

## ON THE RANDOM $n \times m$ ASSIGNMENT PROBLEM

SUNGCHUL LEE\* AND ZHONGGEN SU†

ABSTRACT. Consider the random  $n \times m$  assignment problem with  $m \geq n$ . Let  $u_{i,j}$  and  $t_{i,j}$  be iid uniform random variables on  $[0, 1]$  and exponential random variables with mean 1, respectively, and let  $U_{n,m}$  and  $T_{n,m}$  denote the optimal assignment costs corresponding to  $u_{i,j}$  and  $t_{i,j}$ . In this paper we first give a comparison result about the discrepancy  $ET_{n,m} - EU_{n,m}$ . Using this comparison result with a known lower bound for  $\text{Var}(T_{n,m})$  we obtain a lower bound for  $\text{Var}(U_{n,m})$ . Finally, we study the way that  $EU_{n,m}$  and  $ET_{n,m}$  vary as  $m$  does. It turns out that only when  $m - n$  is large enough, the cost decreases significantly.

### 1. Introduction and main results

Suppose there are  $n$  jobs,  $m$  machines with  $m \geq n$ , and an  $n \times m$  non-negative matrix  $(a_{i,j})$  representing the cost of job  $i$  done using machine  $j$ . An assignment  $\pi$  is a one to one map from  $I = \{1, 2, \dots, n\}$  to  $J = \{1, 2, \dots, m\}$ , indicating that job  $i$  is assigned to machine  $\pi(i)$ . The  $n \times m$  assignment problem is to find  $\min_{\pi} \sum_{i=1}^n a_{i,\pi(i)}$ . In this paper we are mainly interested in the stochastic version of the problem, where the costs  $a_{i,j}$  are independent identically distributed (iid) random variables drawn from a given distribution. The uniform distribution on the unit interval  $[0, 1]$  is the most studied distribution in the literature. However, because of the memoryless property of the exponential distribution one can simplify many calculations under the exponential distribution.

---

Received December 24, 2001.

2000 Mathematics Subject Classification: Primary 05C70, 60C05, 68W40; Secondary 90C27.

Key words and phrases:  $n \times m$  assignment problem, bipartite matching, optimal assignment.

\*This work was supported by the BK21 project of the Department of Mathematics, Yonsei University, the interdisciplinary research program of KOSEF 1999-2-103-001-5 and Com<sup>2</sup>MaC in POSTECH.

†This work was supported by the BK21 project of the Department of Mathematics, Yonsei University, and by NSFC 10071072.

Therefore, the exponential distribution is a good alternative to study. See Aldous [1], [2], Alm and Sorkin [3], Coppersmith and Sorkin [4] for various issues and intriguing results.

Let  $u_{i,j}$  and  $t_{i,j}$  be iid uniform random variables on  $[0, 1]$  and iid exponential random variables with mean 1, and let  $U_{n,m}$  and  $T_{n,m}$  denote the corresponding optimal assignment costs respectively, i.e.,

$$U_{n,m} = \min_{\pi} \sum_{i=1}^n u_{i,\pi(i)}, \quad T_{n,m} = \min_{\pi} \sum_{i=1}^n t_{i,\pi(i)}.$$

If  $m = n$ , we simply write  $U_n$  and  $T_n$  for  $U_{n,m}$  and  $T_{n,m}$ .

A number of authors have investigated the limit behaviours and the exact distributions of  $U_{n,m}$  and  $T_{n,m}$ : For the case  $m = n$ , Aldous [1] first established the existence of the limit of  $ET_n$  and only recently he [2] identifies the limit as  $\zeta(2) = \pi^2/6$ . A tail bound for  $U_n$  is also available by Talagrand [7]. See also Frieze and Sorkin [5] and Coppersmith and Sorkin [4] for interesting results. For the case  $m = \lceil(1+\alpha)n\rceil$ , Talagrand [8] introduces the statistical physics approach to the random  $n \times m$  assignment model and obtains a good description of  $U_{n,m}$  at least at high temperature. In this regards Alm and Sorkin [3] made an intriguing conjecture about  $ET_{n,m}$ , generalizing Parisi's earlier conjecture about  $ET_n$ . Note that this conjecture is only for the exponential distribution and does not hold in the uniform case.

In many studies one obtains some properties under either the uniform or the exponential case. Since only the density at 0 ultimately matters, the properties are automatically valid for the other case. One of our purpose of this paper is to make the discrepancy between  $EU_{n,m}$  and  $ET_{n,m}$  explicit. Here is the result.

**THEOREM 1.** *Let  $m = m(n)$  be a function of  $n$  with  $m \geq n$ . If  $m = O(n)$ , then there are two strictly positive but finite constants  $C_1$  and  $C_2$  such that*

$$C_1 \frac{1}{n} \leq ET_{n,m} - EU_{n,m} \leq C_2 \frac{\log n}{n}.$$

As a direct application of the above comparison theorem, we obtain the lower bound for  $\text{Var}(U_{n,m})$  from the known lower bound for  $\text{Var}(T_{n,m})$ . Note that the below lower bound for  $\text{Var}(U_{n,m})$  is much stronger than the lower bound that we [6] obtained using the direct conditioning argument.

THEOREM 2. Let  $m = m(n)$  be a function of  $n$  with  $m \geq n$ . If  $m = O(n)$ , then there is a strictly positive but finite constant  $C_3$  such that

$$\text{Var}(T_{n,m}) \geq \frac{n}{m^2}, \quad \text{Var}(U_{n,m}) \geq C_3 \frac{1}{n}.$$

In this paper we also investigate how  $EU_{n,m}$  and  $ET_{n,m}$  change as  $m$  does. Intuitively, increasing  $m$  is equivalent to offering more possibilities of assigning each job to machines of smaller cost and so the total cost will decrease. However, our study shows that only when the amount  $m - n$  of spare machines is large enough, the assignment cost will be saved significantly. Our main result in this direction is the following.

THEOREM 3. Let  $m = m(n)$  be a function of  $n$  with  $m \geq n$ .

(i) If  $m/n \rightarrow \infty$ , then  $EU_{n,m} \rightarrow 0$ .

(ii) If  $(m - n) \log n/n \rightarrow 0$ , then  $EU_{n,m} \rightarrow \zeta(2)$ .

(iii) If  $m = [(1 + \alpha)n]$  for some fixed  $0 < \alpha < \infty$ , then  $\limsup_{n \rightarrow \infty} EU_{n,m} < \zeta(2)$ .

The same results hold for  $T_{n,m}$ .

REMARK. Regarding (ii), it would be interesting if one could prove  $EU_{n,m} \rightarrow \zeta(2)$  under the condition  $(m - n)/n \rightarrow 0$ . Indeed, if  $U_n$  has a Gaussian tail, i.e., for any  $\varepsilon > 0$

$$P(|U_n - EU_n| > \varepsilon n^{-1/2}) \leq C_4 e^{-C_5 \varepsilon^2},$$

then a simple calculation shows that  $EU_{n,m} - EU_n \rightarrow 0$ . Hence, by Aldous' identification of the limit of  $ET_n$  and by Theorem 1 we would have  $EU_{n,m} \rightarrow \zeta(2)$  under the condition  $(m - n)/n \rightarrow 0$ . Regarding (iii), we don't know whether the limit exists under the condition  $m = [(1 + \alpha)n]$ .

The proofs are given in Section 2. In the proof of Theorem 1 and 2, we use the following well-known coupling that for the cumulative distribution function  $F(x)$  of the exponential distribution with mean 1,  $\{F(t_{i,j})\}$  has the same distribution as  $\{u_{i,j}\}$  and  $\{F^{-1}(u_{i,j})\}$  has the same distribution as  $\{t_{i,j}\}$ . We will approximate both  $F(x)$  and  $F^{-1}(x)$  by Taylor's expansion. In the proof of Theorem 3, we compare any given assignment cost with the optimal assignment cost.

In this paper, there are lots of strictly positive but finite constants whose specific values are not of interest. We denote them by  $C_i$ .

## 2. Proofs

We start with a lemma of Frieze and Sorkin [5] which tells that the maximum edge cost used in an optimal assignment is of order  $\log n/n$ .

LEMMA 1. Denote by  $U_{\max}$  and  $T_{\max}$  the maximum edge cost used in an optimal assignment for the uniform and the exponential cases, respectively. If  $m = O(n)$ , then

$$P(U_{\max} \geq C_6 \frac{\log n}{n}) \leq \frac{1}{n^8}, \quad P(T_{\max} \geq C_6 \frac{\log n}{n}) \leq \frac{1}{n^8}.$$

PROOF. This lemma was first obtained by Frieze and Sorkin [5] in the case of the uniform cost and  $m = n$ . Since  $(1 - e^{-x})/x \rightarrow 1$  as  $x \rightarrow 0$ , the same argument is valid for the exponential case. Also, their proof can be easily extended to the case  $m = O(n)$ . The details are omitted.  $\square$

PROOF OF THEOREM 1. Let's start with the upper bound. Note that for the cumulative distribution function  $F(x)$  of the exponential distribution with mean 1,  $\{F^{-1}(u_{i,j})\}$  has the same distribution as  $\{t_{i,j}\}$ , where  $F^{-1}(x) = -\ln(1-x)$ . Thus, we have

$$ET_{n,m} - EU_{n,m} = E\left(\min_{\pi} \sum_{i=1}^n F^{-1}(u_{i,\pi(i)}) - \min_{\pi} \sum_{i=1}^n u_{i,\pi(i)}\right).$$

Let  $\Omega_n$  be

$$\Omega_n = \left\{U_{\max} \leq C_6 \log n/n\right\},$$

where  $C_6$  is given in Lemma 1, and let  $\sigma$  be an optimal assignment for  $u_{i,j}$ . Since for  $0 \leq x \leq 0.43$ ,  $-\ln(1-x) - x \leq x^2/2 + x^3/2 \leq x^2$ , we

have then for large enough  $n$  with  $C_6 \log n/n \leq 0.43$

$$\begin{aligned}
 & E\left(\min_{\pi} \sum_{i=1}^n F^{-1}(u_{i,\pi(i)}) - \min_{\pi} \sum_{i=1}^n u_{i,\pi(i)}\right) \mathbf{1}(\Omega_n) \\
 & \leq E \sum_{i=1}^n \left(F^{-1}(u_{i,\sigma(i)}) - u_{i,\sigma(i)}\right) \mathbf{1}(\Omega_n) \\
 (2.1) \quad & \leq E \sum_{i=1}^n u_{i,\sigma(i)}^2 \mathbf{1}(\Omega_n) \\
 & \leq \frac{C_6 \log n}{n} EU_{n,m} \\
 & \leq C_7 \frac{\log n}{n},
 \end{aligned}$$

where the last inequality follows from  $\sup_n EU_{n,m} \leq \sup_n EU_n < \infty$ .

On the other hand, we have by Lemma 1

$$\begin{aligned}
 & E\left(\min_{\pi} \sum_{i=1}^n F^{-1}(u_{i,\pi(i)}) - \min_{\pi} \sum_{i=1}^n u_{i,\pi(i)}\right) \mathbf{1}(\Omega_n^c) \\
 & \leq E\left(\min_{\pi} \sum_{i=1}^n F^{-1}(u_{i,\pi(i)})\right) \mathbf{1}(\Omega_n^c) \\
 (2.2) \quad & \leq E\left(\sum_{i=1}^n F^{-1}(u_{i,i})\right) \mathbf{1}(\Omega_n^c) \\
 & \leq nEF^{-1}(u_{1,1}) \mathbf{1}(\Omega_n^c) \\
 & \leq n(E(F^{-1}(u_{1,1}))^2)^{1/2} P(\Omega_n^c)^{1/2} \\
 & \leq \frac{\sqrt{2}}{n^3}.
 \end{aligned}$$

Combining (2.1) and (2.2), we have the upper bound.

For the lower bound, note that  $\{F(t_{i,j})\}$  has the same distribution as  $\{u_{i,j}\}$ . Thus, we have

$$ET_{n,m} - EU_{n,m} = E\left(\min_{\pi} \sum_{i=1}^n t_{i,\pi(i)} - \min_{\pi} \sum_{i=1}^n F(t_{i,\pi(i)})\right).$$

Let  $\Lambda_n$  be

$$\Lambda_n = \left\{T_{\max} \leq C_6 \log n/n\right\},$$

where  $C_6$  is given in Lemma 1, and let  $\tau$  be an optimal assignment for  $t_{i,j}$ . Since for any  $x$ ,  $x - (1 - e^{-x}) \geq x^2/2 - x^3/6$ , we have then

$$\begin{aligned}
 E\left(\min_{\pi} \sum_{i=1}^n t_{i,\pi(i)} - \min_{\pi} \sum_{i=1}^n F(t_{i,\pi(i)})\right) &\geq E \sum_{i=1}^n \left(t_{i,\tau(i)} - F(t_{i,\tau(i)})\right) \\
 (2.3) \qquad \qquad \qquad &\geq E \sum_{i=1}^n \left(\frac{t_{i,\tau(i)}^2}{2} - \frac{t_{i,\tau(i)}^3}{6}\right) \\
 &= \frac{n}{2} E t_{1,\tau(1)}^2 - \frac{n}{6} E t_{1,\tau(1)}^3.
 \end{aligned}$$

It is easy to see that

$$(2.4) \qquad E t_{1,\tau(1)}^2 \geq E \min_{1 \leq j \leq m} t_{1,j}^2 = \frac{2}{m^2}.$$

Also, with  $\Lambda_n$  by Lemma 1

$$\begin{aligned}
 E t_{1,\tau(1)}^3 &= E t_{1,\tau(1)}^3 1(\Lambda_n) + E t_{1,\tau(1)}^3 1(\Lambda_n^c) \\
 &\leq \left(\frac{C_6 \log n}{n}\right) E t_{1,\tau(1)}^2 + E \sum_{1 \leq j \leq m} t_{1,j}^3 1(\Lambda_n^c) \\
 (2.5) \qquad &\leq \left(\frac{C_6 \log n}{n}\right) E t_{1,\tau(1)}^2 + m (E t_{1,1}^6)^{1/2} P(\Lambda_n^c)^{1/2} \\
 &\leq \left(\frac{C_6 \log n}{n}\right) E t_{1,\tau(1)}^2 + C_8 n^{-3}.
 \end{aligned}$$

Now, the lower bound follows from (2.3)-(2.5).  $\square$

REMARK. It seems that the uniform integrability of the sum of the squared edge costs in an optimal assignment is equivalent to  $ET_{n,m} - EU_{n,m} = O(n^{-1})$ . Indeed, with the notion in the above proof one can easily see that if  $\{nu_{1,\sigma(1)}\}$  is  $L_2$ -bounded, then

$$(2.6) \qquad ET_{n,m} - EU_{n,m} = O(n^{-1}).$$

Also, if (2.6) holds, then  $\{nt_{1,\tau(1)}\}$  is  $L_2$ -bounded. It is natural to expect that both  $\{nt_{1,\tau(1)}\}$  and  $\{nu_{1,\sigma(1)}\}$  are  $L_2$ -bounded but we have no justification.

PROOF OF THEOREM 2. The lower bound of  $\text{Var}(T_{n,m})$  in the case  $m = n$  was first given by Theorem 28 of Alm and Sorkin (1998). The memoryless property of the exponential distribution makes the calculation pretty elementary. Actually their proof with obvious modification is also valid for  $m > n$ . So, we skip the proof of  $\text{Var}(T_{n,m}) \geq n/m^2$ .

For the lower bound of  $\text{Var}(U_{n,m})$ , we use the comparison theorem, Theorem 1 as follows. Let  $T_{n,m}^* = \min_{\pi} \sum_{i=1}^n F^{-1}(u_{i,\pi(i)})$ . A similar calculation to (2.1) and (2.2) gives

$$(2.7) \quad E(T_{n,m}^* - U_{n,m})^2 \leq C_9 \frac{(\log n)^3}{n^2}.$$

Indeed, with  $\Omega_n = \{U_{\max} \leq C_6 \log n/n\}$  and  $\sigma$  an optimal assignment for  $u_{i,j}$  for large enough  $n$  with  $C_6 \log n/n \leq 0.43$  as we did in the proof of Theorem 1

$$(2.8) \quad \begin{aligned} & E\left(\min_{\pi} \sum_{i=1}^n F^{-1}(u_{i,\pi(i)}) - \min_{\pi} \sum_{i=1}^n u_{i,\pi(i)}\right)^2 \mathbf{1}(\Omega_n) \\ & \leq E\left(\sum_{i=1}^n u_{i,\sigma(i)}^2\right)^2 \mathbf{1}(\Omega_n) \\ & \leq \left(C_{10} \frac{(\log n)^2}{n}\right) E\left(\sum_{i=1}^n u_{i,\sigma(i)}^2\right) \mathbf{1}(\Omega_n) \\ & \leq C_{11} \frac{(\log n)^3}{n^2} \end{aligned}$$

and

$$(2.9) \quad \begin{aligned} & E\left(\min_{\pi} \sum_{i=1}^n F^{-1}(u_{i,\pi(i)}) - \min_{\pi} \sum_{i=1}^n u_{i,\pi(i)}\right)^2 \mathbf{1}(\Omega_n^c) \\ & \leq E\left(\sum_{i=1}^n F^{-1}(u_{i,i})\right)^2 \mathbf{1}(\Omega_n^c) \\ & \leq E\left(\sum_{i=1}^n (F^{-1}(u_{i,i}))^2 + \sum_{i \neq j} F^{-1}(u_{i,i})F^{-1}(u_{j,j})\right) \mathbf{1}(\Omega_n^c) \\ & \leq C_{12}n^{-3} + C_{13}n^2 E F^{-1}(u_{1,1})F^{-1}(u_{2,2}) \mathbf{1}(\Omega_n^c) \\ & \leq C_{12}n^{-3} + C_{13}n^2 (E(F^{-1}(u_{1,1})F^{-1}(u_{2,2}))^2)^{1/2} P(\Omega_n^c)^{1/2} \\ & \leq C_{14}n^{-2}. \end{aligned}$$

Therefore, (2.7) follows from (2.8) and (2.9).

Now, the Theorem follows from Theorem 1 and (2.7):

$$\begin{aligned}
& \left(\text{Var}(U_{n,m})\right)^{1/2} \\
&= \left(E\left(U_{n,m} - EU_{n,m}\right)^2\right)^{1/2} \\
&= \left(E\left(T_{n,m}^* + (U_{n,m} - T_{n,m}^*) - ET_{n,m}^* - (EU_{n,m} - ET_{n,m}^*)\right)^2\right)^{1/2} \\
&\geq \left(E\left(T_{n,m}^* - ET_{n,m}^*\right)^2\right)^{1/2} - \left(E\left(U_{n,m} - T_{n,m}^*\right)^2\right)^{1/2} \\
&\quad - \left|EU_{n,m} - ET_{n,m}^*\right| \\
&\geq \left(E\left(T_{n,m}^* - ET_{n,m}^*\right)^2\right)^{1/2} - 2\left(E\left(U_{n,m} - T_{n,m}^*\right)^2\right)^{1/2} \\
&\geq \frac{n^{1/2}}{m} - C_{15} \frac{(\log n)^{3/2}}{n} \\
&= C_{16} n^{-1/2}.
\end{aligned}$$

□

PROOF OF THEOREM 3. (i). Although the case  $m/n \rightarrow \infty$  is unrealistic, its simple analysis is to some extent instructive. Consider the cost  $H_{n,m}$  of the heuristic assignment  $\pi$  given by successively examining each job  $i$ ,  $1 \leq i \leq n$ , and making assignment to the free machine with minimal cost, i.e.,  $u_{i,\pi(i)} = \min\{u_{i,j} : j \neq \pi(k) \text{ for all } 1 \leq k \leq i-1\}$ . Since  $u_{i,j}$  is independent and uniformly distributed on  $[0, 1]$ , the  $i$ -th assignment costs the minimum of  $m - i + 1$  independent uniform random variables and hence the expected cost of the  $i$ -th assignment is exactly  $1/(m - i + 2)$ . Thus, we have

$$EU_{n,m} \leq EH_{n,m} = \sum_{i=1}^n \frac{1}{m - i + 2} \rightarrow 0.$$

By Theorem 1, we also have  $ET_{n,m} \rightarrow 0$ .

(ii). First, note that by definition  $EU_{n,m} \leq EU_n$ . Then, by Aldous [2] we have

$$\limsup_{n \rightarrow \infty} EU_{n,m} \leq \lim_{n \rightarrow \infty} EU_n = \zeta(2).$$

So, it suffices to show that  $\liminf_{n \rightarrow \infty} EU_{n,m} \geq \zeta(2)$ .

Let  $u'_{i,j} = \min\{u_{i,j}, C_6 \log n/n\}$  with  $C_6$  as in Lemma 1. Using  $u'_{i,j}$  we define  $U'_{n,m}$  and  $U'_n$  as we did  $U_{n,m}$  and  $U_n$  with  $u_{i,j}$ . Let  $Q$  be any



subset of  $J$  with  $|Q| = n$ , and for each such  $Q$  we define  $U'_n(Q)$  as

$$U'_n(Q) = \min_{\pi} \sum_{i=1}^n u'_{i,\pi(i)},$$

where the minimum is taken over all one to one maps from  $I$  onto  $Q$ . Then, one can easily see that  $EU'_{n,m} = \min_Q U'_n(Q)$  and that

$$\begin{aligned} (2.10) \quad EU_{n,m} &\geq EU'_{n,m} \\ &= E \min_Q U'_n(Q) \\ &\geq EU'_n - E \max_Q |U'_n(Q) - U'_n|. \end{aligned}$$

Since by Lemma 1,  $EU_n - EU'_n \leq nP(U_{\max} \geq C_6 \log n/n) \rightarrow 0$ , by Theorem 1 with Aldous' identification of the limit  $ET_n$  we have  $EU'_n \rightarrow \zeta(2)$ . Hence, by (2.10) it is enough to show that  $E \max_Q |U'_n(Q) - U'_n| \rightarrow 0$ .

Given any  $Q$ , set  $\Gamma = Q \cap \{1, 2, \dots, n\} \subset J$ . Consider the random assignment problem from  $\Gamma$  to  $I$ , and we let  $X'_n$  be the corresponding optimal cost with edge costs  $u'_{i,j}$ . Since  $Q$  and  $I$  have at most  $m - n$  distinct elements, we have

$$X'_n \leq U'_n(Q) \leq X'_n + C_6(m - n) \frac{\log n}{n}$$

$$X'_n \leq U'_n \leq X'_n + C_6(m - n) \frac{\log n}{n}.$$

Therefore,  $\max_Q |U'_n(Q) - U'_n| \leq C_6(m - n) \log n/n \rightarrow 0$  and hence we have  $EU_{n,m} \rightarrow \zeta(2)$ . By Theorem 1, we also have  $ET_{n,m} \rightarrow \zeta(2)$ .

(iii). Since  $EU_{n,m} \leq ET_{n,m}$ , it suffices to give the proof for  $T_{n,m}$ . Since the other cases are similar, we also focus on the case  $\alpha = 1$ . Let  $\tau$  be an optimal assignment for  $T_n$  as before. For  $0 < \lambda < 1$ , define

$$N(\lambda) = \left| \left\{ 1 \leq i \leq n : t_{i,\tau(i)} > \lambda Et_{i,\tau(i)} \right\} \right|.$$

By Theorem 2 of Aldous [2],  $nt_{1,\tau(1)}$  converges in distribution to a random variable  $Z$  with  $EZ = \zeta(2)$ . So,

$$\frac{EN(\lambda)}{n} = P\left(t_{1,\tau(1)} > \lambda Et_{1,\tau(1)}\right) \rightarrow P\left(Z > \lambda \zeta(2)\right).$$

Without loss of generality, suppose that  $t_{i,\tau(i)} > \lambda Et_{i,\tau(i)}$  for  $i = 1, 2, \dots, N(\lambda)$ . Let  $n_0 = [N(\lambda)/2]$  and we define

$$\begin{aligned} t_{1,j_1} &= \min_{j \in \{n+1, \dots, 2n\}} t_{1,j} \\ t_{2,j_2} &= \min_{j \in \{n+1, \dots, 2n\} \setminus \{j_1\}} t_{2,j} \\ &\dots \\ t_{n_0,j_{n_0}} &= \min_{j \in \{n+1, \dots, 2n\} \setminus \{j_1, \dots, j_{n_0-1}\}} t_{n_0,j}. \end{aligned}$$

Construct now an assignment  $\tau'$  on the basis of  $\tau$ ; for  $1 \leq i \leq n_0$ , let

$$\tau'(i) = \begin{cases} j_i & \text{if } t_{i,j_i} < \frac{\lambda}{2} Et_{i,\tau(i)} \\ \tau(i) & \text{if } t_{i,j_i} \geq \frac{\lambda}{2} Et_{i,\tau(i)} \end{cases}$$

and for  $n_0 + 1 < i \leq n$ , let  $\tau'(i) = \tau(i)$ . Obviously,  $T_{n,2n} \leq \sum_{i=1}^n t_{i,\tau'(i)}$ . Thus, we have

$$\begin{aligned} E(T_n - T_{n,2n}) &\geq E \sum_{i=1}^n t_{i,\tau(i)} - E \sum_{i=1}^n t_{i,\tau'(i)} \\ &= E \sum_{i=1}^{n_0} (t_{i,\tau(i)} - t_{i,j_i}) 1(t_{i,j_i} < \frac{\lambda}{2} Et_{i,\tau(i)}) \\ &\geq E \sum_{i=1}^{n_0} (\frac{\lambda}{2} Et_{i,\tau(i)}) 1(t_{i,j_i} < \frac{\lambda}{2} Et_{i,\tau(i)}). \end{aligned}$$

Since  $n_0$  and  $1(t_{i,j_i} < \frac{\lambda}{2} Et_{i,\tau(i)})$  are independent and since  $n_0 \leq n/2$ ,

$$\begin{aligned} E(T_n - T_{n,2n}) &\geq E \sum_{i=1}^{n_0} (\frac{\lambda}{2} Et_{i,\tau(i)}) 1(t_{i,j_i} < \frac{\lambda}{2} Et_{i,\tau(i)}) \\ (2.11) \quad &\geq \left(\frac{\lambda}{2} Et_{1,\tau(1)}\right) (En_0) \left(1 - \exp(-n\lambda Et_{1,\tau(1)}/4)\right) \\ &= \left(\frac{\lambda}{2} n Et_{1,\tau(1)}\right) \left(\frac{En_0}{n}\right) \left(1 - \exp(-n\lambda Et_{1,\tau(1)}/4)\right). \end{aligned}$$

Since  $En_0/n \rightarrow P(Z > \lambda\zeta(2))/2$  and since  $nEt_{1,\tau(1)} \rightarrow \zeta(2)$ , with

$$c = \frac{\lambda}{4} \zeta(2) P(Z > \lambda\zeta(2)) (1 - e^{-\lambda\zeta(2)/4})$$

we have by (2.11)

$$\limsup_{n \rightarrow \infty} ET_{n,2n} \leq (1 - c)\zeta(2).$$

□

## References

- [1] D. J. Aldous, *Asymptotics in the random assignment problem*. Probab. Theory Relat. Fields **93** (1992), 507–534.
- [2] ———, *The  $\zeta(2)$  limit in the random assignment problem*. Random Struct. Alg., to appear (2001).
- [3] S. E. Alm and G. B. Sorkin, *Exact expectations and distributions for the random assignment problem*. Preprint, (1999).
- [4] D. Coppersmith and G. B. Sorkin, *Constructive bounds and exact expectations for the random assignment problem*. Random Struct. Alg. **15** (1999), 113–144.
- [5] A. M. Frieze, and G. B. Sorkin, *The probabilistic relationship between the assignment and asymmetric traveling salesman problems*. Proceedings of SODA, (2001), 652–660.
- [6] Lee, S. and Su, Z. *On the fluctuation in the random assignment problem*. Comm. Korean Math. Soc., to appear (2001).
- [7] M. Talagrand, *Concentration of measure and isoperimetric inequalities in product spaces*. Publ. Math. IHES. **81** (1995), 73–205.
- [8] ———, *An assignment problem at high temperature*. Preprint, (2001).

Sungchul Lee  
Department of Mathematics  
Yonsei University  
Seoul 120-749, Korea  
*E-mail*: sungchul@mail.yonsei.ac.kr

Zhonggen Su  
Department of Mathematics  
Zhejiang University  
Hangzhou 310028, P. R. China  
*E-mail*: zgsu@mail.hz.zj.cn

and

Department of Mathematics  
Yonsei University  
Seoul 120-749, Korea  
*E-mail*: zgsu2001@yahoo.com