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ON GENERALIZED RICCI-RECURRENT TRANS-SASAKIAN MANIFOLDS

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ABSTRACT. Generalized Ricci-recurrent trans-Sasakian manifolds are studied. Among others, it is proved that a generalized Ricci-recurrent cosymplectic manifold is always recurrent. Generalized Ricci-recurrent trans-Sasakian manifolds of dimension ≥ 5 are locally classified. It is also proved that if M is one of Sasakian, α-Sasakian, Kenmotsu or β-Kenmotsu manifolds, which is generalized Ricci-recurrent with cyclic Ricci tensor and non-zero $A(\xi)$ everywhere; then M is an Einstein manifold.

1. Introduction

A non-flat Riemannian manifold M is called a *generalized Ricci-recurrent manifold* ([3]) if its Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)\,S(Y,Z) + B(X)\,g(Y,Z), \tag{1}$$

where ∇ is Levi-Civita connection of the Riemannian metric g, and A, B are 1-forms on M. In particular, if the 1-form B vanishes identically, then M reduces to the well known *Ricci-recurrent manifold* ([14]).

In [15], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He showed that they can be divided into three classes: (1) homogeneous normal contact Riemannian manifolds with $c > 0$, (2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and (3) a warped product space $\mathbb{R}\times_f\mathbb{C}^n$ if $c < 0$. It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure. Kenmotsu ([8])

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characterized the differential geometric properties of the manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian ([8]).

In the Gray-Hervella classification of almost Hermitian manifolds ([6]), there appears a class, $\mathcal{W}_4$, of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds ([5]). An almost contact metric structure on a manifold M is called a *trans-Sasakian structure* ([13]) if the product manifold $M \times \mathbb{R}$ belongs to the class $\mathcal{W}_4$. The class $\mathcal{C}_6 \oplus \mathcal{C}_5$ ([9]) coincides with the class of trans-Sasakian structures of type (α, β). We note that trans-Sasakian structures include cosymplectic ([1]), α-Sasakian ([7]), Sasakian, β-Kenmotsu ([7]), Kenmotsu and normal locally conformal almost cosymplectic ([10]) or f-Kenmotsu ([11]) structures. In ([16]), it is proved that trans-Sasakian manifolds are always *generalized quasi-Sasakian* ([12]); thus these structures provide a large class of generalized quasi-Sasakian structures also.

Thus motivated sufficiently, in this paper we study generalized Ricci-recurrent trans-Sasakian manifolds. Section 2 contains necessary details about trans-Sasakian manifolds. In Section 3, for generalized Ricci-recurrent trans-Sasakian manifolds, a relation between the 1-forms A and B is established. Among others, it is proved that a generalized Ricci-recurrent cosymplectic manifold is always Ricci-recurrent. Generalized Ricci-recurrent trans-Sasakian manifolds of dimension ≥ 5 are also classified. In the last section, an expression for Ricci-tensor for a generalized Ricci-recurrent trans-Sasakian manifold with cyclic Ricci tensor is obtained. It is also proved that if M is one of Sasakian, α-Sasakian, Kenmotsu or β-Kenmotsu manifolds which is generalized Ricci-recurrent manifold with cyclic Ricci tensor and non-zero $A(\xi)$ everywhere, then M is an Einstein manifold.

2. Trans-Sasakian manifolds

Let M be an almost contact metric manifold ([1]) with an almost contact metric structure (φ, ξ, η, g), that is, φ is a $(1,1)$ tensor field, ξ is a vector field; η is a 1-form and g is a compatible Riemannian metric such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \tag{2}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3}$$

$$g(X, \varphi Y) = -g(\varphi X, Y), \quad g(X, \xi) = \eta(X) \tag{4}$$

for all $X, Y \in TM$.

An almost contact metric structure (φ, ξ, η, g) in M is called a *trans-Sasakian structure* ([13]) if $(M \times \mathbb{R}, J, G)$ belongs to the class $\mathcal{W}_4$ ([6]), where J is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J(X, \lambda d/dt) = (\varphi X - \lambda \xi, \eta(X) d/dt)$$

for all vector fields X on M and smooth functions λ on $M \times \mathbb{R}$ and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition ([2])

$$(\nabla_X \varphi) Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X) \tag{5}$$

for some smooth functions α and β on M, and we say that the trans-Sasakian structure is of type (α, β).

Let (x, y, z) be Cartesian coordinates in $\mathbb{R}^3$, then (φ, ξ, η, g) given by

$$\xi = \partial/\partial z, \qquad \eta = dz - y dx,$$

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -y & 0 \end{pmatrix}, \qquad g = \begin{pmatrix} e^z + y^2 & 0 & -y \\ 0 & e^z & 0 \\ -y & 0 & 1 \end{pmatrix}$$

is a trans-Sasakian structure of type $(-1/(2e^z), 1/2)$ in $\mathbb{R}^3$ ([2]). In fact, in a 3-dimensional K-contact manifold with structure tensors (φ, ξ, η, g), for a non-constant function f, defining $g' = fg + (1-f)\,\eta \otimes \eta$; (φ, ξ, η, g') is a trans-Sasakian structure of type $(1/f, (1/2)\,\xi(\ln f))$ ([9]).

3. Generalized Ricci-recurrent trans-Sasakian manifolds

Let M be a $(2n+1)$-dimensional trans-Sasakian manifold. From (5) it is easy to see that

$$\nabla_X \xi = -\alpha \varphi X + \beta(X - \eta(X)\xi), \tag{6}$$

$$(\nabla_X \eta) Y = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y). \tag{7}$$

In view of (5), (6) and (7), we are able to state the following Lemma.

LEMMA 3.1. ([4]) *In a $(2n+1)$-dimensional trans-Sasakian manifold, we have*

$$\begin{aligned} R(X,Y)\xi \;=\;& (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) \\ &+ 2\alpha\beta(\eta(Y)\varphi X - \eta(X)\varphi Y) - (X\alpha)\varphi Y \\ &+ (Y\alpha)\varphi X - (X\beta)\varphi^2 Y + (Y\beta)\varphi^2 X, \end{aligned} \tag{8}$$

(9) $S(X,\xi) = 2n(\alpha^2 - \beta^2)\eta(X) - (2n-1)X\beta - \eta(X)(\xi\beta) - (\varphi X)\alpha,$

(10) $$Q\xi = 2n(\alpha^2 - \beta^2)\xi - (2n-1)\,\mathrm{grad}\beta - (\xi\beta)\xi + \varphi(\mathrm{grad}\alpha),$$

where R and S are curvature and Ricci curvature tensors, while Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$. In particular, we have

(11) $$S(\xi,\xi) = 2n(\alpha^2 - \beta^2 - \xi\beta).$$

Now, we prove the following

THEOREM 3.2. *Let M be a $(2n+1)$-dimensional generalized Ricci-recurrent trans-Sasakian manifold. Then, the 1-forms A and B are related by*

(12) $$\begin{aligned} B(X) &= 2n\{X(\alpha^2 - \beta^2 - \xi\beta) - (\alpha^2 - \beta^2 - \xi\beta)A(X)\} \\ &\quad -2(2n-1)(\alpha\varphi X + \beta\varphi^2 X)\beta - 2(\alpha\varphi^2 X - \beta\varphi X)\alpha. \end{aligned}$$

In particular, we get

(13) $$B(\xi) = 2n(\xi(\alpha^2 - \beta^2 - \xi\beta) - (\alpha^2 - \beta^2 - \xi\beta)A(\xi)).$$

Proof. Using (1) in

(14) $$(\nabla_X S)(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z),$$

we get

(15)
$$A(X)S(Y,Z) + B(X)g(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z).$$

Putting $Y = Z = \xi$, in the above equation we obtain

$$S(\xi,\xi)A(X) + B(X) = XS(\xi,\xi) - 2S(\nabla_X\xi,\xi),$$

which in view of (11), (4) and (6) yields (12). The equation (13) is obvious from (12). □

Let A^* and B^* be the associated vector fields of A and B, that is, $g(X,A^*) = A(X)$ and $g(X,B^*) = B(X)$.

COROLLARY 3.3. *In a $(2n+1)$-dimensional generalized Ricci-recurrent α-Sasakian (resp. Sasakian) manifold, we have*

(16) $$B = -2n\alpha^2 A \quad (\text{resp. } B = -2nA).$$

Thus, the associated vector fields A^* and B^* are in opposite directions.

Proof. A trans-Sasakian manifold of type $(\alpha, 0)$ is α-Sasakian ([7]). In this case α becomes a constant. If $\alpha = 1$, then α-Sasakian manifold is Sasakian. Thus, from the equation (12), the proof follows immediately. □

COROLLARY 3.4. *In a* $(2n+1)$*-dimensional generalized Ricci-recurrent normal almost cosympectic* f*-structure (or* f*-Kenmotsu) manifold we have*

$$(17)\quad B(X) = 2n\left(\left(f^2+\xi f\right)A(X) - X\left(f^2+\xi f\right)\right) - 2(2n-1)f\left(\varphi^2 X\right)f.$$

Proof. A trans-Sasakian structure with $\alpha = 0$ and $\beta \equiv f$ is a normal almost cosympectic f-structure ([10]) (or f-Kenmotsu structure [11]). Thus, putting $\alpha = 0$ and $\beta \equiv f$ in the equation (12), we get (17). □

COROLLARY 3.5. *For a* $(2n+1)$*-dimensional generalized Ricci-recurrent* β*-Kenmotsu (resp. Kenmotsu) manifold, we have*

$$(18)\qquad B = 2n\beta^2 A \quad (\textit{resp. } B = 2nA).$$

Thus, the associated vector fields A^* *and* B^* *are in same direction.*

Proof. A trans-Sasakian structure is β-Kenmotsu ([7]) if $\alpha = 0$ and $\beta =$ constant. In particular, 1-Kenmotsu structure is a Kenmotsu structure. Putting $f = \beta =$ constant (resp. $f = 1$) in (17), we obtain (18). □

A trans-Sasakian structures of type $(0, 0)$ is cosymplectic ([1]). Thus, putting $\alpha = 0 = \beta$ in (12), we get $B = 0$. Hence, we have the following

THEOREM 3.6. *A generalized Ricci-recurrent cosymplectic manifold* M *is always Ricci-recurrent.*

Now, we give the following classification for generalized Ricci-recurr -ent trans-Sasakian manifold of dimension ≥ 5 locally.

THEOREM 3.7. *Let* M *be a generalized Ricci-recurrent trans-Sasakian manifold of dimension* $(2n+1) \geq 5$. *Then*

1. *either* M *is Ricci-recurrent,*
2. *or* $B + 2n\alpha^2 A = 0$,
3. *or* $B - 2n\beta^2 A = 0$,

where α and β are non-zero constant.

Proof. We know that locally a trans-Sasakian manifold of dimension ≥ 5 is either cosymplectic, or α-Sasakian or β-Kenmotsu manifold ([9]). Hence, in view of Corollaries 3.3, 3.5 and Theorem 3.6, the proof is complete. □

4. Generalized Ricci-recurrent trans-Sasakian manifolds with cyclic Ricci tensor

A Riemannian manifold is said to admit cyclic Ricci tensor if

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0. \tag{19}$$

Now, we prove the following.

THEOREM 4.1. *In a $(2n+1)$-dimensional generalized Ricci-recurrent trans-Sasakian manifolds with cyclic Ricci tensor, the Ricci tensor satisfies*

$$\begin{aligned}
& A(\xi)\, S(X,Y) \\
= \; & 2n\left\{\left(\alpha^2-\beta^2-\xi\beta\right)A(\xi)-\xi\left(\alpha^2-\beta^2-\xi\beta\right)\right\}g(X,Y) \\
& + (2n-1)\{A(X)\,Y\beta + A(Y)\,X\beta\} \\
& - (2n-1)(\xi\beta)\{\eta(Y)\,A(X)+\eta(X)\,A(Y)\} \\
& + A(X)(\varphi Y)\,\alpha + A(Y)(\varphi X)\,\alpha \\
& - 2n\left\{\eta(X)\,Y\left(\alpha^2-\beta^2-\xi\beta\right) + \eta(Y)\,X\left(\alpha^2-\beta^2-\xi\beta\right)\right\} \\
& + 2(2n-1)\left\{\eta(X)\left(\alpha\varphi Y+\beta\varphi^2 Y\right)\beta + \eta(Y)\left(\alpha\varphi X+\beta\varphi^2 X\right)\beta\right\} \\
& + 2\left\{\eta(X)\left(\alpha\varphi^2 Y-\beta\varphi Y\right)\alpha + \eta(Y)\left(\alpha\varphi^2 X-\beta\varphi X\right)\alpha\right\}.
\end{aligned} \tag{20}$$

Proof. Suppose that M is a generalized Ricci symmetric manifold admitting cyclic Ricci tensor. Then in view of (1) and (19), we get

$$\begin{aligned}
0 \;=\; & A(X)\,S(Y,Z)+A(Y)\,S(Z,X)+A(Z)\,S(X,Y) \\
& + B(X)\,g(Y,Z)+B(Y)\,g(Z,X)+B(Z)\,g(X,Y).
\end{aligned}$$

Moreover, if M is trans-Saskian manifold, putting $Z=\xi$, in the above equation we get,

$$\begin{aligned}
A(\xi)\,S(X,Y) \;=\; & -B(\xi)\,g(X,Y)-A(X)\,S(Y,\xi)-A(Y)\,S(X,\xi) \\
& - B(X)\,\eta(Y)-B(Y)\,\eta(X),
\end{aligned}$$

which in view of (13) and (9) gives (20). □

COROLLARY 4.2. *For a $(2n+1)$-dimensional generalized Ricci-recurr -ent manifold M with cyclic Ricci tensor, we have the following stateme- nts:*

1. *If M is an α-Sasakian manifold, then*

$$A(\xi)\,S(X,Y) = 2n\alpha^2 A(\xi)\,g(X,Y).$$

2. *If M is a Sasakian manifold, then*

$$A(\xi)\,S(X,Y) = 2nA(\xi)\,g(X,Y).$$

3. *If M is a f-Kenmotsu manifold, then*

$$\begin{aligned} & A(\xi)\,S(X,Y) \\ = \; & 2n\left\{\xi\left(f^2+\xi f\right) - A(\xi)\left(f^2+\xi f\right)\right\} g(X,Y) \\ & + \; (2n-1)\left\{A(X)\,Yf + A(Y)\,Xf\right\} \\ & - \; (2n-1)\,(\xi f)\left\{\eta(Y)\,A(X) + \eta(X)\,A(Y)\right\} \\ & + 2n\left\{\eta(X)\,Y\left(f^2+\xi f\right) + \eta(Y)\,X\left(f^2+\xi f\right)\right\} \\ & + 2\,(2n-1)\left\{\eta(X)\left(f\varphi^2 Y\right) f + \eta(Y)\left(f\varphi^2 X\right) f\right\}. \end{aligned}$$

4. *If M is a β-Kenmotsu manifold, then*

$$A(\xi)\,S(X,Y) = -2n\beta^2 A(\xi)\,g(X,Y).$$

5. *If M is a Kenmotsu manifold, then*

$$A(\xi)\,S(X,Y) = -2nA(\xi)\,g(X,Y).$$

6. *If M is a cosymplectic manifold, then*

$$A(\xi)\,S(X,Y) = 0.$$

A Riemannian manifold is an Einstein manifold if

$$S(X,Y) = \rho g(X,Y).$$

Therefore, in view of Corollary 4.2, we are able to state the following

THEOREM 4.3. *Let M be generalized Ricci-recurrent manifold with cyclic Ricci tensor. If M is one of Sasakian, α-Sasakian, Kenmotsu and β-Kenmotsu manifolds with non-zero $A(\xi)$ everywhere, then M is Einstein.*

References

[1] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics 509, Springer Verlag, Berlin-New York, 1976.

[2] D. E. Blair and J. A. Oubina, *Conformal and related changes of metric on the product of two almost contact metric manifolds*, Publ. Mat. **34** (1990), no. 1, 199–207.

[3] U. C. De, N. Guha and D. Kamilya, *On generalized Ricci-recurrent manifolds*, Tensor (N.S.) **56** (1995), no. 3, 312–317.

[4] U. C. De and M. M. Tripathi, *Ricci tensor in 3-dimensional trans-Sasakian manifolds*, (preprint).

[5] S. Dragomir and L. Ornea, *Locally conformal Kähler geometry*, Progress in Mathematics 155, Birkhäuser Boston, Inc., Boston, MA, 1998.

[6] A. Gray and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl., **123** (1980), no. 4, 35–58.

[7] D. Janssens, L. Vanhecke, *Almost contact structures and curvature tensors*, Kodai Math. J. **4** (1981), no. 1, 1–27.

[8] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tôhoku Math. J. **24** (1972), 93–103.

[9] J. C. Marrero, *The local structure of trans-Sasakian manifolds*. Ann. Mat. Pura Appl. **162** (1992), no. 4, 77–86.

[10] K. Matsumoto, I. Mihai, and R. Rosca, *A certain locally conformal almost cosymplectic manifold and its submanifolds*, Tensor (N.S.) **51** (1992), no. 1, 91–102.

[11] Z. Olszak, *Locally conformal almost cosymplectic manifolds*, Colloq. Math. **57** (1989), no. 1, 73–87.

[12] R. S. Mishra, *Almost contact metric manifolds*, Monograph 1, Tensor Society of India, Lucknow, 1991.

[13] J. A. Oubina, *New classes of almost contact metric structures*, Publ. Math. Debrecen **32** (1985), no. 3-4, 187–193.

[14] E. M. Patterson, *Some theorems on Ricci-recurrent spaces*, J. London Math. Soc. **27** (1952), 287–295.

[15] S. Tanno, *The automorphism groups of almost contact Riemannian manifolds*, Tôhoku Math. J. **21** (1969), 21–38.

[16] M. M. Tripathi, *Trans-Sasakian manifolds are generalized quasi-Sasakian*, Nepali Math. Sci. Rep. **18** (1999-2000), no. 1-2, 11–14.

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