $i = 1, \ldots, r$. In fact, otherwise for $i = 1, \ldots, r$ we have $2k_i = l_i(l_i+1) > k_i(k_i+1) \geq 2k_i$, which is a contradiction. Thus, we have

$$\sum_{i=1}^{r} l_i \leq \sum_{i=1}^{r} k_i = d, \tag{3.7}$$

and, moreover, since $l_i \leq k_i$ for all $i = 1, \ldots, r$ the equality (3.7) holds if and only if $r = d$ and $k_i = l_i$ for all $i = 1, \ldots, r$, namely, $r = d$ and $k_i = l_i = 1$ for all $i = 1, \ldots, r$. This completes the proof. □

4. Examples

In this section, we give two examples which show that our main results are sharp. First, we will show that the condition $SW_{X,L}(x^{\frac{\dim M_X(L)}{2}}) \neq 0$ is crucial and so cannot be removed from Theorem 3.3.

EXAMPLE 4.1. Fix a positive even number n greater than 2. We denote $E(2n)$ by the simply connected elliptic fibration over $\mathbb{C}P^1$, without multiple fibers and with its $\chi(E(2n)) = 2n$. Let X be the 4-manifold $E(2n)\#2(S^3 \times S^1)$. Let S and F denote a section and a fiber of the elliptic fibration $E(2n)$ such that $S^2 = -2n$. Let Σ_0 denote the symplectic submanifold representing the homology class $S + \frac{n}{2}F$, and let T denote a fiber in the elliptic fibration of the first summand $S^3 \times S^1$. Now, we define Σ to be the internal connected sum $\Sigma_0\#T$. Then, we see that $g(\Sigma) = \frac{n}{2} + 1 > 1$ and $\Sigma \cdot \Sigma = -n$.

Now, consider the spin^c structure L over X induced from the canonical spin^c structure on $E(2n)$. Then, it was shown in [5] that the spin^c structure L is a basic class of non-zero dimension $\dim M_X(L) = 2 = \sum_{i=1}^r l_i(l_i+1)$ such that $SW_{X,L}(b_1 \cdot b_2) = 1$, where $b_i \in H_1(X,\mathbb{Z})$ generate H_1 of the i-th copy of $S^3 \times S^1$. Note also that since $E(2n)$ is of simple type and $H^2(X,\mathbb{Z}) = H^2(E(2n),\mathbb{Z})$, X must be of type 2. Thus, in this case we can take $\sum_{i=1}^r l_i = 1$. On the other hand, since $\langle L, [\Sigma]\rangle = 2n-2$, we see that

$$|\langle L, [\Sigma]\rangle| + \Sigma \cdot \Sigma + 2\sum_{i=1}^{r} l_i = 2n - 2 - n + 2 = n,$$

which equals $2g(\Sigma) - 2 = 2(\frac{n}{2}+1) - 2$. But, if we consider Σ_0 with the induced spin^c structure L on X then $g(\Sigma_0) = \frac{n}{2} > 1$, and the inequality (3.6) does not hold.

We close this paper with another example which shows that the inequality (3.6) is in fact sharp because it is saturated by some examples of 4-manifold which admit basic classes of non-zero dimension.

EXAMPLE 4.2. Fix a positive integer n, and let Y be the 2-sphere bundle over a Riemann surface Σ of genus $g(\Sigma) > 0$, associated to the circle bundle with Euler number n. Then, we have $\Sigma \cdot \Sigma = n$. In the common chamber corresponding to $[\Sigma]$, its anti-canonical line bundle $-K_Y$ is a basic class of zero dimension. Let F be the class of the fiber of Y. Then, it was shown [5] that $SW_{Y,-K_Y+2d[F]}(x^d) \neq 0$, and $\langle -K_Y + 2d[F], [\Sigma]\rangle = 2d$ for $n \geq 2d$. We assume without loss of generality that Y is of the Seiberg-Witten type $2d$.

Now, let X be the blown-up manifold $Y\#(n+d)\overline{\mathbb{C}P}^2$, and let $\tilde{\Sigma}$ be the proper transform of Σ such that $[\tilde{\Sigma}] = [\Sigma] + E_1 + \ldots + E_{n+d}$, where E_i $(1 \leq i \leq n+d)$ are exceptional divisors. Consider the spin^c structure $\tilde{L}$ on X such that $\tilde{L} = -K_Y + 2d[F] + E_1 + \ldots + E_{n+d}$.

Then, it is easy to see that

$$\langle \tilde{L}, [\tilde{\Sigma}] \rangle = 2d - (n+d) = -n + d$$
$$\tilde{\Sigma} \cdot \tilde{\Sigma} = \Sigma \cdot \Sigma - (n+d) = -d < 0.$$

Note also that since $\dim M_X(\tilde{L}) = 2d = \sum_{i=1}^r l_i(l_i+1)$, we can take $\sum_{i=1}^r l_i = d$. Thus, we get

$$|\langle \tilde{L}, [\tilde{\Sigma}] \rangle| + \tilde{\Sigma} \cdot \tilde{\Sigma} + 2\sum_{i=1}^r l_i = n - d - d + 2d = n = 2g(\tilde{\Sigma}) - 2.$$

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Department of Mathematics, KAIST, Kusong-Dong, Yusong-Gu, Daejon 305–701, Korea
E-mail: jinkim@math.kaist.ac.kr