EXISTENCE RESULTS FOR SEMILINEAR DIFFERENTIAL EQUATIONS

R. SAKTHIVEL, Q. H. CHOI, AND T. JUNG

ABSTRACT. In this paper we prove the existence of mild solutions for semilinear differential equations in a Banach space. The results are obtained by using the semigroup theory and the Schaefer fixed point theorem. An example is provided to illustrate the theory.

1. Introduction

The problem of existence of mild solutions for differential and integrodifferential equations in abstract spaces have been studied by several authors [2, 6, 8, 10, 12, 14]. In [4] Frigon and Regan studied the existence of solutions of differential equations by using the Krasnoselski fixed point theorem. Lin and Liu [7] studied the existence of mild solution for autonomous integrodifferential equations in a Banach space by using the resolvent operators and the Banach fixed point theorem. Balachandran and Ilamaran [1] derived the existence of mild and strong solutions for semilinear evolution differential equations in Banach spaces. Dhakne and Pachpatte [3] studied the mild solutions for the functional integrodifferential equations in Banach spaces by using the Schaefer fixed point theorem. In this paper we study the existence of mild solutions for semilinear differential equations in a Banach space by using the semigroup theory and the Schaefer fixed point theorem.

Consider the semilinear differential equation of the form

\[ \frac{dx(t)}{dt} + g(t, x(t)) = A x(t) + f(t, x(t)), \quad t \in J = [0, b], \]

\[ x(0) = x_0, \]

\[ \text{Received December 16, 2001.} \]

\[ \text{2000 Mathematics Subject Classification: 34A05, 34G20, 34A09.} \]

\[ \text{Key words and phrases: semilinear differential equation, mild solution, Schaefer fixed point theorem.} \]

\[ \text{This work was supported by the Korea Research Foundation KRF-2001-015-DP0024.} \]
where the state $x(\cdot)$ takes values in a Banach space $X$ with the norm $\|\cdot\|$, $A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t), t \geq 0$ in the Banach space $X$, $f : J \times X \rightarrow X$, and $g : J \times X \rightarrow X$ are continuous functions. The case $g = 0$ has an extensive literature. The books of Pazy [9], Goldstein [5] and the references contained therein give good account of the most important results.

2. Preliminaries

In order to define the concept of mild solution for (1), by comparison with the abstract Cauchy problem

$$x'(t) = Ax(t) + f(t),$$

whose properties are well known [9], we associate problem (1) to the integral equation

(2) $x(t) = T(t)[x_0 + g(0, x_0)] - g(t, x(t)) - \int_0^t AT(t - s)g(s, x(s))ds$

$$+ \int_0^t T(t - s)f(s, x(s))ds.$$

**Definition.** A function $x \in C([0, b], X)$ is a mild solution of the Cauchy problem (1) if the following holds: $x(0) = x_0$; for each $0 \leq t < b$ and $s \in [0, t)$, the function $AT(t - s)g(s, x(s))$ is integrable and the integral equation (2) is satisfied.

We need the following fixed point theorem due to Schaefer [11]

**Schaefer’s Theorem.** Let $E$ be a normed linear space. Let $F : E \rightarrow E$ be a completely continuous operator, i.e., it is continuous and the image of any bounded set is contained in a compact set, and let

$$\zeta(F) = x \in E; x = \lambda Fx \quad \text{for some} \quad 0 < \lambda < 1.$$

Then either $\zeta(F)$ is unbounded or $F$ has a fixed point.

We assume the following hypotheses:

(i) $A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in $X$ such that

$$\|T(t)\| \leq M_1, \quad \text{for some} \quad M_1 \geq 1 \quad \text{and} \quad \|AT(t)\| \leq M_2, \quad M_2 \geq 0.$$

(ii) For each $t \in J$ the function $f(t, \cdot) : X \rightarrow X$ is continuous, and for each $x \in X$ the function $f(\cdot, x) : J \rightarrow X$ is strongly measurable.
(iii) For every positive integer \(k\) there exists \(\alpha_k \in L^1(0,b)\) such that 
\[
\sup_{\|z\| \leq k} \|f(t,z)\| \leq \alpha_k(t), \text{ for } t \in J \text{ a.e.}
\]

(iv) The function \(g\) is completely continuous and uniformly bounded, there exists a constant \(M_3 > 0\) such that 
\[
\|g(t,x)\| \leq M_3, \quad t \in J, \quad x \in X.
\]

(v) There exists an integrable function \(m : [0,b] \to [0,\infty)\) such that 
\[
\|f(t,x)\| \leq m(t)\Omega(\|x\|), \quad 0 \leq t \leq b, \; x \in X,
\]
where \(\Omega : [0,\infty) \to (0,\infty)\) is a continuous nondecreasing function.

(vi) 
\[
M_1 \int_0^b m(s) ds < \int_c^\infty \frac{ds}{\Omega(s)},
\]
where \(c = M_1[\|x_0\| + M_3] + M_3 + M_2M_3b\).

3. Existence result

**Theorem.** If the assumptions (i) to (vi) are satisfied, then the problem (1) has a mild solution on \([0,b]\).

**Proof.** Consider the space \(C = C(J,X)\), the Banach space of all continuous functions from \(J\) into \(X\) with sup norm.

To prove the existence of a mild solution of (1) we apply Schaefer’s theorem. First we obtain a priori bounds for the solutions of the following problem, as in [8]

\[
(3) \quad \frac{d}{dt}[x(t)+\lambda g(t,x(t))]=Ax(t)+\lambda f(t,x(t)), \quad \lambda \in (0,1), \quad t \in J = [0,b], \\
\quad x(0) = \lambda x_0.
\]

Let \(x\) be a mild solution of the problem (3). From 
\[
x(t) = \lambda T(t)[x_0+g(0,x_0)] - \lambda g(t,x(t)) - \lambda \int_0^t AT(t-s)g(s,x(s))ds \\
\quad \quad \quad + \lambda \int_0^t T(t-s)f(s,x(s))ds.
\]

From the assumptions, we have 
\[
\|x(t)\| \leq M_1[\|x_0\| + M_3] + M_3 + M_2M_3b \\
\quad \quad \quad + M_1 \int_0^t m(s)\Omega(\|x(s)\|)ds.
\]
Denoting by \( u(t) \) the right-hand side of the above inequality we have
\[
c = u(0) = M_1 \|x_0\| + M_3 + M_2 M_3 b, \quad \|x(t)\| \leq u(t), \quad 0 \leq t \leq b
\]
and
\[
u'(t) = M_1 m(t) \omega(\|x(t)\|) \\
\leq M_1 m(t) \omega(u(t)).
\]
This implies
\[
\int_{u(0)}^{u(t)} \frac{ds}{\Omega(s)} \leq M_1 \int_0^b m(s) ds < \int_c^\infty \frac{ds}{\Omega(s)}, \quad 0 \leq t \leq b.
\]
This inequality implies that \( u(t) < \infty \). So, there is a constant \( K \) such that \( u(t) \leq K, t \in [0, b] \) and hence \( \|x(t)\| \leq K, t \in [0, b] \), where \( K \) depends only on \( b \) and on the functions \( m \) and \( \Omega \).

In the second step we must prove that the operator \( F : C \rightarrow C \)
defined by
\[
(Fx)(t) = T(t)[x_0 + g(0, x_0)] - g(t, x(t)) - \int_0^t AT(t-s)g(s, x(s)) ds \\
+ \int_0^t T(t-s)f(s, x(s)) ds
\]
is a completely continuous operator.

Let \( B_k = \{ x \in C : \|x\| \leq k \} \) for some \( k \geq 1 \). We first show that \( F \) maps \( B_k \) into an equicontinuous family. Let \( x \in B_k \) and \( t_1, t_2 \in J \). Then if 0 < \( t_1 < t_2 \leq b \),
\[
\| (Fx)(t_1) - (Fx)(t_2) \| \\
\leq \| T(t_1) - T(t_2) \| \| x_0 + g(0, x_0) \| + \| g(t_1, x(t_1)) - g(t_2, x(t_2)) \| \\
+ \| \int_0^{t_1} AT(t_1-s) - T(t_2-s)g(s, x(s)) ds \| \\
+ \| \int_{t_1}^{t_2} AT(t_2-s)g(s, x(s)) ds \| \\
+ \| \int_0^{t_1} [T(t_1-s) - T(t_2-s)]f(s, x(s)) ds \| \\
+ \| \int_{t_1}^{t_2} T(t_2-s)f(s, x(s)) ds \|
\]
\[
\leq \|T(t_1) - T(t_2)\| \|x_0 + g(0, x_0)\| + \|g(t_1, x(t_1)) - g(t_2, x(t_2))\|
\]
\[
+ \int_0^{t_1} \|A[T(t_1 - s) - T(t_2 - s)]\| M_3 ds
\]
\[
+ \int_{t_1}^{t_2} \|AT(t_2 - s)\| M_3 ds
\]
\[
+ \int_0^{t_1} \|T(t_1 - s) - T(t_2 - s)\| \alpha_k(s) ds
\]
\[
+ \int_{t_1}^{t_2} \|T(t_2 - s)\| \alpha_k(s) ds.
\]

The right hand side is independent of \(x \in B_k\) and tends to zero as \(t_2 - t_1 \to 0\), since \(g\) is completely continuous and the compactness of \(T(t)\) for \(t > 0\) implies the continuity in the uniform operator topology. Thus \(F\) maps \(B_k\) into an equicontinuous family of functions.

Notice that we considered here only the case \(0 < t_1 < t_2\), since the other cases \(t_1 < t_2 < 0\) or \(t_1 < 0 < t_2\) are very similar.

It is easy to see that the family \(FB_k\) is uniformly bounded. Next, we show \(FB_k\) is compact. Since we have shown \(FB_k\) is equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that \(F\) maps \(B_k\) into a precompact set in \(X\).

Let \(0 < t \leq b\) be fixed and \(\epsilon\) a real number satisfying \(0 < \epsilon < t\). For \(x \in B_k\) we define

\[
(F, x)(t) = T(t)[x_0 + g(0, x_0)] - g(t, x(t))
\]
\[
- \int_0^{t-\epsilon} AT(t - s)g(s, x(s)) ds
\]
\[
+ \int_0^{t-\epsilon} T(t - s)f(s, x(s)) ds
\]
\[
= T(t)[x_0 + g(0, x_0)] - g(t, x(t))
\]
\[
- T(\epsilon) \int_0^{t-\epsilon} AT(t - s - \epsilon)g(s, x(s)) ds
\]
\[
+ T(\epsilon) \int_0^{t-\epsilon} T(t - s - \epsilon)f(s, x(s)) ds.
\]

Since \(T(t)\) is a compact operator, the set \(Y_\epsilon(t) = \{(F, x)(t) : x \in B_k\}\) is precompact in \(X\) for every \(\epsilon, 0 < \epsilon < t\). Moreover for every \(x \in B_k\) we
have
\[
\|(F_x)(t) - (F_x)(t)\| \leq \int_{t-\epsilon}^{t} \|AT(t-s)g(s, x(s))\| ds \\
+ \int_{t-\epsilon}^{t} \|T(t-s)f(s, x(s))\| ds \\
\leq \int_{t-\epsilon}^{t} \|AT(t-s)\| M_3 ds \\
+ \int_{t-\epsilon}^{t} \|T(t-s)\| \alpha_k(s) ds.
\]

Therefore there are precompact sets arbitrarily close to the set \{(F_x)(t) : x \in B_k\}. Hence the set \{(F_x)(t) : x \in B_k\} is precompact in X.

It remains to show that \( F : C \to C \) is continuous. Let \( \{x_n\}_{n=0}^{\infty} \subseteq C \) with \( x_n \to x \) in C. Then there is an integer \( r \) such that \( \|x_n(t)\| \leq r \) for all \( n \) and \( t \in J \), so \( x_n \in B_r \) and \( x \in B_r \). By (ii) \( f(t, x_n(t)) \to f(t, x(t)) \) for each \( t \in J \) and since \( \|f(t, x_n(t)) - f(t, x(t))\| \leq 2\alpha_r(t) \), and also \( g \) is completely continuous, we have by dominated convergence theorem

\[
\|F x_n - F x\| = \sup_{t \in J} \|g(t, x_n(t)) - g(t, x(t))\| \\
+ \int_{0}^{t} AT(t-s)\|g(s, x_n(s)) - g(s, x(s))\| ds \\
+ \int_{0}^{t} T(t-s)\|f(s, x_n(s)) - f(s, x(s))\| ds \\
\leq \|g(t, x_n(t)) - g(t, x(t))\| \\
+ \int_{0}^{b} \|AT(t-s)\|\|g(s, x_n(s)) - g(s, x(s))\| ds \\
+ \int_{0}^{b} \|T(t-s)\|\|f(s, x_n(s)) - f(s, x(s))\| ds \\
\to 0 \text{ as } n \to \infty.
\]

Thus \( F \) is continuous. This completes the proof that \( F \) is completely continuous.

Finally the set \( \zeta(F) = \{x \in C : x = \lambda Fx, \lambda \in (0, 1)\} \) is bounded, as we proved in the first step. Consequently by Schaefer's theorem the operator \( F \) has a fixed point in \( C \). This means that the problem (1) has a mild solution on \( J \). \( \square \)
4. Example

Consider the following partial differential equation of the form

$$\frac{\partial}{\partial t}[z(t,y) + \mu_1(t,z(t,y))] = \frac{\partial^2}{\partial y^2}z(t,y) + \mu_2(t,z(t,y)),$$

$$z(t,0) = z(t,1) = 0,$$

$$z(0,y) = z_0(y),\ 0 < y < 1,\ t \in J = [0,1].$$

Take $X = L^2(J)$ and let $f(t,w)(y) = \mu_2(t,w(y))$ and $g(t,w)(y) = \mu_1(t,w(y)).$

Let $A : X \to X$ be defined by $Aw = w''$ with domain $D(A) = \{w \in X : w, w' \text{ are absolutely continuous}, \ w'' \in X, \ w(0) = w(1) = 0\}.$

Then

$$Aw = \sum_{n=1}^{\infty} (-n^2)(w,w_n)w_n, \ w \in D(A),$$

where $w_n(y) = \sqrt{2} \sin ny \ (n = 1, 2, 3,...)$ is the orthogonal set of eigenvectors of $A.$

It is well known that $A$ is the infinitesimal generator of an analytic semigroup $T(t), \ t \geq 0$ in $X$ and is given by $[13]$

$$T(t)w = \sum_{n=1}^{\infty} \exp(-n^2t)(w,w_n)w_n, \ w \in X.$$  

Since the analytic semigroup $T(t)$ is compact, there exist constants $N \geq 1$ and $N_1 > 0$ such that $\|T(t)\| \leq N$ and $\|AT(t)\| \leq N_1$ for each $t \geq 0.$

Further the function $\mu_1 : J \times X \to X$ is completely continuous and there exists a constant $k_1 > 0$ such that

$$\|\mu_1(t,w)\| \leq k_1,$$

and also there exists an integrable function $l : J \to [0,\infty)$ such that

$$\|\mu_2(t,w)\| \leq l(t)\Omega_1(\|w\|),$$

where $\Omega_1 : [0,\infty) \to (0,\infty)$ is continuous and nondecreasing and

$$N \int_0^1 l(s)ds < \int_c^{\infty} \frac{ds}{\Omega_1(s)},$$

where $c = N[\|z_0\| + k_1] + k_1 + k_1N_1.$

Further, all the conditions of the above theorem are satisfied. Hence, the equation (4) has a mild solution on $J.$
References


R. Sakthivel and Q. H. Choi  
Department of Mathematics  
Inha University  
Inchon 402-751, Korea

T. Jung  
Department of Mathematics  
Kunsan National University  
Kunsan 573-701, Korea  
E-mail: krsakthivel@yahoo.com