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TORSION IN THE COHOMOLOGY OF FINITE H-SPACES

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ABSTRACT. We study torsion phenomena in the integral cohomology of finite H-spaces X through the Eilenberg–Moore spectral sequence converging to $H^*(\Omega X; Z_p)$. We also investigate how the difference between the Z_p-filtration length $f_p(X)$ and the Z_p-cup length $c_p(X)$ on a simply connected finite H-space X is reflected in the Eilenberg–Moore spectral sequence converging to $H^*(\Omega X; Z_p)$. Finally we get the following result: Let p be an odd prime and X an n–connected finite H–space with dim $QH^*(X; Z_p) \leq m$. Then $H^*(X; Z)$ is p–torsion free if $p \geq \frac{m-1}{n}$. .

0. Introduction

In this paper we study the behavior of the Eilenberg–Moore spectral sequence for a path–loop fibration converging to $H^*(\Omega X; Z_p)$ to get the information about p-torsion in the integral cohomology of a simply connected finite H-space X. First, we investigate how the difference between the Z_p-filtration length and the Z_p-cup length, $f_p(X) - c_p(X)$, is reflected in the Eilenberg–Moore spectral sequence converging to $H^*(\Omega X; Z_p)$. We show that $f_p(X)$ is equal to $c_p(X)$ if and only if the Eilenberg–Moore spectral sequence converging to $H^*(\Omega X; Z_p)$ collapses at the E_2-term. We obtain that $f_2(Y)$ and $c_2(Y)$ are equal for any simply connected finite H–space Y with associative mod 2 homology ring. Hence the Eilenberg–Moore spectral sequence converging to $H^*(\Omega Y; Z_2)$ collapses at the E_2–term.

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We also show that for an odd prime p, $H^*(X;Z)$ is p–torsion free if and only if $f_p(X)$ is equal to $c_p(X)$ if and only if the Eilenberg–Moore spectral sequence converging to $H^*(\Omega X; Z_p)$ collapses at the E_2–term.

We study the suspension map and find the condition under which the suspension map is injective. Finally exploiting the suspension map, we get the following theorem.

THEOREM 3.2. *Let p be an odd prime and X an n–connected finite H–space with dim $QH^*(X;Z_p) \leq m$. Then $H^*(X;Z)$ is p–torsion free if $p \geq \frac{m-1}{n}$.*

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1. Preliminaries

Let X be a simply connected finite H-space and R a commutative ring with unity. Given a path–loop fibration, $\Omega X \to PX \to X$, there is a Eilenberg–Moore spectral sequence of $C_*(\Omega X;R)$, that is, a first quadrant spectral sequence of bicommutative and biassociative Hopf algebras $\{E^r, d^r\}$ converging to $H_*(X;R)$, where

$$E^2 = \mathrm{Tor}^{H_*(\Omega X;R)}(R,R)\,.$$

In this Eilenberg–Moore spectral sequence, the R–filtration length of X is defined in [14] as follows:

DEFINITION. The R–filtration length of X, $f_R(X)$, is at most k if $E^{\infty}_{p,*} = 0$ for $p \geq k+1$. The R–filtration length of X is equal to k if $f_R(X) \leq k$ but $f_R(X) \leq k-1$ is false.

That is, $f_R(X)$ is the smallest integer k such that $E^{\infty}_{p.*} = 0$ for all $p \geq k+1$. The R-cup length of X is defined as follows:

DEFINITION. The R-cup length of X, $c_R(X)$, is defined as the largest integer k such that there exist $x_i \in H^*(X;R)$, $i = 1, \cdots, k$ with a nontrivial cup product $x_1 \cdots x_k$.

When $R = Z_p$, we will use the notation $f_p(X)$ and $c_p(X)$ for $f_{Z_p}(X)$ and $c_{Z_p}(X)$, respectively.

Given a path–loop fibration, $\Omega X \to PX \to X$, there also exists a second quadrant Eilenberg–Moore spectral sequence $\{E_r, d_r\}$ of bicommutative and biassociative Hopf algebras, where

(1) $E_2 = Tor_{H^*(X;R)}(R,R)$ as Hopf algebras,

(2) $E_\infty = E_0(H^*(\Omega X; R))$ as Hopf algebras,
(3) d_r has bidegree $(r, -r+1)$.

It is shown in [5, 9] that the primitives of $E_0(H^*(\Omega X; R))$ have external degree -1 or -2 and $E_\infty = H^*(\Omega X; R)$ as coalgebras. Hence the primitives of E_∞ corresponds to the primitives of $H^*(\Omega X; R)$.

We denote the primitives and the indecomposables of $H^*(X; Z_p)$ by $PH^*(X; Z_p)$ and $QH^*(X; Z_p)$, respectively. In the Eilenberg–Moore spectral sequence, we have a suspension map

$$\sigma : QH^*(X; Z_p) \cong Tor^{-1,*}_{H^*(X;Z_p)}(Z_p, Z_p)$$
$$= E_2^{-1,*} \to E_\infty^{-1,*} \subset H^{*-1}(\Omega X; Z_p).$$

Since the elements of $Tor^{-1,*}_{H^*(X;Z_p)}(Z_p, Z_p)$ are primitive and permanent cycles in the Eilenberg–Moore spectral sequence, the above map induces the suspension homomorphism $\sigma : QH^*(X; Z_p) \to PH^{*-1}(\Omega X; Z_p)$.

THEOREM 1.1. [8] *Let X be a simply connected H-space. Then the following is true.*
(a) *The suspension $\sigma : QH^{odd}(X; Z_p) \to PH^{even}(\Omega X; Z_p)$ is injective.*
(b) *The suspension $\sigma : QH^{even}(X; Z_p) \to PH^{odd}(\Omega X; Z_p)$ is surjective.*
(c) *The quotient $PH^{even}(\Omega X; Z_p)/\sigma(QH^{odd}(X; Z_p))$ is obtained by tran -spotence.*
(d) *The elements in $\ker \sigma$ are dual to elements in the image of the homology transpotence.*

The filtration length $f_p(X)$ is defined in terms of homology and the cup length $c_p(X)$ is defined in terms of cohomology. But $f_p(X)$ can be compared with $c_p(X)$ by duality between homology and cohomology.

First, we interpret $f_p(X)$ in terms of the cohomology spectral sequence. Let $\{E_r\}$ be the spectral sequence dual to the Eilenberg–Moore spectral sequence $\{E^r\}$ converging to $H_*(X; Z_p)$, that is, $E_r^{s,t} = (E^r_{s,t})^*$. Then $\{E_r\}$ is a spectral sequence of bicommutative biassociative Hopf algebras converging to $H^*(X; Z_p)$, $E_\infty = E_0(H^*(X; Z_p))$, and the product in E_∞ corresponds to the cup product in $H^*(X; Z_p)$. If $E^\infty_{s,*} = 0$ for all $s > s_0$, then $E_\infty^{s,*} = 0$ for all $s > s_0$. Hence $f_p(X)$ is the same as the smallest number s_0 such that $E_\infty^{s,t} = 0$ for all $s > s_0$.

Next, we can define the filtration degree of an element x of $H^*(X; Z_p)$ as follows. Each $x \in H^*(X; Z_p)$ can be considered as an element of $E_\infty^{s,t}$ since $E_\infty = E_0(H^*(X; Z_p))$. The number s is called the filtration degree of x. The filtration degree of an element $x \in H_*(X; Z_p)$ can be defined in the same way.

The difference $f_p(X) - c_p(X)$ comes from the generator of $H^*(X; Z_p)$ of filtration degree greater than 1, which corresponds to a primitive element in $H_*(X; Z_p)$ through dualizing process. Since the primitives of $H_*(X; Z_p)$ is of filtration degree either 1 or 2, the difference $f_p(X) - c_p(X)$ comes only from those primitives in $H_*(X; Z_p)$ of even filtration degree, which are transpotence elements. So in terms of cohomology, the difference comes from the elements in $\ker \sigma$ by Theorem 1.1 (d).

THEOREM 1.2. [14] *Let X be a finite H-space. If $\ker \sigma$ is generated by $x_1, \dots, x_n$, which have heights $p^{h_1}, \dots, p^{h_n}$ in $H^*(X; Z_p)$, then*

$$f_p(X) - c_p(X) = \sum_{i=1}^{n} (p^{h_i} - 1).$$

The above theorem is originally expressed in terms of σ^* where it is induced by the canonical map $\Sigma\Omega X \to X$ which is the adjoint of the identity on ΩX. Then we have the following commutative diagram,

$$\begin{array}{ccc} H^*(X; Z_p) & \xrightarrow{\sigma^*} & H^{*-1}(\Omega X; Z_p) \\ {\scriptstyle q}\downarrow & & \uparrow{\scriptstyle \text{inclusion}} \\ QH^*(X; Z_p) & \xrightarrow{\sigma} & PH^{*-1}(\Omega X; Z_p), \end{array}$$

where q is the quotient map. From the diagram, we may identify the elements of $QH^*(X; Z_p)$ with generators of $H^*(X; Z_p)$ by the abuse of language.

2. Eilenberg–Moore spectral sequence and $f_p(X) - c_p(X)$

From now on, unless stated otherwise, the Eilenberg–Moore spectral sequence converging to $H^*(\Omega X; Z_p)$ is the Eilenberg–Moore spectral sequence for a path–loop fibration converging to $H^*(\Omega X; Z_p)$ with

$$E_2 = \mathrm{Tor}_{H^*(X;Z_p)}(Z_p, Z_p).$$

Similarly, the Eilenberg–Moore spectral sequence converging to $H_*(\Omega X; Z_p)$ is the Eilenberg–Moore spectral sequence with

$$E^2 = \mathrm{Cotor}^{H_*(X;Z_p)}(Z_p, Z_p).$$

THEOREM 2.1. *Let X be a simply connected finite H-space. Then the following conditions are equivalent.*

(1) $f_p(X) - c_p(X) = 0$;
(2) $\ker\sigma = 0$;
(3) *The Eilenberg–Moore spectral sequence converging to $H^*(\Omega X; Z_p)$ collapses at the E_2-term;*
(4) *The Eilenberg–Moore spectral sequence converging to $H_*(\Omega X; Z_p)$ collapses at the E^2-term.*

Proof. (3) is equivalent to (4) by duality.

(1) implies (2). It follows from Theorem 1.2.

(2) implies (3). Consider the Eilenberg–Moore spectral sequence converging to $H^*(\Omega X; Z_p)$. In the spectral sequence of Hopf algebras, the source of first nontrivial differential is an indecomposable element and its target is a primitive element. Moreover, every primitive element has either -1 or -2 external degree. Hence the spectral sequence collapses at the E_2-term if and only if every primitive element of -1 or -2 external degree survives permanently. Since $\ker\sigma = 0$, any generator of the external degree -1 can not be a target of first nontrivial differential. The primitive elements of the external degree -2 are transpotence elements of even dimension. The generators of external degree less than -2 is of the form, $\gamma_{p^r}(\cdot), r \geq 2$, which is also even dimensional. Therefore a primitive element of external degree -2 can not be a target of the first nontrivial differential, either. Hence the Eilenberg–Moore spectral sequence collapses from the E_2-term.

(3) implies (1). Assume that $E_2 = E_\infty$. Then there exist only trivial differentials in the Eilenberg–Moore spectral sequence. Hence every suspension element survives permanently. Since $H^*(\Omega X; Z_p)$ is evenly generated for any finite H-space X [10,11], there does not exist any even dimensional generator in $H^*(X; Z_p)$. Since $QH^{even}(X; Z_p) = PH_{even}(X; Z_p)$ by duality, there is no even dimensional primitive element in $H_*(X; Z_p)$. This implies that there is no transpotence element. Since the difference between the filtration length and the cup length comes only from transpotence elements in $H_*(X; Z_p)$, we obtain that $f_p(X) - c_p(X) = 0$. □

First we consider the case of an even prime.

THEOREM 2.2. [12] *In any simply connected finite H-space with associative mod 2 homology ring, any generator in a degree of the form*

$2^r + 2^{r+1}k - 1$ *for* $k > 0$ *lies in the image of the Steenrod operation* $Sq^{2^r k}$.

When $r = 0$, any $2k$ dimensional generator lie in the image of Sq^k and $Sq^k(x_k) = x_k^2$. Hence there is no even dimensional generator in mod 2 cohomology. Hence we can get the following.

COROLLARY 2.3. *In any simply connected finite H–space X with associative mod 2 homology ring, we have* $f_2(X) - c_2(X) = 0$.

Proof. Since the suspension on $QH^{odd}(X; Z_2)$ is injective and there is no even dimensional generator by Theorem 2.2, $ker\sigma = 0$. Hence $f_2(X) - c_2(X) = 0$ by Theorem 2.1. □

The above is a generalization of the result in [14] that $f_2(X) - c_2(X) = 0$ in every compact, simply connected, simple Lie group. From Theorem 2.1, we get the following corollary.

COROLLARY 2.4. *The followings hold for every simply connected finite H–space with associative mod 2 homology ring.*

(1) $f_2(X) - c_2(X) = 0$;
(2) $ker\ \sigma = 0$;
(3) *The Eilenberg–Moore spectral sequence converging to* $H^*(\Omega X; Z_2)$ *collapses from the* E_2*–term;*
(4) *The Eilenberg–Moore spectral sequence converging to* $H_*(\Omega X; Z_2)$ *collapses from the* E^2*–term.*

Hence we have that $H^*(\Omega G; Z_2) = \mathrm{Tor}_{H^*(G;Z_2)}(Z_2, Z_2)$ as a coalgebra for every Lie group G.

Now we turn to the case of odd primes p.

THEOREM 2.5. *Let X be a simply connected finite H–space and p an odd prime. Then* $H^*(X; Z)$ *is p–torsion free if and only if* $f_p(X) - c_p(X) = 0$

Proof. Assume that $H^*(X; Z)$ is p–torsion free for an odd prime p. For a finite H–space X, $H^*(X; Z)$ is p–torsion free if and only if $H^*(X; Z_p)$ is an exterior algebra on odd dimensional generators [1]. Hence $\ker\sigma = 0$ by Theorem 1.1 (a). Therefore we have $f_p(X) - c_p(X) = 0$ by Theorem 2.1.

Assume that the cohomology of X has p–torsion. Then by the universal coefficient theorem, there should exist an even element in $H^*(X; Z_p)$. Since $x^2=0$ for any odd dimensional generator x, the even element must be an indecomposable element, say y. Since $H^*(\Omega X; Z_p)$ is even dimensional for any finite H-space X, we have that $\sigma(y) = 0$ in $H^*(\Omega X; Z_p)$. Hence $\ker \sigma \neq 0$, so that $f_p(X) - c_p(X) \neq 0$ by Theorem 2.1. □

COROLLARY 2.6. *Let p be an odd prime and X a simply connected finite H-space. If $H^*(X; Z)$ has p-torsion, then $cat(X)$ is strictly greater than $c_p(X)$.*

Proof. If $H^*(X; Z)$ has p-torsion for an odd prime p, then $f_p(X) - c_p(X) \neq 0$. So $c_p(X) \lneq f_p(X)$. Since $f_p(X) \leq cat(X)$ [14], we have that $c_p(X) \lneq cat(X)$. □

From Theorem 2.1 and Theorem 2.5, we get the following result, which may be also derived from the results of [9,12].

COROLLARY 2.7. *Let X be a simply connected finite H-space and p an odd prime. Then $H^*(X; Z)$ is p-torsion free if and only if the Eilenberg-Moore spectral sequence converging to $H^*(\Omega X; Z_p)$ collapses at the E_2-term.*

From above results, we can detect the behavior of the Eilenberg–Moore spectral sequence converging to the loop space of compact, simply connected, simple Lie groups. For odd primes p, the only following compact, simply connected, simple Lie groups have p-torsions in their integral cohomology [13]:

$$\begin{aligned} p = 3: &\quad G = F_4, E_6, E_7, E_8 \\ p = 5: &\quad G = E_8. \end{aligned}$$

Hence for each of the above pairs (G, p), the Eilenberg–Moore spectral sequence converging to $H^*(\Omega G; Z_p)$ does not collapse at the E_2-term and $f_p(G) - c_p(G) \neq 0$ by Theorem 2.5 and Corollary 2.7. But for the other pairs (G, p) including every classical Lie group, the Eilenberg–Moore spectral sequence to $H^*(\Omega G; Z_p)$ collapses at the E_2-term and $f_p(G) - c_p(G) = 0$.

For even prime, compact, simply connected, simple Lie groups have 2-torsions in their integral cohomology only for the following cases [13]:

$$G = Spin(n), n \geq 7, G_2, F_4, E_6, E_7, E_8.$$

But the Eilenberg-Moore spectral sequence converging to $H^*(\Omega G; Z_2)$ collapses at the E_2-term by Corollary 2.4.

EXAMPLE 2.8. For each odd prime p, there exist a simply connected finite complex X whose localization $X(p)$ at p is an H-space [7] with

$$H^*(X(p); Z_p) = Z_p[\beta\mathcal{P}^1 x_3]/((\beta\mathcal{P}^1 x_3)^p) \otimes E(x_3, \mathcal{P}^1 x_3).$$

Since the action of β on $\mathcal{P}^1 x_3$ is non trivial, the cohomology of $X(p)$ has p–torsion. So by Corollary 2.7, the Eilenberg–Moore spectral sequence converging to $H^*(\Omega X(p); Z_p)$ does not collapse at the E_2–term. In fact, $\sigma(\beta\mathcal{P}^1 x_3) \neq 0$, $f_p(X(p)) - c_p(X(p)) = p-1$, and there is the differential from $[x_3|\cdots|x_3]$ (p factors) to $[\beta\mathcal{P}^1 x_3]$ in the E_{p-1} term of the Eilenberg–Moore spectral sequence.

But the above results do not hold if the finiteness condition is omitted.

EXAMPLE 2.9. If X is torsion free for a prime p, then $H^*(\Omega^2\Sigma^2 X; Z)$ has p–torsion of order p [6]. But the Eilenberg–Moore spectral sequence converging to $H^*(\Omega^3\Sigma^2 X; Z_p)$ collapses at the E_2–term.

By considering Theorem 2.5 for a finite complex with certain property, we get the following result.

THEOREM 2.10. *Let X be a finite complex such that $H^*(X; Q)$ is generated by odd degree generators. If $H^*(X; Z)$ is p–torsion free for an odd prime p, then $f_p(X) - c_p(X) = 0$.*

Proof. Assume that $f_p(X) - c_p(X) \neq 0$. Since the difference between the filtration length and the cup length comes only from transpotence elements, there is an even dimensional generator in $H^*(X; Z_p)$. Consider the Bockstein spectral sequence converging to

$$(H^*(X; Z)/\text{torsion}) \otimes Z_p \ \text{ with } E_0^* = H^*(X; Z_p).$$

In this Bockstein spectral sequence, the differential can be interpreted in terms of the higher Bockstein operators. Since $H^*(X; Q)$ is generated by odd degree generators, even dimensional transpotence can not survive permanently. Hence it should be target of some differential in the Bockstein spectral sequence. This implies that $H^*(X; Z)$ has a p–torsion, which is a contradiction. Therefore $f_p(X) - c_p(X) = 0$. □

The real Stiefel manifolds of type $SO(m)/SO(2n+1)$, $m > 2n+1$, all complex Stiefel manifolds $SU(m)/SU(n)$, $m > n$, and all quaternionic Stiefel manifolds $Sp(m)/Sp(n)$, $m > n$ satisfy the hypothesis of above theorem, so the Z_p-filtration lengths are equal to the Z_p-cup lengths for these spaces for any odd prime p. In fact, we have that $f_p(X) - c_p(X) = 0$

for all real, complex, quaternionic Stiefel manifolds X for all primes p and the Eilenberg–Moore spectral sequence converging to $H^*(\Omega X; Z_p)$ collapses at the E_2-term. Hence $H^*(\Omega X; Z_p) = \mathrm{Tor}_{H^*(X;Z_p)}(Z_p, Z_p)$ as a graded vector space.

3. Torsion in finite H-space

Now we study the suspension map to get the main result.

THEOREM 3.1. *Let X be an n-connected H-space. Then*

$$\sigma : QH^i(X; Z_p) \to PH^{i-1}(\Omega X; Z_p)$$

is injective when $i \le 4n+1$ for $p = 2$, and when $i \le pn+1$ for any odd prime p.

Proof. We consider the Eilenberg–Moore spectral sequence converging to $H^*(\Omega X; Z_p)$. Since $H^i(X; Z_p) = 0$ for $0 \le i \le n$ and $QH^*(X; Z_p) \cong Tor^{-1,*}_{H^*(X;Z_p)}(Z_p, Z_p)$, we have that $E_2^{-1,t} = 0$ for $0 \le t \le n$. So from the bar construction, we have

$$E_2^{-s,t} = Tor^{-s,t}_{H^*(X;Z_p)}(Z_p, Z_p) = 0 \text{ for } t \le ns\,.$$

Since this spectral sequence is a spectral sequence of a Hopf algebra, the source of the first non zero differential is a generator and the target is a primitive element.

This Eilenberg–Moore spectral sequence is a second quadrant spectral sequence and $d_r : E_r^{-s,t} \to E_r^{-s+r,t-r+1}$ for $r \ge 2$, so that generators of the external degree -1 or -2 can not be the source of non zero differentials. From the Borel decomposition, $H^*(X; Z_p) = \otimes_i A_i$ as an algebra where each A_i is a Hopf algebra on a single generator, that is, an exterior algebra, a polynomial algebra, or a truncated polynomial algebra on a single generator. Hence $Tor_{H^*(X;Z_p)}(Z_p, Z_p) = \otimes_i B_i$ where each B_i is an exterior algebra or a divided power algebra on a single generator. Moreover, the divided power algebra $\Gamma(x)$ on a generator x is

$$\Gamma(x) = \otimes_{k \ge 0} \gamma_{2^k}(x) \quad \text{as an algebra,}$$

where $\gamma_{2^k}(x) = [x|\cdots\cdots|x]$ (2^k times) on the level of the bar resolution. The external degree of $\gamma_{2^k}(x)$ is -2^k and only $\gamma_{2^k}(x)$, $k \ge 1$ are generators of external degree less than -1 in $Tor_{H^*(X;Z_p)}(Z_p, Z_p)$.

Therefore the source of first non zero differential is a generator of the external degree less than or equal to -4 for $p = 2$, and less than or equal to $-p$ for any odd prime p. So the possible first non zero differential is

$$d_3 : E_3^{-4,4n+4} \to E_3^{-1,4n+2} \text{ for } p = 2,$$
$$d_{p-1} : E_{p-1}^{-p,pn+p} \to E_{p-1}^{-1,pn+2} \text{ for odd primes } p.$$

Hence when $t \leq 4n+1$ for $p = 2$, no element of $E_2^{-1,t}$ is in the image of any non trivial differential. Similarly when $t \leq pn+1$ for any odd prime p, no element of $E_2^{-1,t}$ is in the image of any non trivial differential. Since the suspension map is defined by $\sigma : QH^i(X; Z_p) \cong Tor_{H^*(X;Z_p)}^{-1,i}(Z_p, Z_p) = E_2^{-1,i} \to E_\infty^{-1,i}$, σ is injective if $i \leq 4n+1$ for $p = 2$, and if $i \leq pn+1$ for any odd prime p. □

Using above theorem, we prove the following main result of this paper.

THEOREM 3.2. *Let p be an odd prime and X an n–connected finite H–space with dim $QH^*(X; Z_p) \leq m$. Then $H^*(X; Z)$ is p–torsion free if $p \geq \frac{m-1}{n}$.*

Proof. Consider the Eilenberg–Moore spectral sequence converging to $H^*(\Omega X; Z_p)$. By Theorem 3.1, $\sigma : QH^i(X; Z_p) \to PH^{i-1}(\Omega X; Z_p)$ is injective if $i \leq pn+1$ for any odd prime p. So if $m \leq pn+1$, then σ is an injective map, that is, ker $\sigma = 0$. By Theorem 2.1, the Eilenberg–Moore spectral sequence converging to $H^*(\Omega X; Z_p)$ collapses at the E_2–term. Hence by Corollary 2.7, $H^*(X; Z)$ is p–torsion free. □

We conclude this paper with the following question about any simply connected H–space without the finiteness condition.

QUESTION. Let p be a prime and Y a simply connected H–space such that $H^*(Y; Z)$ is p–torsion free. Then does the Eilenberg–Moore spectral sequence converging to $H^*(\Omega Y; Z_p)$ collapse at the E_2–term ?

For each prime p, $H^*(\Omega X; Z)$ is p–torsion free for any simply connected finite H–space X [10, 11]. If the question is positive, the Eilenberg –Moore spectral sequence converging to $H^*(\Omega^2 X; Z_p)$ collapses at the E_2–term. In fact, for any compact, simply connected, simple Lie group G, the Eilenberg–Moore spectral sequence converging to $H^*(\Omega^2 G; Z_p)$ collapses at the E_2–term [2, 3, 4, 15] unlike one converging to the cohomology of single loop space.

References

[1] W. Browder, *Torsion in H- spaces*, Ann. of Math. **74** (1961), 24–51.

[2] Y. Choi, *Homology of the double and triple loop space of $SO(n)$*, Math. Z. **222** (1996), 59–80.

[3] Y. Choi and S. Yoon, *Homology of the triple loop space of the exceptional Lie group F_4*, J. Korean Math. Soc. **35** (1998), 149–164.

[4] ———, *Homology of the double and the triple loop space of E_6, E_7, and E_8*, Manuscripta Math. **103** (2000), 101–116.

[5] A. Clark, *Homotopy commutativity and the Moore spectral sequence*, Pacific J. Math. **15** (1965), 65–74.

[6] F. R. Cohen, *The homology of C_{n+1}-spaces, $n \geq 0$,* The homology of iterated loop spaces, 207–351, Lecture Notes in Mathematics Vol. 533, Springer, Berlin, 1976.

[7] J. Harper, *On the construction of mod p H-space,* Algebraic and geometric topology, 207–214, Proc. Sym. Pure Math. **32**, Amer. Math. Soc., Providence, R.I., 1978.

[8] S. Gitler, *Nota sobre la transpotencia de Cartan*, Bol. Soc. Mat. Mex. **7** (1962), 85–91.

[9] R. Kane, *On loop spaces without p torsion*, Pacific J. Math. **60** (1975), 189–201.

[10] ———, *The homology of Hopf spaces*, North-Holland Math. Library **40**, North-Holland, Amsterdam, 1988.

[11] J. P. Lin, *Two torsion and the loop space conjecture*, Ann. of Math. **115** (1982), 35–91.

[12] ———, *Steenrod connections and connectivity in H-spaces*, Mem. Amer. Math. Soc. **68** (1987).

[13] M. Mimura and H. Toda, *Topology of Lie groups, I and II*, Trans. Math. Monographs **91**, Amer. Math. Soc., New York, 1991.

[14] G. H. Toomer, *Lusternik-Schnirelmann category and the Moore spectral sequence*, Math. Z. **138** (1974), 123–143.

[15] A. Yamaguchi, *The homology of double loop spaces of complex Stiefel manifolds*, Publ. Res, Inst. Math. Sci. **22** (1986), 767–800.

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