Bull. Korean Math. Soc. **39** (2002), No. 3, pp. 471–478

STATE EXTENSIONS OF STATES ON UHF_n ALGEBRA TO CUNTZ ALGEBRA

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ABSTRACT. Let $\eta = \{\eta_m\}_m$ be an eventually constant sequence of unit vectors η_m in $\mathbb{C}^n$ and let ρ_η be the pure state on UHF_n algebra which is defined by $\rho_\eta(v_{i_1} \cdots v_{i_k} v_{j_k}^* \cdots v_{j_1}^*) = \overline{\eta_1^{i_1} \cdots \eta_k^{i_k}} \eta_k^{j_k} \cdots \eta_1^{j_1}$. We find infinitely many state extensions of ρ_η to Cuntz algebra $\mathcal{O}_n$ using representations and unitary operators. Also, we present their concrete expressions.

1. Introduction

For each integer $n \geq 2$, J. Cuntz [4] introduced the universal C^*-algebra $\mathcal{O}_n$ generated by n isometries $s_1, s_2, \cdots, s_n$ satisfying the Cuntz relations

$$s_i^* s_j = \delta_{ij} 1 \quad \text{and} \quad \sum_{i=1}^{n} s_i s_i^* = 1.$$

It is called the *Cuntz algebra* and the isomorphic type of this simple C^*-algebra does not depend on the choice of isometries but on the number of isometries. A UHF_n *algebra* is a uniformly hyperfinite algebra $\overset{\infty}{\underset{i=1}{\otimes}} M_n$ which is the completion of finite linear combinations of operators of the form $A_1 \otimes A_2 \otimes \cdots$, where each A_i is an $n \times n$ matrix and all but finitely many of the A_i's are the identity and we understand UHF_n algebra as a subalgebra of $\mathcal{O}_n$. We note that the linear span of operators of the form $s_{i_1} s_{i_2} \cdots s_{i_k} s_{j_l}^* s_{j_{l-1}}^* \cdots s_{j_1}^*$ for $k, l = 1, 2, \cdots$ is dense in the Cuntz algebra $\mathcal{O}_n$ and a subalgebra UHF_n of $\mathcal{O}_n$ is the closure of the linear span of operators of the form $s_{i_1} s_{i_2} \cdots s_{i_k} s_{j_k}^* s_{j_{k-1}}^* \cdots s_{j_1}^*$.

Received April 4, 2002.
2000 Mathematics Subject Classification: Primary 46L30; Secondary 46L35.
Key words and phrases: Cuntz algebra, UHF_n algebra, state, extension.
This work was supported by UOS-2000.

One of the topics of the study of the Cuntz algebra $\mathcal{O}_n$ is about it's representations. In detail, there is a correspondence between representations of the Cuntz algebra $\mathcal{O}_n$ and endomorphisms of $\mathcal{B}(\mathcal{H})$ of Powers index n up to unitary action, where $\mathcal{B}(\mathcal{H})$ is the set of all bounded linear operators on a Hilbert space $\mathcal{H}$.

So, it is one of main concerns to study the representations of the Cuntz algebra $\mathcal{O}_n$. Since states on the Cuntz algebra $\mathcal{O}_n$ give representations of $\mathcal{O}_n$ by GNS constructions, it is natural that our attention is to study states on $\mathcal{O}_n$.

For a sequence $\eta = \{\eta_m\}_m$ of unit vectors $\eta_m = (\eta_m^1, \cdots, \eta_m^n) \in \mathbb{C}^n$, the *associated linear functional* ω_η on $\mathcal{O}_n$ which is defined by

$$\omega_\eta(s_{i_1} \cdots s_{i_k} s_{j_l}^* \cdots s_{j_1}^*) = \overline{\eta_1^{i_1} \cdots \eta_k^{i_k}} \eta_l^{j_l} \cdots \eta_1^{j_1}$$

is a state if and only if it is a constant sequence $\{\eta_m\}_m$ of a unit vector $\eta_m = \eta_1$ for all m. While, for any sequence $\eta = \{\eta_m\}_m$ of unit vectors $\eta_m = (\eta_m^1, \cdots, \eta_m^n) \in \mathbb{C}^n$, the *associated linear functional* $\rho_\eta = \omega_\eta|_{\mathrm{UHF}_n}$ on UHF_n algebra is a pure state(see [5], [8]).

In this paper, we find infinitely many state extensions of ρ_η on UHF_n algebra to Cuntz algebra $\mathcal{O}_n$ using representations and unitary operators. Also, we present their concrete expressions.

2. State extensions by representations

Throughout this section, let the sequence $\eta = \{\eta_m\}_m$ of unit vectors $\eta_m = (\eta_m^1, \cdots, \eta_m^n) \in \mathbb{C}^n$ be fixed and eventually constant, that is, there is a positive integer k such that $l, m \geq k$ implies that $\eta_l = \eta_m$.

The pure state ρ_k on M_n is defined by $\rho_k(a) = \langle a\eta_k, \eta_k \rangle$, where $\langle , \rangle$ is the usual inner product on $\mathbb{C}^n$. Then for an eventually constant sequence $\eta = \{\eta_m\}_m$ of unit vectors $\eta_m \in \mathbb{C}^n$, the pure state ρ_η on UHF_n algebra is the product state $\otimes\rho_k$ on $\otimes M_n$, so that the corresponding GNS representation $(\pi_\eta, \mathcal{H}_\eta, \Omega_\eta)$ for ρ_η is irreducible and $\rho_\eta(x) = \langle \pi_\eta(x)\Omega_\eta, \Omega_\eta \rangle$.

We review some important facts about $\mathcal{H}_\eta$ and $\mathcal{B}(\mathcal{H}_\eta)$(see [3], [6]).

Consider the set of all formal tensor products of vectors $x_1 \otimes x_2 \otimes \cdots$, where $x_k = \eta_k$ for all but finitely many of the vectors x_k. Then there is a natural inner product which is defined on finite linear combinations of those vectors, satisfying

$$\langle \otimes x_k, \otimes y_k \rangle = \prod_{k=1}^{\infty} \langle x_k, y_k \rangle.$$

Then $\mathcal{H}_\eta$ is the Hilbert space completion of the inner product space. Also note that the cyclic unit vector Ω_η in the GNS representation for ρ_η is $\otimes\eta_k$.

Let $\{\xi_m : m \in \mathbb{N}\}$ be a sequence of unit vectors in $\mathbb{C}^n$ which satisfies $\sum_{m=1}^{\infty} ||\xi_m - \eta_m|| < \infty$ and for each positive integer m, let ξ^m be the vector in $\mathcal{H}_\eta$ given by $\xi^m = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_m \otimes \eta_{m+1} \otimes \eta_{m+2} \otimes \cdots$. Then $\{\xi^m\}$ is a Cauchy sequence(see,[3], [6]). Hence it makes sense to represent the limit of such a Cauchy sequence by the symbol $\otimes\xi_m := \otimes_{m=1}^{\infty}\xi_m$.

For each $m \in \mathbb{N}$, let $\{\eta_{m1}, \cdots, \eta_{mn}\}$ be an orthonormal basis for $\mathbb{C}^n$ selected so that $\eta_{m1} = \eta_m$. Let $\mathbb{I}$ be the set of all ordered sequence $P = \{p_1, p_2, \cdots\}$ where $p_m \in \{1, 2, \cdots, n\}$ for each m, and $p_m = 1$ for all m but finitely many of p_m. We define $\delta_{P,Q}, P, Q \in \mathbb{I}$, to be 1 if $p_m = q_m$ for all k, and otherwise 0. We use the notation $\eta(P)$ to represent the unit vector $\otimes\eta_{mp_m}$ in $\mathcal{H}_\eta$. From discussion above, linear combinations of the vectors $\eta(P)$ are dense in H_η and furthermore, $\langle\eta(P), \eta(Q)\rangle = \delta_{P,Q}, P, Q \in \mathbb{I}$ and the set $\{\eta(P) : P \in \mathbb{I}\}$ forms an orthonormal basis for $\mathcal{H}_\eta$ (see [3], [6]).

PROPOSITION 2.1. *Let $\{e_1, \cdots, e_n\}$ be the standard orthonormal basis for $\mathbb{C}^n$. Define linear operators $v_1, v_2, \cdots, v_n$ in $\mathcal{B}(\mathcal{H}_\eta)$ by $v_i(\eta(P)) = e_i \otimes \eta(P)$. Then the following hold:*

(1) $v_i^*(h \otimes \eta(P)) = \langle h, e_i\rangle\eta(P)$ *for each* $h \in \mathbb{C}^n$.
(2) $v_k^* v_l = \delta_{k,l} I$ *and* $\sum_{i=1}^{n} v_i v_i^* = I$.
(3) $v_{i_1} v_{i_2} \cdots v_{i_k} v_{j_l}^* \cdots v_{j_1}^* (h_1 \otimes h_2 \otimes \cdots)$
$= \langle h_1, e_{j_1}\rangle \cdots \langle h_l, e_{j_l}\rangle e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes h_{l+1} \otimes h_{l+2} \otimes \cdots$.

Proof. (1) Since for any $Q \in \mathbb{I}$,

$$\begin{aligned}\langle v_i^*(h \otimes \eta(P)), \eta(Q)\rangle &= \langle h \otimes \eta(P), v_i(\eta(Q))\rangle \\ &= \langle h \otimes \eta(P), e_i \otimes \eta(Q)\rangle \\ &= \langle h, e_i\rangle\langle\eta(P), \eta(Q)\rangle \\ &= \langle\langle h, e_i\rangle\eta(P), \eta(Q)\rangle\end{aligned}$$

and the set $\{\eta(Q) : Q \in \mathbb{I}\}$ forms an orthonormal basis for $\mathcal{H}_\eta$, we have $v_i^*(h \otimes \eta(P)) = \langle h, e_i\rangle\eta(P)$ for each $h \in \mathbb{C}^n$.

(2) Since $v_k^*(h \otimes \eta(P)) = \langle h, e_k\rangle\eta(P)$ for each $h \in \mathbb{C}^n$ and any $P \in \mathbb{I}$, we have $v_k^* v_l(\eta(P)) = v_k^*(e_l \otimes \eta(P)) = \langle e_l, e_k\rangle\eta(P)$. Therefore for any $i \in \{1, 2, \cdots, n\}$ we have $v_k^* v_l = \delta_{k,l} I$. Since for any $k \in \{1, 2, \cdots, n\}$

and any $P \in \mathbb{I}$, we have

$$\begin{aligned}(\sum_{i=1}^{n} v_i v_i^*)(e_k \otimes \eta(P)) &= \sum_{i=1}^{n} v_i(v_i^*(e_k \otimes \eta(P)) \\ &= \sum_{i=1}^{n} v_i(\langle e_k, e_i\rangle \eta(P)) \\ &= v_k(\eta(P)) \\ &= e_k \otimes \eta(P).\end{aligned}$$

Therefore we have $\sum_{i=1}^{n} v_i v_i^* = I$.

(3) We have the following.

$$\begin{aligned}&v_{i_1} v_{i_2} \cdots v_{i_k} v_{j_l}^* \cdots v_{j_1}^*(h_1 \otimes h_2 \otimes \cdots) \\ &= v_{i_1} v_{i_2} \cdots v_{i_k} v_{j_l}^* \cdots v_{j_2}^*(v_{j_1}^*(h_1 \otimes h_2 \otimes \cdots)) \\ &= v_{i_1} v_{i_2} \cdots v_{i_k} v_{j_l}^* \cdots v_{j_2}^*(\langle h_1, e_{j_1}\rangle h_2 \otimes h_3 \otimes \cdots) \\ &=\langle h_1, e_{j_1}\rangle v_{i_1} v_{i_2} \cdots v_{i_k} v_{j_l}^* \cdots v_{j_2}^*(h_2 \otimes h_3 \otimes \cdots) \\ &=\langle h_1, e_{j_1}\rangle \cdots \langle h_l, e_{j_l}\rangle v_{i_1} v_{i_2} \cdots v_{i_k}(h_{l+1} \otimes h_{l+2} \otimes \cdots) \\ &=\langle h_1, e_{j_1}\rangle \cdots \langle h_l, e_{j_l}\rangle e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes h_{l+1} \otimes h_{l+2} \otimes \cdots .\end{aligned}$$

□

By (2) of the previous Proposition 2.1, we see that the operators $v_1, \cdots, v_n$ are isometries and generate the Cuntz algebra $\mathcal{O}_n$.

In this section, let $\eta = \{\eta_m\}_m$ be an eventually constant sequence of unit vectors $\eta_m \in \mathbb{C}^n$.

THEOREM 2.2. *Let ρ_η be the pure state on UHF_n algebra which is defined by $\rho_\eta(v_{i_1} \cdots v_{i_k} v_{j_k}^* \cdots v_{j_1}^*) = \overline{\eta_1^{i_1} \cdots \eta_k^{i_k}} \eta_k^{j_k} \cdots \eta_1^{j_1}$. Then there is a vector state ψ_η on $\mathcal{B}(\mathcal{H}_\eta)$ which is an extension of ρ_η.*

Proof. Let Ω be the ordered sequence given by $\Omega = \{p_1, p_2, \cdots\}$ with $p_k = 1$ for all k and let ψ_η be the vector state on $\mathcal{B}(\mathcal{H}_\eta)$ given by $\psi_\eta(s) = \langle s\eta(\Omega), \eta(\Omega)\rangle$. Then $\eta(\Omega) = \eta_{11} \otimes \eta_{21} \otimes \cdots = \eta_1 \otimes \eta_2 \otimes \cdots$ and

$$\begin{aligned}&\psi_\eta(v_{i_1} v_{i_2} \cdots v_{i_k} v_{j_l}^* \cdots v_{j_1}^*) \\ &=\langle v_{i_1} v_{i_2} \cdots v_{i_k} v_{j_l}^* \cdots v_{j_1}^* \eta(\Omega), \eta(\Omega)\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle v_{j_l}^* \cdots v_{j_1}^* \eta(\Omega), v_{i_k}^* \cdots v_{i_1}^* \eta(\Omega) \rangle \\
&= \langle \langle \eta_1, e_{j_1} \rangle \cdots \langle \eta_l, e_{j_l} \rangle \eta_{l+1} \otimes \eta_{l+2} \otimes \cdots, \\
&\quad \langle \eta_1, e_{i_1} \rangle \cdots \langle \eta_k, e_{i_k} \rangle \eta_{k+1} \otimes \eta_{k+2} \otimes \cdots \rangle \\
&= \langle \eta_1, e_{j_1} \rangle \cdots \langle \eta_l, e_{j_l} \rangle \overline{\langle \eta_1, e_{i_1} \rangle \cdots \langle \eta_k, e_{i_k} \rangle} \\
&\quad \langle \eta_{l+1} \otimes \eta_{l+2} \otimes \cdots, \eta_{k+1} \otimes \eta_{k+2} \otimes \cdots \rangle \\
&= \overline{\eta_1^{i_1} \cdots \eta_k^{i_k}} \eta_l^{j_l} \cdots \eta_1^{j_1} \langle \eta_{l+1}, \eta_{k+1} \rangle \langle \eta_{l+2}, \eta_{k+2} \rangle \cdots .
\end{aligned}
$$

In particular, $\psi_\eta(v_{i_1} v_{i_2} \cdots v_{i_k} v_{j_k}^* \cdots v_{j_1}^*) = \rho_\eta(v_{i_1} \cdots v_{i_k} v_{j_k}^* \cdots v_{j_1}^*)$. Hence the vector state ψ_η on $\mathcal{B}(\mathcal{H}_\eta)$ given by $\psi_\eta(s) = \langle s\eta(\Omega), \eta(\Omega) \rangle$ is an extension of ρ_η. □

EXAMPLE 2.3. In case of $\eta_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\eta_m = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $m \geq 2$, let ψ_η be the state extension of ρ_η in Theorem 2.2. Then

$$\psi_\eta(v_{i_1} v_{i_2} \cdots v_{i_k} v_{j_l}^* \cdots v_{j_1}^*) = \overline{\eta_1^{i_1} \cdots \eta_k^{i_k}} \eta_k^{j_k} \cdots \eta_1^{j_1} \langle \eta_{l+1}, \eta_{k+1} \rangle \cdots .$$

Hence

$$\psi_\eta(v_{i_1} v_{i_2} \cdots v_{i_k} v_{j_l}^* \cdots v_{j_1}^*) = \begin{cases} 1, & \text{if } i_1 = j_1 = 2,\ i_2 = \cdots = j_2 = \cdots = 1 \\ 0, & \text{otherwise.} \end{cases}$$

REMARK 2.4. Let λ be a complex number with $|\lambda| = 1$ and let $\lambda\eta = \{\xi_m\}_m$ be the sequence with $\xi_m = \lambda\eta_m$. Then $\rho_\eta = \rho_{\lambda\eta}$ for each complex number λ with $|\lambda| = 1$, but $\psi_{\lambda_1\eta} = \psi_{\lambda_2\eta}$ if and only if $\lambda_1 = \lambda_2$. Hence ρ_η has infinitely many vector state extensions to $\mathcal{B}(\mathcal{H}_\eta)$. Since vector states are pure on $\mathcal{B}(\mathcal{H}_\eta)$, we see that ρ_η has infinitely many pure state extensions.

3. State extensions by unitary operators

Throughout this section, we fix a unit vector h in $\mathbb{C}^n$. Let $h_c = \{h_m\}_m$ be the constant sequence with $h_m = h$ for all m , let l be a fixed positive integer and let $\eta = \{\eta_m\}_m$ be a fixed sequence of unit vectors $\eta_m = (\eta_m^1, \cdots, \eta_m^n) \in \mathbb{C}^n$ with $\eta_k = h$ for $k > l$.

We identify a vector y in $\mathbb{C}^n$ with an $n \times 1$ matrix. For $1 \leq i \leq l$, let u_i be an $n \times n$ unitary matrix satisfying $u_i(\eta_i) = h$ and let $u = u_1 \otimes u_2 \otimes \cdots \otimes$

$u_l \otimes I_n \otimes I_n \otimes \cdots$, where I_n is the $n \times n$ identity matrix. Then $u \in \bigotimes_{i=1}^{\infty} M_n$ and u is unitary. Let $\{e_{ij}\}_{i,j}$ be the standard matrix units for M_n. Identifying $e_{i_1 j_1} \otimes \cdots \otimes e_{i_k j_k} \otimes I_n \otimes \cdots$ in $\bigotimes_{i=1}^{\infty} M_n$ with $s_{i_1} s_{i_2} \cdots s_{i_k} s_{j_k}^* \cdots s_{j_1}^*$ in $\mathcal{O}_n$ we understand UHF$_n$ algebra as a subalgebra of $\mathcal{O}_n$.

Recall that for a sequence $\eta = \{\eta_m\}_m$ of unit vectors $\eta_m = (\eta_m^1, \cdots, \eta_m^n) \in \mathbb{C}^n$, the pure state ρ_η on UHF$_n$ algebra is defined by

$$\begin{aligned}\rho_\eta(s_{i_1} \cdots s_{i_k} s_{j_k}^* \cdots s_{j_1}^*) &= \overline{\eta_1^{i_1} \cdots \eta_k^{i_k}} \eta_k^{j_k} \cdots \eta_1^{j_1} \\ &= (\eta_1^* e_{i_1 j_1} \eta_1) \cdots (\eta_k^* e_{i_k j_k} \eta_k).\end{aligned}$$

PROPOSITION 3.1. *In the above notations, $\rho_\eta(\cdot) = \rho_{h_c}(u \cdot u^*)$.*

Proof. Let $u_i = I_n$ for $i \geq l$. Then for any $x = e_{i_1 j_1} \otimes \cdots \otimes e_{i_k j_k} \otimes I_n \otimes \cdots = s_{i_1} s_{i_2} \cdots s_{i_k} s_{j_k}^* \cdots s_{j_1}^*$ in UHF$_n$ algebra and $h^* u_i = (u_i \eta_i)^* u_i = \eta_i^*$, we have

$$\begin{aligned}\rho_{h_c}(uxu^*) &= \rho_{h_c}((u_1 \otimes \cdots \otimes u_l \otimes I_n \otimes \cdots)(e_{i_1 j_1} \otimes \cdots \otimes e_{i_k j_k} \otimes I_n \otimes \cdots) \\ &\quad (u_1^* \otimes \cdots \otimes u_l^* \otimes I_n \otimes \cdots)) \\ &= \rho_{h_c}(u_1 e_{i_1 j_1} u_1^* \otimes \cdots \otimes u_k e_{i_k j_k} u_k^* \otimes I_n \otimes I_n \otimes \cdots) \\ &= (h^* u_1 e_{i_1 j_1} u_1^* h) \cdots (h^* u_k e_{i_k j_k} u_k^* h) \\ &= (\eta_1^* e_{i_1 j_1} \eta_1) \cdots (\eta_k^* e_{i_k j_k} \eta_k) \\ &= (\overline{\eta_1^{i_1}} \eta_1^{j_1}) \cdots (\overline{\eta_k^{i_k}} \eta_k^{j_k}) \\ &= \overline{\eta_1^{i_1} \cdots \eta_k^{i_k}} \eta_1^{j_1} \cdots \eta_k^{j_k} \\ &= \rho_\eta(s_{i_1} s_{i_2} \cdots s_{i_k} s_{j_k}^* \cdots s_{j_1}^*)\end{aligned}$$

and the subalgebra UHF$_n$ of $\mathcal{O}_n$ is the closure of the linear span of operators of the form $s_{i_1} s_{i_2} \cdots s_{i_k} s_{j_k}^* s_{j_{k-1}}^* \cdots s_{j_1}^*$. Hence we have $\rho_\eta(\cdot) = \rho_{h_c}(u \cdot u^*)$. □

THEOREM 3.2. *Let ρ_η be the pure state on UHF$_n$ algebra which is defined by $\rho_\eta(v_{i_1} \cdots v_{i_k} v_{j_k}^* \cdots v_{j_1}^*) = \overline{\eta_1^{i_1} \cdots \eta_k^{i_k}} \eta_k^{j_k} \cdots \eta_1^{j_1}$. Then there is a unitary operator u in UHF$_n$ algebra such that $\psi(\cdot) = \omega_{h_c}(u \cdot u^*)$ is an extension of ρ_η to the Cuntz algebra $\mathcal{O}_n$.*

Proof. Let $u = u_1 \otimes u_2 \otimes \cdots \otimes u_l \otimes I_n \otimes I_n \otimes \cdots$ be unitary in $\bigotimes_{i=1}^{\infty} M_n$ as above. Since ω_{h_c} is a state on the Cuntz algebra $\mathcal{O}_n$ and u is unitary in UHF$_n$ algebra, $\psi(\cdot) = \omega_{h_c}(u \cdot u^*)$ is a state on the Cuntz algebra $\mathcal{O}_n$. By Proposition 3.1, ψ is an extension of ρ_η to the Cuntz algebra $\mathcal{O}_n$. □

REMARK 3.3. Let λ be a complex number with $|\lambda| = 1$ and let $\lambda h_c = \{\xi_m\}_m$ be the sequence with $\xi_m = \lambda h$. Then $\rho_{h_c} = \rho_{\lambda h_c}$ and $\psi_\lambda(\cdot) = \omega_{\lambda h_c}(u \cdot u^*)$ is a state on the Cuntz algebra $\mathcal{O}_n$. Furthermore $\psi_{\lambda_1} = \psi_{\lambda_2}$ if and only if $\lambda_1 = \lambda_2$. Hence ρ_η has infinitely many state extensions to the Cuntz algebra $\mathcal{O}_n$.

EXAMPLE 3.4. In case of $\eta_1 = \eta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $h = \eta_m = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $m \geq 3$, let $u_1 = u_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $u_1(\eta_1) = u_2(\eta_2) = h$,

$$\begin{aligned} u &= u_1 \otimes u_2 \otimes I_2 \otimes I_2 \otimes \cdots \\ &= (e_{12} + e_{21}) \otimes (e_{12} + e_{21}) \otimes I_2 \otimes I_2 \otimes \cdots \\ &= (e_{12} \otimes e_{12} + e_{12} \otimes e_{21} + e_{21} \otimes e_{12} + e_{21} \otimes e_{21}) \otimes I_2 \otimes I_2 \otimes \cdots \\ &= s_1^2 (s_2^*)^2 + s_1 s_2 s_1^* s_2^* + s_2 s_1 s_2^* s_1^* + s_2^2 (s_1^*)^2 \end{aligned}$$

and $\omega_{h_c}(u \cdot u^*)$ is a state extension of $\rho_\eta$to the Cuntz algebra $\mathcal{O}_2$.

EXAMPLE 3.5. In case of $\eta_1 = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}$, $\eta_2 = \begin{pmatrix} 0 \\ \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$ and $h = \eta_m = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for $m \geq 3$, let $u_1 = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & -\frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & \frac{4}{5} & -\frac{3}{5} \\ 1 & 0 & 0 \end{pmatrix}$.

Then $u_1(\eta_1) = u_2(\eta_2) = h$ and

$$\begin{aligned} & u_1 \otimes u_2 \\ &= (\frac{3}{5} e_{11} + \frac{4}{5} e_{12} + \frac{4}{5} e_{21} - \frac{3}{5} e_{22} + e_{33}) \\ &\quad \otimes (\frac{3}{5} e_{12} + \frac{4}{5} e_{13} + \frac{4}{5} e_{22} - \frac{3}{5} e_{23} + e_{31}) \\ &= \frac{9}{25} e_{11} \otimes e_{12} + \frac{12}{25} e_{11} \otimes e_{13} + \frac{12}{25} e_{11} \otimes e_{22} - \frac{9}{25} e_{11} \otimes e_{23} + \frac{3}{5} e_{11} \otimes e_{31} \\ &\quad + \frac{12}{25} e_{12} \otimes e_{12} + \frac{16}{25} e_{12} \otimes e_{13} + \frac{16}{25} e_{12} \otimes e_{22} - \frac{12}{25} e_{12} \otimes e_{23} + \frac{4}{5} e_{12} \otimes e_{31} \\ &\quad + \frac{12}{25} e_{21} \otimes e_{12} + \frac{16}{25} e_{21} \otimes e_{13} + \frac{16}{25} e_{21} \otimes e_{22} - \frac{12}{25} e_{21} \otimes e_{23} + \frac{4}{5} e_{21} \otimes e_{31} \\ &\quad - \frac{9}{25} e_{22} \otimes e_{12} - \frac{12}{25} e_{22} \otimes e_{13} - \frac{12}{25} e_{22} \otimes e_{22} + \frac{9}{25} e_{22} \otimes e_{23} - \frac{3}{5} e_{22} \otimes e_{31} \\ &\quad + \frac{3}{5} e_{33} \otimes e_{12} + \frac{4}{5} e_{33} \otimes e_{13} + \frac{4}{5} e_{33} \otimes e_{22} - \frac{3}{5} e_{33} \otimes e_{23} + e_{33} \otimes e_{31}. \end{aligned}$$

Hence

$$\begin{aligned}
u &= u_1 \otimes u_2 \otimes I_3 \otimes I_3 \otimes \cdots \\
&= \frac{9}{25} s_1^2 s_2^* s_1^* + \frac{12}{25} s_1^2 s_3^* s_1^* + \frac{12}{25} s_1 s_2 s_2^* s_1^* - \frac{9}{25} s_1 s_2 s_3^* s_1^* + \frac{3}{5} s_1 s_3 (s_1^*)^2 \\
&+ \frac{12}{25} s_1^2 (s_2^*)^2 + \frac{16}{25} s_1^2 s_3^* s_2^* + \frac{16}{25} s_1 s_2 (s_2^*)^2 - \frac{12}{25} s_1 s_2 s_3^* s_2^* + \frac{4}{5} s_1 s_3 s_1^* s_2^* \\
&+ \frac{12}{25} s_2 s_1 s_2^* s_1^* + \frac{16}{25} s_2 s_1 s_3^* s_1^* + \frac{16}{25} s_2^2 s_2^* s_1^* - \frac{12}{25} s_2^2 s_3^* s_1^* + \frac{4}{5} s_2 s_3 (s_1^*)^2 \\
&- \frac{9}{25} s_2 s_1 (s_2^*)^2 - \frac{12}{25} s_2 s_1 s_3^* s_2^* - \frac{12}{25} s_2^2 (s_2^*)^2 + \frac{9}{25} s_2^2 s_3^* s_2^* - \frac{3}{5} s_2 s_3 s_1^* s_2^* \\
&+ \frac{3}{5} s_3 s_1 s_2^* s_3^* + \frac{4}{5} s_3 s_1 (s_3^*)^2 + \frac{4}{5} s_3 s_2 s_2^* s_3^* - \frac{3}{5} s_3 s_2 (s_3^*)^2 + s_3^2 s_1^* s_3^*
\end{aligned}$$

and $\omega_{h_c}(u \cdot u^*)$ is a state extension of ρ_η to the Cuntz algebra $\mathcal{O}_3$.

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