NOTE ON GOOD IDEALS IN
GORENSTEIN LOCAL RINGS

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ABSTRACT. Let $I$ be an ideal in a Gorenstein local ring $A$ with the maximal ideal $m$ and $d = \dim A$. Then we say that $I$ is a good ideal in $A$, if $I$ contains a reduction $Q = (a_1, a_2, \cdots, a_d)$ generated by $d$ elements in $A$ and $G(I) = \oplus_{n \geq 0} I^n/I^{n+1}$ of $I$ is a Gorenstein ring with $a(G(I)) = 1 - d$, where $a(G(I))$ denotes the $a$-invariant of $G(I)$. Let $S = A[Q/a_1]$ and $P = mS$. In this paper, we show that the following conditions are equivalent.

1. $I^2 = QI$ and $I = Q : I$.
2. $I^2S_a = a_1IS$ and $IS = a_1S : SIS$.
3. $I^2Sp = a_1ISp$ and $ISp = a_1Sp : SpISp$.

We denote by $\mathcal{X}_A(Q)$ the set of good ideals $I$ in $\mathcal{X}_A$ such that $I$ contains $Q$ as a reduction. As a Corollary of this result, we show that

$$I \in \mathcal{X}_A(Q) \iff ISp \in \mathcal{X}_S(Q_p).$$

1. Introduction

Let $A$ be a Gorenstein local ring with the maximal ideal $m$ and $d = \dim A$. Let $I$ denote an $m$-primary ideal in $A$. Then we say that $I$ is a good ideal in $A$ if $I$ contains a parameter ideal $(c_1, c_2, \cdots, c_d)$ in $A$ as a reduction and the associated graded ring $G(I) = \oplus_{n \geq 0} I^n/I^{n+1}$ of $I$ is a Gorenstein ring with $a(G(I)) = 1 - d$ ([3]), where $a(G(I))$ denotes the $a$-invariant of $G(I)$ ([4], Definition 3.1.4)). We denote by $\mathcal{X}_A$ the set of good ideals $I$ in $A$. The concept of good ideals was first introduced by S. Goto, S. Iai, and K. Watanabe and they intensively studied $m$-primary
good ideals in a given Gorenstein local ring and gave many inspiring
results ([3]).

Let \( Q = (a_1, a_2, \cdots, a_d) \) be a fixed parameter ideal in \( A \). Let \( S = A[Q/a_1] \) and \( P = mS \). We denote by \( \mathcal{X}_A(Q) \) the set of good ideals \( I \) in \( \mathcal{X}_A \) such that \( I \) contains \( Q \) as a reduction. With this notation the main result of this paper is stated as follows.

**Theorem 1.1.** Let \( I (\neq A) \) be an ideal in \( A \). Suppose that \( I \) contains a parameter ideal \( Q = (a_1, \cdots, a_d) \) as a reduction. Then the following conditions are equivalent.

1. \( I^2 = QI \) and \( I = Q : I \).
2. \( I^2S = a_1IS \) and \( IS = a_1IS : SIS \).
3. \( I^2SP = a_1ISP \) and \( ISP = a_1SP : SISP \).

**Corollary 1.2.** Let \( I (\neq A) \) be an ideal in \( A \). Suppose that \( I \) contains a parameter ideal \( Q = (a_1, \cdots, a_d) \) as a reduction. Then the following conditions are equivalent.

1. \( I \in \mathcal{X}_A(Q) \).
2. \( ISP \in \mathcal{X}_{SP}(QS_P) \).

In what follows, let \((A, m)\) be a Gorenstein local ring and \( d = \dim A \). Let \( K = Q(A) \) be the total quotient ring of \( A \). We denote by \( \mu_A(*) \) the number of generators and \( \ell_A(*) \) the length.

Let \( B = \oplus_{n \in \mathbb{Z}} B_n \) be a Noetherian graded ring and assume that \( B \) contains a unique graded maximal ideal \( \mathfrak{M} \). We denote by \( H^i_{\mathfrak{M}}(*) \) \((i \in \mathbb{Z})\) the \( i \)th local cohomology functor of \( B \) with respect to \( \mathfrak{M} \). For each graded \( B \)-module \( E \) and \( n \in \mathbb{Z} \), let \( H^i_{\mathfrak{M}}(E)_n \) denote the homogeneous component of the graded \( B \)-module \( H^i_{\mathfrak{M}} \) of degree \( n \). Let \( E \) be a graded \( B \)-module. For each \( n \in \mathbb{Z} \) let \( E(n) \) stand for the graded \( B \)-module, whose underlying \( B \)-module coincides with that of \( E \) and whose graduation is given by \( [E(n)]_i = E_{n+i} \) for all \( i \in \mathbb{Z} \). We refer the reader to [5], [1], or [6] for any unexplained notation or terminology.

**2. Preliminaries**

Let \((A, m)\) be a \( d \)-dimensional Gorenstein local ring with \( d \geq 2 \) and \( K = Q(A) \) be the total quotient ring of \( A \). Let \( Q = (a_1, \cdots, a_d) \) be a fixed parameter ideal for \( A \). Let \( S = A[Q/a_1](= \cup_{n \geq 0} Q^n/a_1^n) \) and
\[ P = \mathfrak{m}S. \] Then \( A \subseteq S \subseteq K \) and we have the isomorphism

\[
S \cong \frac{A[T_2, T_3, \cdots, T_d]}{(a_1T_2 - a_2, a_1T_3 - a_3, \cdots, a_1T_d - a_d)},
\]

where \( T_2, T_3, \cdots, T_d \) denote indeterminates over \( A \). Hence \( S \) is a \( d \)-dimensional Gorenstein ring, since \( a_1T_2 - a_2, a_1T_3 - a_3, \cdots, a_1T_d - a_d \) is a regular sequence ([2]). Moreover \( P \) is a height 1 prime ideal of \( S \), because \( S/P \cong (A/m)[T_2, T_3, \cdots, T_d] \) is a \((d-1)\)-dimensional regular domain, whence \( S_P \) is a 1-dimensional Gorenstein local ring. For the proof of our result we need the following lemmas.

**Lemma 2.1.** Let \( I \neq A \) be an ideal in \( A \). Suppose that \( I \) contains \( Q \) as a reduction. Then

1. \( IS \) is a \( P \)-primary ideal in \( S \).
2. \( IS_P \cap A = I \).
3. \( IS \cap A = I \).
4. \( \ell_{S_P}(S_P/IS_P) = \ell_A(A/I) \) and \( \ell_{S_P}(S_P/QS_P) = \ell_A(A/Q) \).

**Proof.** Notice that \( QS = a_1S \) and \( \sqrt{QS} = \sqrt{IS} = P \).

1. \( S/IS \cong (A/I)[T_2, T_3, \cdots, T_d] \), since \( IA[T_2, T_3, \cdots, T_d] \supseteq (a_1T_2 - a_2, a_1T_3 - a_3, \cdots, a_1T_d - a_d) \). Hence \( \text{Ass}(S/IS) = \{\mathfrak{m}S\} \), because \( \text{Ass}(A[T_2, \cdots, T_d]/IA[T_2, \cdots, T_d]) = \{mA[T_2, \cdots, T_d]\} \). Thus \( IS \) is a \( P \)-primary ideal in \( S \).

2. \( IS_P \cap S = I \) by (1). Hence we have \( IS_P \cap A = (IS_P \cap S) \cap A = I \cap A = I \).

3. Let \( \alpha \in IS \cap A \) and write \( \alpha = \beta \frac{a_1}{a_1} \) with \( \beta \in I \) and \( g \in Q^\ell \) for some \( \ell \geq 0 \). Since \( \alpha \in A \), we get \( \omega a_1^\ell = \beta g \in IQ^\ell = I(a_1^\ell + (a_2, a_3, \cdots, a_d)Q^{\ell-1}) \). Now we write \( \omega a_1^\ell = \omega (a_1^\ell + f \sum_{i=2}^d x_ia_i) \) with \( \omega \in I, f \in Q^{\ell-1} \), and \( x_i \in A \) for \( i = 2, \cdots, d \). Then \( a_1^\ell (\alpha - \omega) = \omega f \sum_{i=2}^d x_ia_i \in (a_2, a_3, \cdots, a_d) \) so that \( \alpha - \omega \in (a_2, a_3, \cdots, a_d) : a_1^\ell = (a_2, a_3, \cdots, a_d) \), since \( a_1, a_2, \cdots, a_d \) is a regular sequence. Hence \( \alpha \in \omega + (a_2, a_3, \cdots, a_d) \in I \). The other inclusion is obvious and hence \( IS \cap A = I \).

4. We have the following isomorphisms

\[
\frac{S_P}{IS_P} \cong \left( \frac{A[T_2, T_3, \cdots, T_d]}{IA[T_2, T_3, \cdots, T_d]} \right) \frac{mA[T_2, T_3, \cdots, T_d]}{mA[T_2, T_3, \cdots, T_d]} \\
\cong \frac{A[T_2, T_3, \cdots, T_d]}{IA[T_2, T_3, \cdots, T_d] \frac{mA[T_2, T_3, \cdots, T_d]}{mA[T_2, T_3, \cdots, T_d]},}
\]
where \( mA[T_2,T_3,\ldots,T_d] = mA[T_2,T_3,\ldots,T_d] \). Hence \( \ell_{S_P}(S_P/IS_P) = \ell_A(A/I) \), because \( \text{Ass}_S(S/a_1S) \) is faithfully flat over \( A \). Similarly, we have \( \ell_{S_P}(S_P/QS_P) = \ell_A(A/Q) \). This completes the proof of Lemma (2.1). \( \square \)

**Lemma 2.2.** ([3], Proposition (2.2)) Let \( I \) be an \( m \)-primary ideal in \( A \) and assume that \( I \) contains \( Q \) as a reduction. Then the following conditions are equivalent.

1. \( I \in \mathcal{X}_A \).
2. \( I^2 = QI, I = Q : I \).
3. \( I^2 = QI, \ell_A(A/I) = \frac{1}{2}\ell_A(A/Q) \).
4. \( I^2 \subseteq Q^2 \) and \( I = Q : I \).
5. The algebra \( R'(I) = \oplus_{n \geq 0} I^n t^n \) is a Gorenstein ring and \( K_{R'(I)} \cong R'(I)(2 - d) \) as graded \( R'(I) \)-modules, where \( K_{R'(I)} \) denotes the canonical module of \( R'(I) \).

If \( d \geq 1 \), we may add the following.

6. \( I^n = Q^n : I \) for all \( n \in \mathbb{Z} \).

When this is the case, we have \( r(A/I) = \mu_A(I/Q) = \mu_A(I) - d + 1 \) and \( e_I(A) = 2\ell_A(A/I) \), where \( r(A/I) \) denotes the Cohen-Macaulay type of \( A/I \) and \( e_I(A) \) denotes the multiplicity of \( A \) with respect to \( I \).

3. **Proof of Theorem 1.1**

**Proof of Theorem 1.1.** (1)\( \Rightarrow \)(2) Since \( QS = a_1S \), we have \( I^2S = QIS = a_1IS \). Let \( f \in a_1S : SIS \) with \( f \in S \). Then \( fx \in a_1S \) with \( x \in I \) and write \( fx = a_1(Q^\ell/a_1^\ell) \) for some \( \ell \geq 0 \), since \( S = A[Q/a_1] = \bigcup_{n \geq 0} Q^n/a_1^n \). Since \( f \in S \), we have \( x = a_1^\ell h \) with \( h \in Q^u \) and \( g \in Q^\ell \) for some \( u \geq 0 \). We may assume that \( \ell = u \). Hence \( xh = a_1^\ell g \in Q^{\ell+1} \). Since \( x \in I \), we have \( h \in Q^{\ell+1} : I = I^{\ell+1} = Q^{\ell+1} \) by Lemma 2.2.(6), whence \( f = a_1^\ell \in I^{Q^\ell}/a_1^\ell \subseteq IS \). Thus \( IS = a_1S : SIS \).

(2)\( \Rightarrow \)(3) This is clear.

(3)\( \Rightarrow \)(2) Suppose that \( I^2S \not\subseteq a_1IS \). Then there exists a prime ideal \( p \in \text{Ass}_S(S/a_1IS) \) such that \( I^2S_p \not\subseteq a_1IS_p \). If \( p = P \), then \( I^2S_p = a_1IS_p \), which is impossible. Hence \( p \supsetneq P \), whence \( h_tsp \geq 2 \). We look at the exact sequences

\[
(*) \quad 0 \rightarrow (IS)_p \xrightarrow{a_1} S_p \rightarrow (S/a_1IS)_p \rightarrow 0,
\]
of $S_p$-modules. Apply functors $H_m^l(-)$ to (***) and we have $\text{depth}(IS)_p \geq 2$, because $S_p$ is a Gorenstein local ring of $\dim S_p \geq 2$ and $\text{depth}(S/IS)_p \geq 1$, since $p \supseteq P$ and $IS$ is a $P$-primary ideal. Now apply functors $H_m^l(-)$ to (*) and we have $\text{depth}(S/a_1IS)_p \geq 1$, when $p \notin \text{Ass}_S(S/a_1IS)$. This is impossible, because $p \in \text{Ass}_S(S/a_1IS)$ by our assumption. Thus $I^2S = a_1IS$. Suppose that $IS \subseteq a_1S :_S IS$. Then there exists a prime ideal $q \in \text{Ass}_S(S/IS)$ such that $IS_q \subseteq a_1S_q :_q IS_q$. Since $\text{Ass}_S(S/IS) = \{p\}$, we have $q = p$. This is a contradiction to our assumption. Hence $IS = a_1S :_S IS$.

(2)$\Rightarrow$(1) $I^2 \subseteq I^2S \cap A = a_1IS \cap A \subseteq a_1S \cap A = QS \cap A = Q$, by the similar reason of Lemma 2.1.(3). Hence $I \subseteq Q :_A I$. By Lemma 2.1 (3), we have

$$I = IS \cap A = (a_1S :_S IS) \cap A$$
$$\supseteq (Q :_A I)^{cc}$$
$$\supseteq Q :_A I.$$
where \( \alpha = (\alpha_1, \ldots, \alpha_d) | \alpha_1 + \cdots + \alpha_d = l \) and \( 0 \leq \alpha_i \in I \). Then 
\[
y^l = \sum c_\alpha (a_1 t)^{\alpha_1} (a_2 t)^{\alpha_2} \cdots (a_d t)^{\alpha_d},
\]
whence 
\[
y^l = \sum c_\alpha t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_d^{\alpha_d}
\]
and hence 
\[
c_1 t_1^l + c_2 t_2^{l-1} t_1 + \cdots + c_d t_d^{l-1} = \sum c_\alpha t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_d^{\alpha_d}.
\]

Thus we have \( c_i = c_{\alpha_i} \) for some \( \alpha = (\alpha_1, \ldots, \alpha_d). \) Since \( c_i \in A/Q \) and \( c_{\alpha_i} \in I/Q, \) we have \( c_i - c_{\alpha_i} \in Q, \) whence \( c_i \in c_{\alpha_i} + Q \subseteq I \) and hence 
\[
x = \sum_{i=1}^d c_i a_i \in QI.
\]
Therefore \( I^2 = QI. \) This completes the proof of Theorem 1.1. \( \square \)

Proof of Corollary 1.2. Let \( I \) contain \( Q \) as a reduction. Hence \( I \) contains \( Q \) as a reduction if and only if \( IS_p \) contains \( QS_p \) as a reduction. Thus 
\[
I \in \mathcal{X}_A(Q) \iff IS_p \in \mathcal{X}_S(QS_p)
\]
by Theorem 1.1. \( \square \)

References


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