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GENERALIZED HERMITE INTERPOLATION AND SAMPLING THEOREM INVOLVING DERIVATIVES

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ABSTRACT. We derive the generalized Hermite interpolation by using the contour integral and extend the generalized Hermite interpolation to obtain the sampling expansion involving derivatives for band-limited functions f, that is, f is an entire function satisfying the following growth condition

$$|f(z)| \le A \exp(\sigma|y|) \text{ for some } A,\, \sigma > 0 \text{ and any } z = x + iy \in \mathbb{C}.$$

1. Introduction

If p_n is a polynomial of degree n and $z_0, z_1, \cdots, z_n$ are distinct points in $\mathbb{C}$, then p_n can be constructed with the values $p_n(z_i), i = 0, \cdots, n$ and polynomials independent of p_n. The polynomial $p_n(z)$ is called the *Lagrange interpolation* ([1]) and is given by

$$p_n(z) = \sum_{k=0}^{n} p_n(z_k) \frac{G_n(z)}{(z - z_k)G_n'(z_k)}, \tag{1.1}$$

where $G_n(z) = \prod_{i=0}^{n}(z - z_i)$. Let $\sigma > 0$ and let B_σ^p be the set of entire functions such that there exists $A > 0$ satisfying

$$|f(z)| \le A \exp(\sigma|y|) \quad \text{for any} \quad z = x + iy \in \mathbb{C} \tag{1.2}$$

and

$$f|_{\mathbb{R}} \in L^p(\mathbb{R}).$$

The function f in B_σ^p for some $1 \le p \le \infty$ is called a *band-limited function* with band-limit σ. The following inclusions (p.24, [11]) hold

$$B_1 \subseteq B_p \subseteq B_q \subseteq B_\infty, \qquad 1 \le p \le q \le \infty. \tag{1.3}$$

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The Lagrange interpolation can be extended so that we obtain celebrated Shannon's sampling theorem ([2], [11]) which is the following:

THEOREM 1.1. *For any* $f \in B^p_\sigma (1 \le p < \infty)$,

$$f(z) = \sum_{n=-\infty}^{\infty} f(k_n) \frac{G(z)}{(z-k_n)G'(k_n)}, \tag{1.4}$$

where $k_n = \frac{n\pi}{\sigma}, n \in \mathbb{Z}$,

$$G(z) = \frac{\sin \sigma z}{\sigma} = z \prod_{n=1}^{\infty} (1 - \frac{z^2}{k_n^2}) = \lim_{m \to \infty} \tilde{G}_m(z),$$

$$\tilde{G}_m(z) = z \prod_{n=1}^{m} (1 - \frac{z^2}{k_n^2})$$

and the series converges uniformly on any compact subsets of $\mathbb{C}$.

In the expansion (1.4), k_n's are called the *sampling points*, the sampling frequency or the Nyquist rate ([11]) is $\frac{\sigma}{\pi}$ and distance between two nearest sampling points is $\frac{\pi}{\sigma}$.

D. A. Linden [4] and M. D. Rawn [9] generalized Shannon's sampling theorem by expanding $f \in B^2_\sigma$ with values of f and its derivatives. M. D. Rawn [9] used the Riesz basis to obtain the sampling expansion of f. The aim of this paper is to derive a sampling expansion for $f \in B^p_\sigma (1 \le p \le \infty)$ in an easier way by using the contour integral and express f in terms of values of f and derivatives at sampling points. The sampling expansion of f in this paper is simpler than Linden's and Rawn's results. Moreover, we regard the sampling expansion of f as a generalization of generalized Hermite interpolation.

Let $p_{\bar{N}}(z)$ be a polynomial of degree $\bar{N}$, $z_1 \cdots, z_n$ be n distinct points, $\alpha_1, \cdots, \alpha_n$ be n positive integers and $\bar{N} = \alpha_1 + \cdots + \alpha_n - 1$. The polynomial $p_{\bar{N}}(z)$ can be constructed with the values $p^{(i)}_{\bar{N}}(z_j), i = 0, \cdots, \alpha_i - 1, j = 1, \cdots, n$, and some polynomials independent of $p_{\bar{N}}$. The polynomial $p_{\bar{N}}(z)$ is called the *generalized Hermite interpolation* ([1], [5]).

In this paper, using the contour integral we derive the generalized Hermite interpolation $p_{\bar{N}}(z)$ which is given by

$$p_{\bar{N}}(z) = \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} \sum_{k=0}^{\alpha_i-1-j} p^{(j)}_{\bar{N}}(z_i) \frac{w(z)}{j!k!} (z-z_i)^{-\alpha_i+j+k} \frac{d^k}{dz^k} \left[\frac{(z-z_i)^{\alpha_i}}{w(z)} \right]_{z=z_i}, \tag{1.5}$$

where $w(z) = \prod_{i=1}^{n}(z-z_i)^{\alpha_i}$. Note that the expression of the generalized Hermite interpolation in page 37 of [1] is incorrect. Clearly the Lagrange interpolation is the special case of the generalized Hermite interpolation. In fact, if $\alpha_i = 1, i = 1, \cdots, n$ and $\bar{N} = n-1$, then the generalized Hermite interpolation $p_{\bar{N}}$ in (1.5) equals the Lagrange interpolation $p_n(z)$ in (1.1).

We also extend the generalized Hermite interpolation to obtain the sampling expansion involving values of a function and its derivatives at sampling points for entire functions f satisfying

$$(1.6) \qquad (1+|x|)|f(z)| \le A\exp(\sigma|y|) \text{ for some } A>0,\ \sigma>0 \text{ and any } z = x+iy \in \mathbb{C}.$$

For $N \in \mathbb{N}$ and any entire function f satisfying the growth condition (1.6), we obtain the following sampling expansion

$$(1.7) \quad f(z) = \sum_{n\in\mathbb{Z}}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1-j} f^{(j)}(\beta_n)\frac{(\sin\sigma z/N)^N}{j!k!} \times (z-\beta_n)^{-N+j+k}\frac{d^k}{dz^k}\left[\left(\frac{z-\beta_n}{\sin\sigma z/N}\right)^N\right]_{z=\beta_n},$$

where $\beta_n = \frac{Nn\pi}{\sigma}, n \in \mathbb{Z}$ and the series converges uniformly on any compact subsets of $\mathbb{C}$. In the expansion (1.7) the set of sampling points is $\{\beta_n : n \in \mathbb{Z}\}$ and the distance between two nearest sampling points is $\frac{N\pi}{\sigma}$, whereas distance between two nearest sampling points in Shannon's sampling theorem is $\frac{\pi}{\sigma}$. However, we need the values of $f, f', \cdots, f^{(N-1)}$ at each sampling point. Note that when $N = 1$, the expansion (1.7) equals the expansion (1.4).

Moreover, we show that for any $f \in B_\sigma^p(1 \le p \le \infty)$ and $\sigma' > \sigma$, the following sampling expansion holds

$$(1.8) \quad f(z) = \sum_{n\in\mathbb{Z}}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1-j} f^{(j)}(\tilde{\beta}_n)\frac{(\sin\sigma' z/N)^N}{j!k!} \times (z-\tilde{\beta}_n)^{-N+j+k}\frac{d^k}{dz^k}\left[\left(\frac{z-\tilde{\beta}_n}{\sin\sigma' z/N}\right)^N\right]_{z=\tilde{\beta}_n},$$

where $\tilde{\beta}_n = \frac{Nn\pi}{\sigma'}, n \in \mathbb{Z}$ and the series converges uniformly on any compact subsets of $\mathbb{C}$.

2. Generalized Hermite interpolation

Let $p_{\bar{N}}$ be a polynomial. Then $p_{\bar{N}}$ can be expressed in terms of values of $p_{\bar{N}}$, its derivatives and some polynomials independent of $p_{\bar{N}}$.

THEOREM 2.1 (**Generalized Hermite Interpolation**). *Let $p_{\bar{N}}$ be a polynomial of degree $\bar{N}$ and let $z_1 \cdots, z_n$ be n distinct points in $\mathbb{C}$, $\alpha_1, \cdots, \alpha_n$ be n positive integers and $\bar{N} = \alpha_1 + \cdots + \alpha_n - 1$. Then the polynomial $p_{\bar{N}}(z)$is given by*

(2.1)

$$p_{\bar{N}}(z) = \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} \sum_{k=0}^{\alpha_i-1-j} p_{\bar{N}}^{(j)}(z_i) \frac{w(z)}{j!k!}(z-z_i)^{-\alpha_i+j+k} \frac{d^k}{dz^k}\left[\frac{(z-z_i)^{\alpha_i}}{w(z)}\right]_{z=z_i},$$

where $w(z) = \prod_{i=1}^{n}(z-z_i)^{\alpha_i}$. The polynomial $p_{\bar{N}}(z)$ is called the generalized Hermite interpolation.

PROOF. Let z be a complex number distinct from z_i, $i = 1, \cdots, n$. We define a function g on $\mathbb{C}$ by

$$g(\zeta) = \frac{p_{\bar{N}}(\zeta)}{(\zeta - z)w(\zeta)}.$$

The function g is a meromorphic function having a simple pole at z and poles of order α_i at $z_i, i = 1, \cdots, n$. Let C_n be a circular path centered at the origin with radius n. We denote by $\mathrm{Res}(g : \bar{z})$ the residue of g at $\bar{z}$. Since $|p_{\bar{N}}(\zeta)| = O(|\zeta|^{\bar{N}})$, $|w(\zeta)| = O(|\zeta|^{\bar{N}+1})$ and $|g(\zeta)| = O(|\zeta|^{-2})$ as $|\zeta| \to \infty$, we can easily see that

$$\lim_{n\to\infty} \int_{C_n} g(\zeta) d\zeta = 0$$

and

$$\mathrm{Res}(g : z) + \sum_{i=1}^{n} \mathrm{Res}(g : z_i) = 0. \tag{2.2}$$

The residue at each pole is the following:

$$\mathrm{Res}(g : z) = \frac{p_{\bar{N}}(z)}{w(z)}$$

and for $i = 1, \cdots, n$, by Leibniz formula

$$\begin{aligned}
\mathrm{Res}(g : z_i) &= \lim_{\zeta \to z_i} \frac{1}{(\alpha_i - 1)!} \frac{d^{\alpha_i - 1}}{d\zeta^{\alpha_i - 1}} [(\zeta - z_i)^{\alpha_i} g(\zeta)] \\
&= \lim_{\zeta \to z_i} \frac{(-1)}{(\alpha_i - 1)!} \frac{d^{\alpha_i - 1}}{d\zeta^{\alpha_i - 1}} \left[\frac{(\zeta - z_i)^{\alpha_i}}{w(\zeta)} \frac{p_{\bar{N}}(\zeta)}{(z - \zeta)} \right] \\
&= \lim_{\zeta \to z_i} \frac{(-1)}{(\alpha_i - 1)!} \sum_{j=0}^{\alpha_i - 1} \binom{\alpha_i - 1}{j} p_{\bar{N}}^{(j)}(\zeta) \\
&\quad \times \frac{d^{\alpha_i - 1 - j}}{d\zeta^{\alpha_i - 1 - j}} \left[\frac{(\zeta - z_i)^{\alpha_i}}{w(\zeta)} \frac{1}{(z - \zeta)} \right] \\
&= \lim_{\zeta \to z_i} \frac{(-1)}{(\alpha_i - 1)!} \sum_{j=0}^{\alpha_i - 1} \binom{\alpha_i - 1}{j} p_{\bar{N}}^{(j)}(\zeta) \sum_{k=0}^{\alpha_i - 1 - j} \binom{\alpha_i - 1 - j}{k} \\
&\quad \times (\alpha_i - 1 - j - k)!(z - \zeta)^{-\alpha_i + j + k} \frac{d^k}{d\zeta^k} \frac{(\zeta - z_i)^{\alpha_i}}{w(\zeta)} \\
&= \sum_{j=0}^{\alpha_i - 1} \sum_{k=0}^{\alpha_i - 1 - j} \frac{(-1)}{j!k!} p_{\bar{N}}^{(j)}(z_i)(z - z_i)^{-\alpha_i + j + k} \\
&\quad \times \frac{d^k}{d\zeta^k} \left[\frac{(\zeta - z_i)^{\alpha_i}}{w(\zeta)} \right]_{\zeta = z_i}.
\end{aligned}$$

In view of (2.2) we have that
(2.3)

$$p_{\bar{N}}(z) = \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} \sum_{k=0}^{\alpha_i - 1 - j} p_{\bar{N}}^{(j)}(z_i) \frac{w(z)}{j!k!} (z - z_i)^{-\alpha_i + j + k} \frac{d^k}{d\zeta^k} \left[\frac{(\zeta - z_i)^{\alpha_i}}{w(\zeta)} \right]_{\zeta = z_i}$$

for $z \neq z_i, i = 1, \cdots, n$. Since both sides of (2.3) are polynomials and the equality (2.3) holds for $z \neq z_i, i = 1, \cdots, n$, (2.3) holds for any $z \in \mathbb{C}$. □

The general form of the generalized Hermite interpolation is usually given in the following way.

COROLLARY 2.2. *Let $z_1 \cdots, z_n$ be n distinct points in $\mathbb{C}$, $\alpha_1, \cdots, \alpha_n$ be n positive integers and $\bar{N} = \alpha_1 + \cdots + \alpha_n - 1$. The polynomial $p_{\bar{N}}$ of* *degree $\bar{N}$ satisfying*

(2.4)

$$\begin{array}{llll} p_{\bar{N}}(z_1) = r_{10}, & p'_{\bar{N}}(z_1) = r_{11}, & \cdots & p_{\bar{N}}^{(\alpha_1-1)}(z_1) = r_{1\alpha_1-1} \\ \vdots & \vdots & \vdots & \vdots \\ p_{\bar{N}}(z_n) = r_{n0}, & p'_{\bar{N}}(z_n) = r_{n1}, & \cdots & p_{\bar{N}}^{(\alpha_n-1)}(z_n) = r_{n\alpha_n-1} \end{array}$$

is given by

(2.5)

$$p_{\bar{N}}(z) = \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i-1} \sum_{k=0}^{\alpha_i-1-j} r_{ij} \frac{w(z)}{j!k!} (z - z_i)^{-\alpha_i+j+k} \frac{d^k}{dz^k} \left[\frac{(z - z_i)^{\alpha_i}}{w(z)} \right]_{z=z_i}.$$

3. Sampling expansion of entire functions

In this section we extend the generalized Hermite interpolation derived in Section 2 to obtain the sampling expansion involving derivatives for entire functions.

THEOREM 3.1. *Let $\sigma > 0, N \in \mathbb{N}$ and let f be an entire function such that there exists $A > 0$ satisfying*

$$(3.1) \qquad (1 + |x|)|f(z)| \leq A \exp(\sigma|y|) \quad \text{for any } z = x + iy \in \mathbb{C}.$$

Then f can be expanded as

$$\begin{aligned} (3.2) \quad f(z) &= \sum_{n\in\mathbb{Z}} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1-j} f^{(j)}(\beta_n) \frac{(\sin \sigma z/N)^N}{j!k!} \\ &\quad \times (z - \beta_n)^{-N+j+k} \frac{d^k}{dz^k} \left[\left(\frac{z - \beta_n}{\sin \sigma z/N} \right)^N \right]_{z=\beta_n}, \end{aligned}$$

where $\beta_n = \frac{Nn\pi}{\sigma}, n \in \mathbb{Z}$, and the series converges uniformly on any compact subsets of $\mathbb{C}$.

PROOF. Let $M > 1$ and

$$S = \{z \in \mathbb{C} \ : \ |z| \leq M \quad \text{and} \quad z \neq \beta_n, \quad n \in \mathbb{Z}\}.$$

Choose $z \in S$ and define a function g on the complex plane by

$$g(\zeta) = \frac{f(\zeta)}{(\zeta - z)(\sin \frac{\sigma\zeta}{N})^N}.$$

Let $\tau_n = \frac{N\pi}{\sigma}(n+\frac{1}{2})$ and let C_n be a rectangular path whose vertices are $\pm\tau_n \pm i\tau_n$. Choose $N_1 \in \mathbb{N}$ so large that $\tau_{N_1} \geq 2M$. Let $n \geq N_1$ and for $\delta > 0$ let $\alpha_\delta = \frac{e^{2\delta}-1}{2e^{2\delta}}$. Observing that for $y \geq \delta > 0$, $\xi, \eta \in \mathbb{R}$

$$\sinh y \geq \alpha_\delta e^y,$$

$$\begin{aligned}|\sin\frac{\sigma}{N}(\xi + i\tau_n)| &= \sqrt{\sin^2\frac{\sigma}{N}\xi + \sinh^2\pi(n+\frac{1}{2})}\\ &\geq \alpha_{\pi(N_1+\frac{1}{2})}\exp(\pi(n+\frac{1}{2})),\end{aligned}$$

$$|\sin\frac{\sigma}{N}(\tau_n + i\eta)| = \sqrt{\sin^2\pi(n+\frac{1}{2}) + \sinh^2\frac{\sigma}{N}\eta} \geq 1$$

and for $|\eta| \geq \delta$,

$$|\sin\frac{\sigma}{N}(\tau_n + i\eta)| \geq \alpha_{\frac{\alpha\delta}{N}}\exp(\frac{\sigma}{N}|\eta|),$$

by a direct computation or following the proof of Theorem 2.1 of [10] we can see that

$$\lim_{n\to\infty}\int_{C_n} g(\zeta)d\zeta = 0$$

uniformly on S. Thus the sum of residues of g equals zero. Note that the function g has a simple pole at $\zeta = z$ and poles of order N at $\zeta = \beta_n, n \in \mathbb{Z}$. Following the computation in Section 2, we have

$$\text{Res}(g:z) = \lim_{\zeta\to z}(\zeta - z)g(\zeta) = \frac{f(z)}{(\sin\frac{\sigma}{N}z)^N},$$

and for each $n \in \mathbb{Z}$

$$\begin{aligned}\text{Res}(g:\beta_n) &= \lim_{\zeta\to\beta_n}\frac{1}{(N-1)!}\frac{d^{N-1}}{d\zeta^{N-1}}\left[(\zeta-\beta_n)^N g(\zeta)\right]\\ &= (-1)\sum_{j=0}^{N-1}\sum_{k=0}^{N-1-j}\frac{1}{j!k!}f^{(j)}(\beta_n)(z-\beta_n)^{-N+j+k}\\ &\quad\times\frac{d^k}{d\zeta^k}\left[\left(\frac{\zeta-\beta_n}{\sin\frac{\sigma}{N}\zeta}\right)^N\right]_{\zeta=\beta_n}.\end{aligned}$$

Since $\text{Res}(g:z)+\sum_{n\in\mathbb{Z}}\text{Res}(g:\beta_n)=0$, it follows that for any $z\in S$

$$\text{(3.3)}\quad f(z) = \sum_{n\in\mathbb{Z}}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1-j} f^{(j)}(\beta_n)\frac{(\sin\frac{\sigma}{N}z)^N}{j!k!}(z-\beta_n)^{-N+j+k} \times\frac{d^k}{d\zeta^k}\left[\left(\frac{\zeta-\beta_n}{\sin\frac{\sigma}{N}\zeta}\right)^N\right]_{\zeta=\beta_n},$$

where the series converges uniformly on S. Since $M>1$ is an arbitrary number, the series converges uniformly on any compact subsets of $\mathbb{C}\setminus\cup_{n\in\mathbb{Z}}\{\beta_n\}$.

We next show that the equality (3.3) holds for each $z=\beta_n$, $n\in\mathbb{Z}$. For convenience, we write $h(z)=(\sin\frac{\sigma}{N}z)^N$. Let $F(z)$ be the right side of (3.3). Then $f(z)=F(z)$ for $z\neq\beta_n$, $n\in\mathbb{N}$. We show that for any $n\in\mathbb{Z}, f(\beta_n)=F(\beta_n)$. Since $h(z)$ has zeros at each β_n with multiplicity N, the function defined by

$$\tilde{h}_n(z)=\begin{cases}\frac{h(z)}{(z-\beta_n)^N} & \text{if } z\neq\beta_n,\\ \frac{h^{(N)}(\beta_n)}{N!} & \text{if } z=\beta_n,\end{cases}$$

is an entire function and $\tilde{h}_n(\beta_n)\neq 0$. Thus $\frac{1}{\tilde{h}_n(z)}$ is analytic on some neighborhood of β_n and for any $\ell\in\mathbb{N}$ and $1\leq k\leq N-1$,

$$\lim_{z\to\beta_n}(z-\beta_n)^\ell\frac{d^k}{dz^k}\frac{1}{\tilde{h}_n(z)}=0.$$

Since for $m\neq n$, $h(\beta_n)(\beta_n-\beta_m)^{-N+j+k}=0$, we have

$$\begin{aligned}F(\beta_n) &= \lim_{z\to\beta_n}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1-j}\frac{h(z)}{j!k!}f^{(j)}(\beta_n)(z-\beta_n)^{-N+j+k}\frac{d^k}{dz^k}\frac{1}{\tilde{h}_n(z)}\\ &= \lim_{z\to\beta_n}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1-j}\frac{\tilde{h}_n(z)}{j!k!}f^{(j)}(\beta_n)(z-\beta_n)^{j+k}\frac{d^k}{dz^k}\frac{1}{\tilde{h}_n(z)}\\ &= \tilde{h}_n(\beta_n)f(\beta_n)\frac{1}{\tilde{h}_n(\beta_n)}=f(\beta_n)\end{aligned}$$

since when $j\neq 0$ or $k\neq 0$, the limits vanish. Hence the equality (3.3) holds for any $z\in\mathbb{C}$ and the right side of (3.3) converges uniformly on any compact subsets of $\mathbb{C}$. □

Following proof of the above theorem, we can replace the growth condition (3.1) by (3.5) and obtain the same result.

REMARK 3.2. Let $\sigma > 0, N \in \mathbb{N}$ and let f be an entire function such that there exists $A > 0$ satisfying

$$(3.4) \qquad (1+|y|)|f(z)| \le A\exp(\sigma|y|) \quad \text{for any } z = x+iy \in \mathbb{C}.$$

Then f can be expanded as (3.2).

Next, for any $f \in B^p_\sigma, 1 \le p \le \infty$, we can expand f into a sampling series of the form (3.2).

THEOREM 3.3. *Let $\sigma' > \sigma$ and let f be a band-limited function, that is, there exists $A > 0$ satisfying*

$$(3.5) \qquad |f(z)| \le A\exp(\sigma|y|) \quad \text{for any } z = x+iy \in \mathbb{C}.$$

The following sampling expansion holds

$$(3.6) \quad f(z) = \sum_{n\in\mathbb{Z}}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1-j} f^{(j)}(\tilde{\beta}_n)\frac{(\sin\sigma' z/N)^N}{j!k!}(z-\tilde{\beta}_n)^{-N+j+k} \times \frac{d^k}{dz^k}\left[\left(\frac{z-\tilde{\beta}_n}{\sin\sigma' z/N}\right)^N\right]_{z=\tilde{\beta}_n},$$

where $\tilde{\beta}_n = \frac{Nn\pi}{\sigma'}, n \in \mathbb{Z}$ and the series converges uniformly on any compact subsets of $\mathbb{C}$.

PROOF. Let $\sigma' > \sigma$. Since f satisfies (3.5), there exists $A' > 0$ satisfying

$$(1+|y|)|f(z)| \le A'\exp(\sigma'|y|) \quad \text{for any } z = x+iy \in \mathbb{C}.$$

By Remark 3.2, the sampling expansion (3.6) holds and the series converges uniformly on any compact subsets of $\mathbb{C}$. □

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