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DERIVATIONS ON PRIME AND SEMI-PRIME RINGS

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ABSTRACT. In this paper we will show that if there exist derivations D, G on a $n!$-torsion free semi-prime ring R such that the mapping $D^2 + G$ is n-commuting on R, then D and G are both commuting on R. And we shall give the algebraic conditions on a ring that a Jordan derivation is zero.

1. Introduction

Throughout this paper R will be represent an associative ring with center $Z(R)$. The commutator $xy - yx$ (resp. the Jordan product $xy + yx$) will be denoted by $[x, y]$ (resp. $\langle x, y\rangle$). We make extensive use of the basic identities $[xy, z] = [x, z]y + x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$. Let $rad(A)$ denote the *(Jacobson) radical* of an algebra A. Recall that R is *prime* if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is *semi-prime* if $aRa = (0)$ implies $a = 0$. An additive mapping D from R to R is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. A derivation D is *inner* if there exists $a \in R$ such that $D(x) = [a, x]$ holds for all $x \in R$. And also, an additive mapping D from R to R is called a *Jordan derivation* if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$. An additive mapping F from R to R is said to be a *commuting* (resp. *centralizing*) if $[F(x), x] = 0$ (resp. $[F(x), x] \in Z(R)$) holds for all $x \in R$. More generally, for a positive integer n, we define a mapping F to be *n-commuting* if $[F(x), x^n] = 0$ for all $x \in R$.

The underlying idea of our research is Posner's second theorem [7, Theorem 2] which is the beginning of the study concerning centralizing and commuting mappings, which states that the existence of a nonzero

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centralizing derivation on a prime ring forces the ring to be commutative. It is our aim in this note to present some results which can be considered as a contribution to the theory of mappings in prime and semi-prime rings.

2. Derivations on prime and semi-prime rings

Our first result was inspired by Posner's first theorem [7, Theorem 1], which asserts that if R is a 2-torsion free prime ring, and D, G are nonzero derivations on R, then DG cannot be a derivation.

THEOREM 2.1. *Let n be a fixed positive integer. Let R be a $n!$-torsion free semi-prime ring. Suppose that there exist derivations $D, G : R \to R$ such that the mapping $D^2 + G$ is n-commuting on R. Then D and G are both commuting on R.*

Proof. For the convenience, let us write F instead of $D^2 + G$. From the assumption of the theorem, the mapping F is n-commuting on R. That is,

$$[F(x), x^n] = 0 \tag{1}$$

for all $x \in R$. Consider an integer k with $1 \leq k \leq n$. Replacing $x + ky$ for x in (1), we obtain

$$kQ_1(x,y) + k^2Q_2(x,y) + \cdots + k^nQ_n(x,y) = 0 \tag{2}$$

for all $x, y \in R$, where $Q_i(x,y)$ denotes the sum of these terms in which y appears as a term in the product i times. By [3, Lemma 1], we have

$$\begin{aligned} Q_1(x,y) =&[F(y), x^n] + [F(x), x^{n-1}y] \\ &+ [F(x), x^{n-2}yx] + [F(x), x^{n-3}yx^2] \\ &+ \cdots + [F(x), yx^{n-1}] = 0 \end{aligned} \tag{3}$$

for all $x, y \in R$. Substituting xy for y in (3), we get

$$\begin{aligned} 0 =&x[F(x), x^{n-1}y] + [F(x), x]x^{n-1}y \\ &+ x[F(x), x^{n-2}yx] + [F(x), x]x^{n-2}yx \\ &+ x[F(x), x^{n-3}yx^2] + [F(x), x]x^{n-3}yx^2 \\ &+ \cdots + x[F(x), yx^{n-1}] + [F(x), x]yx^{n-1} + F(x)[y, x^n] \\ &+ 2[D(x)D(y), x^n] + x[F(y), x^n] \end{aligned} \tag{4}$$

for all $x, y \in R$. Left multiplication of (3) by x gives

$$\begin{aligned}0 = x[F(y),\ x^n] + x[F(x),\ x^{n-1}y] \\ + x[F(x),\ x^{n-2}yx] + x[F(x),\ x^{n-3}yx^2] \\ + \cdots + x[F(x),\ yx^{n-1}]\end{aligned} \tag{5}$$

for all $x, y \in R$. Subtracting (5) from (4), it follows that

$$\begin{aligned}0 =& [F(x), x]x^{n-1}y + [F(x), x]x^{n-2}yx \\ &+ [F(x), x]x^{n-3}yx^2 + \cdots + [F(x), x]yx^{n-1} \\ &+ F(x)[y, x^n] + 2[D(x)D(y), x^n]\end{aligned} \tag{6}$$

for all $x, y \in R$. Substituting yx for y in (6), we obtain

$$\begin{aligned}0 =& [F(x), x]x^{n-1}yx + [F(x), x]x^{n-2}yx^2 \\ &+ [F(x), x]x^{n-3}yx^3 + \cdots + [F(x), x]yx^n + F(x)[y, x^n]x \\ &+ 2[D(x)D(y), x^n]x + 2[D(x)yD(x), x^n]\end{aligned} \tag{7}$$

for all $x, y \in R$. Right multiplication of (6) by x gives

$$\begin{aligned}0 =& [F(x), x]x^{n-1}yx + [F(x), x]x^{n-2}yx^2 \\ &+ [F(x), x]x^{n-3}yx^3 + \cdots + [F(x), x]yx^n \\ &+ F(x)[y, x^n]x + 2[D(x)D(y), x^n]x\end{aligned} \tag{8}$$

for all $x, y \in R$. Subtracting (8) from (7), we get

$$D(x)yD(x)x^n - x^nD(x)yD(x) = 0, \tag{9}$$

for all $x, y \in R$, since R is 2-torsion free. Replacing y by $yD(x)z$ in (9), we obtain

$$D(x)yD(x)zD(x)x^n - x^nD(x)yD(x)zD(x) = 0 \tag{10}$$

for all $x, y, z \in R$. By (9), we can write in the relation (10) $x^nD(x)zD(x)$ for $D(x)zD(x)x^n$ and $D(x)yD(x)x^n$ instead of $x^nD(x)yD(x)$, which gives

$$D(x)y[D(x), x^n]zD(x) = 0 \tag{11}$$

for all $x, y, z \in R$. Putting $x^n y$ for y in (11), we have

$$D(x)x^n y[D(x), x^n]zD(x) = 0 \tag{12}$$

for all $x, y, z \in R$. Left multiplication of (11) by x^n leads to

$$x^n D(x)y[D(x), x^n]zD(x) = 0 \tag{13}$$

for all $x, y, z \in R$. Subtracting (13) from (12), we obtain

$$[D(x), x^n]y[D(x), x^n]zD(x) = 0 \tag{14}$$

for all $x, y, z \in R$. Replacing z by zx^n in (14), we get

$$[D(x), x^n]y[D(x), x^n]zx^n D(x) = 0 \tag{15}$$

for all $x, y, z \in R$. Right multiplication of (14) by x^n gives

$$[D(x), x^n]y[D(x), x^n]zD(x)x^n = 0 \tag{16}$$

for all $x, y, z \in R$. Subtracting (16) from (15), we obtain

$$[D(x), x^n]y[D(x), x^n]z[D(x), x^n] = 0 \tag{17}$$

for all $x, y, z \in R$. Putting $z = w[D(x), x^n]y$ in the above relation, we get

$$[D(x), x^n]y[D(x), x^n]w[D(x), x^n]y[D(x), x^n] = 0$$

for all $x, y, w \in R$. Since R is semi-prime, we arrive at

$$[D(x), x^n] = 0$$

for all $x \in R$. Now, the latter half of the proof in [4, Theorem 2] shows that

$$[D(x), x] = 0 \tag{18}$$

for all $x \in R$. The linearization of (18) gives

$$[D(x), y] + [D(y), x] = 0$$

for all $x, y \in R$. And in particular for $y = D(x)$, we have

$$[D^2(x), x] = 0$$

for all $x \in R$. This means that for a positive integer n,

$$[D^2(x), x^n] = 0$$

for all $x \in R$. Since $F = D^2 + G$ is n-commuting, we also obtain

$$[G(x), x^n] = 0$$

for all $x \in R$. Again, by inspecting the latter half of the proof in [4, Theorem 2], we get

$$[G(x), x] = 0$$

for all $x \in R$, which completes the proof of the theorem. □

Since in noncommutative semi-prime rings there exist nonzero commuting derivations, the assumptions of Theorem 2.1 do not imply that both $D = 0$ and $G = 0$. However, in the special case when either D or G is an inner derivation, we can prove the following result.

COROLLARY 2.2. *Let n be a fixed positive integer. Let R be a $n!$-torsion free semi-prime ring. Suppose that there exist derivations $D, G : R \rightarrow R$ such that the mapping $D^2 + G$ is n-commuting on R. If D is inner we have that $D = 0$. If G is inner then $G = 0$.*

Proof. It is an immediate consequence of Theorem 2.1 and [2, Corollary 5]. □

In 2000, Park and Kim [6] proved that if there exist derivations D, G on a 3!-torsion free noncommutative ring R such that $[D(x), x]G(x) = 0$ for all $x \in R$, then $D = 0$ and $G = 0$. In this paper we shall also give the result due to Park and Kim's theorem [6].

THEOREM 2.3. *Let R be a noncommutative 3!-torsion free prime ring. Suppose that there exist Jordan derivations $D, G : R \rightarrow R$ such that*

$$\langle D(x), x\rangle G(x) = 0, \ G(D(x))D(x) = 0$$

for all $x \in R$. Then we have $D = 0$ or $G = 0$.

Proof. By [1, Theorem 1], it is clear that D and G are derivations on R. Suppose that

$$\langle D(x), x\rangle G(x) = 0 \tag{19}$$

for all $x \in R$. Replace x by $x + ky$ in (19). Then we have

$$kQ_1(y) + k^2Q_2(y) = 0,$$

where

$$Q_i(y) = Q_i(x, y),\ Q_i(ky) = k^iQ_i(y),\ i = 1, 2$$

for all $x, y \in R$. By [3, Lemma 1], we get

$$\begin{aligned} Q_1(y) =& D(x)yG(x) + D(y)xG(x) + D(x)xG(y) \\ &+xD(y)G(x) + yD(x)G(x) + xD(x)G(y) \\ =&0 \end{aligned} \tag{20}$$

for all $x, y \in R$. Replace y by xy in (20). Then we have by (19) and (20)

$$D(x)xyG(x) + D(x)yxG(x) + [D(x), x^2]G(y) = 0 \tag{21}$$

for all $x, y \in R$. Substituting $yG(x)$ for y in (21), we obtain

$$\begin{aligned} &D(x)xyG(x)^2 + D(x)yG(x)xG(x) + [D(x), x^2]G(y)G(x) \\ &+ [D(x), x^2]yG^2(x) = 0 \end{aligned} \tag{22}$$

for all $x, y \in R$. Right multiplication of (21) by $G(x)$ leads to

$$D(x)xyG(x)^2 + D(x)yxG(x)^2 + [D(x), x^2]G(y)G(x) = 0 \tag{23}$$

for all $x, y \in R$. Subtracting (23) from (22), we obtain

$$D(x)y[G(x), x]G(x) + [D(x), x^2]yG^2(x) = 0 \tag{24}$$

for all $x, y \in R$. Let $y = G(D(x))y$ in (24). Then we have by (19) and (21)

$$D(x)G(D(x))y[G(x), x]G(x) = 0 \tag{25}$$

for all $x, y \in R$. Replacing x by $x + tz$ in (25), we obtain

$$tU_1(z) + t^2U_2(z) + t^3U_3(z) + t^4U_4(z) = 0,$$

where

$$U_i(z) = U_i(x, y, z),\ U_i(tz) = t^iU_i(z),\ i = 1, 2, 3, 4$$

for all $x, y, z \in R$. By [3, Lemma 1], we arrive at

$$\begin{aligned} U_1(z) =&(D(z)G(D(x)) + D(x)G(D(z)))y[G(x), x]G(x) \\ &+ D(x)G(D(x))y([G(z), x]G(x) + [G(x), z]G(x) \\ &+ [G(x), x]G(z)) = 0 \end{aligned} \tag{26}$$

for all $x, y, z \in R$. Substituting $y[G(x), x]G(x)r$ for y in (26), we have by (25)

$$(D(z)G(D(x)) + D(x)G(D(z)))y[G(x), x]G(x)r[G(x), x]G(x) = 0 \tag{27}$$

for all $x, y, z, r \in R$. Putting $r(D(z)G(D(x)) + D(x)G(D(z)))y$ instead of r in (27), we obtain by the primeness

$$(D(z)G(D(x)) + D(x)G(D(z)))y[G(x), x]G(x) = 0 \tag{28}$$

for all $x, y, z \in R$. By the same method as above, and starting from (28), we get

$$(D(z)G(D(u)) + D(u)G(D(z)))y[G(x), x]G(x) = 0 \tag{29}$$

for all $x, y, u, z \in R$. Since R is prime, the relation (29) gives

$$(D(z)G(D(u)) + D(u)G(D(z)) = 0 \tag{30}$$

or

$$[G(x), x]G(x) = 0 \tag{31}$$

for all $x \in R$. If (31) holds, then by [9, Lemma] we get $G = 0$. So it suffices the cases that (30) holds. Replacing z by x and u by $D(x)y$ in (30) and using (30), we arrive at

$$D^2(x)[y, G(D(x))] + D(x)D^2(x)G(y) = 0 \tag{32}$$

for all $x, y \in R$. Substituting $yD(x)$ for y in (32), we have by the assumption and (32)

$$D(x)D^2(x)yG(D(x)) = 0 \tag{33}$$

for all $x, y \in R$. Now also, by the same method as above, and starting from (33), we have

$$D(z)D^2(z)yG(D(x)) = 0$$

for all $x, y \in R$. Since R is prime, $D(z)D^2(z) = 0$ or $G(D(x)) = 0$. By [9, Lemma] and Posner's first theorem, $D = 0$ or $G = 0$. This completes the proof. □

Also we obtain the following result.

THEOREM 2.4. *Let R be a noncommutative 3!-torsion free prime ring. Suppose that there exist Jordan derivations $D, G : R \to R$ such that*

$$G(x)\langle D(x), x\rangle = 0, \ D(x)G(D(x)) = 0$$

for all $x \in R$. Then we have $D = 0$ or $G = 0$.

Proof. It is the similar argument as in the proof to Theorem 2.3. □

3. Jordan derivations on a noncommutative Banach algebras

Now, let us prove the following results on a noncommutative Banach algebra in using the preceding algebraic result.

THEOREM 3.1. *Let A be a noncommutative Banach algebras. Suppose that there exist continuous linear Jordan derivations $D, G : A \to A$ such that*

$$\langle D(x), x\rangle G(x) \in rad(A), \ G(D(x))D(x) \in rad(A)$$

for all $x \in A$. Then we have either $D(A) \subseteq rad(A)$ or $G(A) \subseteq rad(A)$.

Proof. Let P be a primitive ideal of A. Since D, G are continuous, by [8, Lemma 3.2], we have $D(P) \subseteq P$ and $G(P) \subseteq P$. Then we can define Jordan derivations D_P, G_P on A/P by

$$D_P(x + P) = D(x) + P, \ G_P(x + P) = G(x) + P$$

for all $x \in A$. The factor algebra A/P is prime and semisimple, since P is a primitive ideal. By [1, Theorem 1], it is obvious that D_P, G_P are derivations on a prime Banach algebra A/P. Johnson and Sincliar [5] have proved that every derivation on a semisimple Banach algebra is continuous. Combining this result with Singer-Wermer theorem, we obtain that there are no nonzero derivations on a commutative Banach algebra. Hence in case A/P is commutative, we have $D_P = 0$ and $G_P = 0$. It remains to show that $D_P = 0$ and $G_P = 0$ in the case when A/P is noncommutative. Note that the intersection of all primitive ideals is the radical. The assumption of the theorem

$$\langle D(x), x\rangle G(x) \in rad(A),\ G(D(x))D(x) \in rad(A)\ (x \in A)$$

gives

$$\langle D_P(x+P), x+P\rangle G_P(x+P)$$
$$= P,\ G_P(D_P(x+P))D_P(x+P) = P\ (x \in A).$$

All the assumptions of Theorem 2.3 are fulfilled. Thus we have $D_P = 0$ or $G_P = 0$. Hence we see that $D(A) \subseteq P$ or $G(A) \subseteq P$, since J is a primitive ideal. Therefore since P was arbitrary, $D(A) \subseteq rad(A)$ or $G(A) \subseteq rad(A)$. □

THEOREM 3.2. *Let A be a noncommutative semisimple Banach algebras. Suppose that there exist linear Jordan derivations $D, G : A \to A$ such that*

$$\langle D(x), x\rangle G(x) = 0,\ G(D(x))D(x) = 0$$

for all $x \in A$. Then we have either $D = 0$ or $G = 0$.

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