Periodic Properties of a Lyapunov Functional of State Delay Systems

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Abstract - This paper is concerned with properties of a Lyapunov functional of state delay systems. It is shown that if a state delay system has a pure imaginary pole for some state delay, then no Lyapunov functional satisfying a Lyapunov condition exists periodically with respect to change of the state delay. This periodic property is unique in state delay systems and has been well known in the frequency domain stability conditions. However, in the time domain stability conditions using a Lyapunov functional, the periodic property is not known explicitly.
Keywords - time delay systems, Lyapunov functional

1. Introduction

Stability conditions of state delay systems [1] can be classified into frequency domain methods and time domain methods. In a frequency domain method [2- 3] , stability is checked based on pole location of a system. In a time domain method, stability is checked based on the unique existence of a Lyapunov functional [4]. A frequency domain method is easier to apply for some cases, and for other cases, a time domain method is easier to apply. Since these two stability conditions play complementary roles, their relationship is important.

Of particular interest is when a system lies on the borderline of stable and unstable regions: i.e., the right-most pole of a system lies on the imaginary axis. It is known that there exist no unique Lyapunov functional satisfying a Lyapunov condition if a state delay system has a pure imaginary pole. For systems without state delay, this relationship can be explicitly shown using a simple algebraic manipulation [5]. For state delay systems, the relationship is only partially known: periodic properties of a Lyapunov functional are not known explicitly. Periodicity is a unique feature in state delay systems: if a state delay system has a pure imaginary pole for some state delay h^*, then the system has pure imaginary poles for all state delay h^*+p*i, $i=1,2,\cdots$ for some $p \geq 0$. Since there does not exist a unique Lyapunov functional when a system has a pure imaginary pole, there should exist no Lyapunov functional periodically with respect to change of state delay h. In this paper, this periodic nature of a Lyapunov equation is explicitly shown, which will clarify the relationship between a frequency domain method and a time domain method of state delay systems.

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This paper is organized as follows. In Section II, after a frequency domain stability condition and a time domain stability condition is briefly reviewed, a main result of this paper (periodic properties of a Lyapunov functional) is given in Theorem 3. In Section III, Theorem 3 is proved. A numerical example is given in Section IV and conclusion is given in Section V.

2. A Pure Imaginary Pole of a State Delay System

Definition: for a matrix $M \in \mathrm{C}^{n\times n}$ given by

$$M=\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix},$$

$\overline{M}$ denotes the complex conjugate transpose of M ; the column string cs M is defined by

$$\mathrm{cs}\ M \equiv [\ m_{11}m_{21}\cdots m_{n1} \mid m_{12}m_{22}\cdots m_{n2} \mid \cdots \\ \mid m_{1n}m_{2n}\cdots m_{nn}]' \in \mathrm{C}^{n^2\times 1}.$$

Consider the following system:

$$\dot{x}(t)=A_0x(t)+A_1x(t-h), \qquad (1)$$

where $x(t)\in\mathbb{R}^n$ is a state and $h \geq 0$ is a state delay. The characteristic equation of Eq. (1) is given by

$$\det(sI-A_0-A_1e^{-sh})=0. \qquad (2)$$

A root of Eq. (2) will be also called a pole of Eq. (1).

First, the frequency domain stability condition [1] is given as follows.

Theorem 1: System (1) is stable if and only if all the roots of its characteristic equation are in the open left half of the complex plane.

Secondly, for the time domain stability condition, consider a Lyapunov functional $V(x(t), P, h)$ for Eq. (1) defined by

$$
\begin{aligned}
&V(x(t),\ P,h)\equiv\\
&x(t)'P_{00}x(t)+x(t)'\int_0^h P_{01}(s)x(t-h+s)ds \\
&+\int_0^h x(t-h+r)'P_{10}(r)dr,x(t)\\
&+\int_0^h\int_0^h x(t-h+r)'P_{11}(r,s)x(t-h+s)dsdr.
\end{aligned}
\tag{3}
$$

The following theorem [6] states the stability of Eq. (1) using the Lyapunov functional of Eq. (3).

Theorem 2: The system (1) is stable if and only if given $Q=Q'\in\mathbb{R}^{n\times n}>0$, there exists a unique self-adjoint P and $\varepsilon > 0$ such that

$$V(x(t),P,h)\ \geq\ \varepsilon x(t)'x(t) \tag{4}$$

$$\frac{d}{dt}V(x(t),P,h)=-x(t)'Qx(t) \tag{5}$$

for all $x(t)$.

Remark 1: The Lyapunov conditions of Eqs. (4) and (5) are not in the standard form [4]. In the standard form, for example, Eq. (4) should be changed to

$$
\begin{aligned}
V(x(t),\ \ &P,h)\geq\\
&\varepsilon_1 x(t)'x(t)+\\
&\varepsilon_2\int_0^h x(t-h+r)'x(t-h+r)dr
\end{aligned}
$$

where $\varepsilon_1>0$ and $\varepsilon_2>0$. However modified forms in Eqs. (4) and (5) are more convenient in many cases as in [7]. An explicit expression of the periodic properties of a Lyapunov functional is derived based on these modified forms.

A unique feature of a pure imaginary pole of Eq.(1) is its periodicity with respect to h: suppose Eq. (1) has a pure imaginary pole jw, $w>0$ for $h=h^*$, that is

$$\det(jwI-A_0-A_1e^{-jwh^*})=0. \tag{6}$$

Then defining

$$h_i\equiv\ h^*+2i\pi/w,\ \ i=1,2,\cdots, \tag{7}$$

we can show that Eq. (1) has the same pure imaginary pole jw for all h_i:

$$
\begin{aligned}
&\det(jwI-A_0-A_1e^{-jw(h^*+2i\pi/w)})=\\
&\quad\det(jwI-A_0-A_1e^{-jwh^*})=0.
\end{aligned}
\tag{8}
$$

In the next section, we will prove the following relationship, which is a main result of the paper.

Theorem 3: If Eq. (1) has a pure imaginary pole jw for $h=h^*$, there exists no Lyapunov functional $V(x(t),P,h_i)$ satisfying the Lyapunov conditions of Eqs. (4) and (5) for all h_i defined by Eq. (7).

Remark 2: The proof of Theorem 3 can also be obtained without the results in the next section. It is known that if Eq. (1) has a pure imaginary pole jw for h, then Lyapunov functional $V(x(t),P,h)$ does not exist uniquely. Combining this fact with Eq. (8), we can prove Theorem 3. However, this proof provides no insight into the periodic properties of a Lyapunov functional V. Our proof given in the next section will explicitly clarify the periodicity of a Lyapunov functional V.

3. Proof of Theorem 3

First, a way to characterize the solution of Eq. (5) is given in the next theorem [8].

Theorem 4: Given $Q=Q'$, a solution P to Eq. (5) is given by

$$
\begin{aligned}
&P_{00}=K(0),\\
&P_{01}(r)=K(r)A_1,\ \ 0\leq r\leq h,\\
&P_{11}(r,s)=\begin{cases}A_1'K(r-s)'A_1, & 0\leq s\leq r\leq h\\ A_1'K(s-r)A_1, & 0\leq r\leq s\leq h\end{cases}
\end{aligned}
\tag{9}
$$

where $K(r)$ is given by the following two equations:

$$\text{cs}\ \dot{K}(0)+\text{cs}\ \dot{K}(0)'=-\text{cs}Q \tag{10}$$

$$
\begin{bmatrix}\frac{d}{dr}\text{cs}K(r)\\ \frac{d}{dr}\text{cs}K(h-r)\end{bmatrix}=H\begin{bmatrix}\text{cs}K(r)\\ \text{cs}K(h-r)\end{bmatrix}
\qquad 0\leq r\leq h.
\tag{11}
$$

Matrix H is defined as follows. Let E_{ij} denote an $n\times n$ matrix with (i,j)-entry equal to 1 and all other entries equal to zero, and let $E\in R^{n^2\times n^2}$ be the block matrix $E\equiv[E_{ji}]$ (i.e., the (i,j)-block of E is E_{ji}). Let $D_0\in R^{n^2\times n^2}$ and $D_1\in R^{n^2\times n^2}$ be defined by

$$
\begin{aligned}
D_0\ &\equiv\ \text{diag}(A_0',\cdots,A_0')\\
D_1\ &\equiv\ \text{diag}(A_1',\cdots,A_1')
\end{aligned}
$$

respectively. The matrix $H\in R^{2n^2\times 2n^2}$ is defined as follows:

$$H\equiv\begin{bmatrix}D_0 & D_1E\\ -D_1E & -D_0\end{bmatrix}. \tag{12}$$

Remark 3: A solution of Eq. (5) can be found by solving the linear equation of Eq. (10) and the differential equation of Eq. (11) simultaneously.

The next lemma relates a pure imaginary pole of Eq. (1) and a Lyapunov equation of Eq. (11) by showing that a pure imaginary pole of (1) is an eigenvalue of the matrix H.

Lemma 1: If jw, $w \in \mathbb{R}$ is a pure imaginary pole of Eq. (1), then it is an eigenvalue of H.

Proof: We will show that there exists $v \neq 0 \in C^{2n^2}$ such that $(jwI - H)v = 0$. Since jw is a pure imaginary pole of Eq. (1), there is $x \neq 0 \in C^n$ such that

$$(jwI - A_0 - A_1 e^{-jwh})x = 0. \tag{13}$$

Let $\alpha \in C^n$ be defined by

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \equiv x e^{-\frac{jwh}{2}}, \tag{14}$$

where α_i, $1 \leq i \leq n$ is a complex number. Let v be defined by

$$v \equiv \begin{bmatrix} u \\ \overline{u} \end{bmatrix}, \tag{15}$$

where

$$u \equiv \begin{bmatrix} \overline{\alpha}_1 x \\ \overline{\alpha}_2 x \\ \vdots \\ \overline{\alpha}_n x \end{bmatrix} \in C^{n^2}. \tag{16}$$

It is obvious $v \neq 0$ from the construction of v and the fact $x \neq 0$. We will show that this v satisfies $(jwI - H)v = 0$. From the definition of H, we obtain

$$\begin{aligned}(jwI - H)v &= \begin{bmatrix} jw - D_0 & -D_1 E \\ D_1 E & jwI + D_0 \end{bmatrix} v \\ &= \begin{bmatrix} (jwI - D_0)u - D_1 E \overline{u} \\ (jwI + D_0)\overline{u} + D_1 E u \end{bmatrix}. \end{aligned} \tag{17}$$

Partition $(jwI - H)v$ into $2n$ complex vectors and let the i-th block of $(jwI - H)v$ be denoted by $r_i \in C^n$. Then r_i, $1 \leq i \leq n$ is given by

$$\begin{aligned} r_i &= (jwI - A_0)\overline{\alpha}_i x \\ &\quad - A_1(E_{1i}\alpha_1 + E_{2i}\alpha_2 + \cdots + E_{ni}\alpha_n)\overline{x}. \end{aligned}$$

Noting the following relationship

$$\begin{aligned} &(E_{1i}\alpha_1 + E_{2i}\alpha_2 + \cdots + E_{ni}\alpha_n)\overline{x} \\ &= (E_{1i}\alpha_1 + E_{2i}\alpha_2 + \cdots + E_{ni}\alpha_n)\overline{\alpha} e^{-\frac{jwh}{2}} \\ &= e^{-\frac{jwh}{2}} \overline{\alpha}_i \alpha. \end{aligned}$$

we obtain

$$\begin{aligned} r_i &= (jwI - A_0)\overline{\alpha}_i \alpha e^{\frac{jwh}{2}} - \overline{\alpha}_i A_1 \alpha e^{-\frac{jwh}{2}} \\ &= \overline{\alpha}_i e^{\frac{jwh}{2}}(jwI - A_0 - A_1 e^{-jwh})\alpha \\ &= \overline{\alpha}_i (jwI - A_0 - A_1 e^{-jwh})x = 0, 1 \leq i \leq n. \end{aligned}$$

The last equality is from Eq. (13). Since $r_{i+n} = -\overline{r}_i$, $1 \leq i \leq n$ (see Eq. (17)), we have $r_i = 0, n+1 \leq i \leq 2n$. Hence, $(jwI - H)v = 0$ where $v \neq 0$.

Remark 4: In Lemma 4, we obtain a method to compute a pure imaginary pole of Eq. (1), which can be used to derive a delay-dependent stability condition [9]. In [10], similar methods for computing a pure imaginary pole are proposed by investigating properties of the polynomial in Eq. (2). It is interesting to note that the imaginary poles of Eq. (2) can also be computed from a constant matrix, which is used to compute a Lyapunov functional.

The next lemma explicitly shows that no unique Lyapunov functional satisfying Eq. (5) exists when Eq. (1) has a pure imaginary pole.

Lemma 2: If Eq. (1) has a pure imaginary pole jw, then no unique Lyapunov functional $V(x(t), P, h)$ satisfying Eq. (5) exists.

Proof: No unique Lyapunov functional satisfying Eq. (5) exists if $K(r)$ satisfying Eqs. (10) and (11) does not exist uniquely. A solution $K(r)$ to Eqs. (10) and (11) can be formulated using a linear relation between Q and $K(0)$ as follows:

$$Q = L_1(h)(K(0)), \tag{18}$$

where $L_1(h)$ denotes a linear operator.

Eq. (18) has a unique solution if and only if the equation

$$0 = L_1(K(0)) \tag{19}$$

has a unique solution $K(0) = 0$. Hence if Eq. (19) has a nonzero solution $K(0)$, then no unique $K(r)$ satisfying Eqs. (10) and (11) exists. A nonzero solution will be constructed in the next. Using the definitions of x and α in the proof of Lemma 1, let v_1 be defined by

$$v_1 \equiv v e^{-\frac{jwh}{2}},$$

then

$$v_1 = \begin{bmatrix} u_1 \\ \overline{u_1} e^{-jwh} \end{bmatrix} \text{ and } (jwI - H)v_1 = 0$$

where

$$u_1 \equiv \begin{bmatrix} \bar{\alpha}_1 \alpha \\ \vdots \\ \bar{\alpha}_n \alpha \end{bmatrix}.$$

Let cs $K(r)$ be defined by

$$\text{cs } K(r) \equiv e^{jwr} u_1 + e^{-jwr} \bar{u}_1, \tag{20}$$

then cs $K(r)$ satisfies Eq. (11). To see this, first note that

$$\begin{aligned} \begin{bmatrix} \text{cs } K(r) \\ \text{cs } K(h-r) \end{bmatrix} &= \begin{bmatrix} e^{jwr} \\ e^{jw(h-r)} \end{bmatrix} u_1 \\ &\quad + \begin{bmatrix} e^{-jwr} \\ e^{-jw(h-r)} \end{bmatrix} \bar{u}_1 \\ &= e^{jwr} v_1 + e^{-jwr} \bar{v}_1. \end{aligned}$$

Then the left side of Eq. (11) is given by

$$\begin{bmatrix} \frac{d}{dr} \text{cs } K(r) \\ \frac{d}{dr} \text{cs } K(h-r) \end{bmatrix} = jwe^{jwr} v_1 - jwe^{-jwr} \bar{v}_1,$$

while the right side of (11) is given by

$$H(e^{jwr} v_1 + e^{-jwr} \bar{v}_1) = e^{jwr} H v_1 + e^{-jwr} H \bar{v}_1. \tag{21}$$

Since H is a real matrix, we have $(-jwI - H)\bar{v}_1 = 0$. Hence the left side and the right side of Eq. (11) coincide; that is, cs $K(r)$ defined by Eq. (20) satisfies Eq. (11). Now if we show $K(r)$ satisfies Eq. (10) with $Q=0$, then $K(0)$ is a solution to Eq. (19). That is, we have to show the following:

$$\text{cs } 0 = \text{cs } \dot{K}(0) + \text{cs } \dot{K}(0)'. \tag{22}$$

Note that (from Eqs. (10) and (21))

$$\begin{aligned} \text{cs } \dot{K}(r) &= [I \ 0] H \begin{bmatrix} \text{cs } K(r) \\ \text{cs } K(h-r) \end{bmatrix} \\ &= jw(e^{jwr} u_1 - e^{-jwr} \bar{u}_1), \end{aligned}$$

and thus

$$\text{cs } \dot{K}(0) = jw(u_1 - \bar{u}_1). \tag{23}$$

From the definition of cs and the definition of u_1, we have

$$\text{cs } \dot{K}(0)' = jw(\bar{u}_1 - u_1). \tag{24}$$

From Eqs. (23) and (24), we obtain Eq. (22). Therefore cs $K(0) = u_1 + \bar{u}_1$ is a solution to Eq. (19). It remains to show that $K(0) \neq 0$. From $\alpha \neq 0$, at least one diagonal element of $K(0)$ is not zero since the i-th diagonal element of $K(0)$ is $2\alpha_i \bar{\alpha}_i$.

Now we are ready to prove Theorem 3.

Proof of Theorem 3: In the proof of Lemma 2, all the terms dependent on h are in the form of e^{jwh}. Hence if Lemma 2 is satisfied for $h = h^*$, then Lemma 2 is also satisfied for all h_i defined by Eq. (7) from the fact that $e^{-jw(h^* + 2i\pi/w)} = e^{-jwh^*}$.

4. Numerical example

Example 1: Consider the following system:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -0.5 & -0.5 \\ 0.9 & -1.5 \end{bmatrix} x(t) \\ &\quad + \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} x(t-h). \end{aligned} \tag{25}$$

Constructing H as defined in Eq. (12), we obtain

$$H = \begin{bmatrix} -0.5 & 0.9 & 0 & 0 & -2 & 0 & -1 & 0 \\ -0.5 & -1.5 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -0.5 & 0.9 & 0 & -2 & 0 & -1 \\ 0 & 0 & -0.5 & -1.5 & 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0.5 & -0.9 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0.5 & 1.5 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0.5 & -0.9 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0.5 & 1.5 \end{bmatrix}.$$

Eigenvalues of H are computed as

$$\{\pm 1.3333j, 1.1035 \pm 0.3458j, -1.1035 \pm 0.3458j, \pm 1.3346\}$$

and the following is satisfied with $x = [0.8683, 0.4726 - 0.1510j]'$ and $h^* = 1.6834$:

$$(1.3333j - A_0 - A_1 e^{-j1.3333h^*})x = 0.$$

This verifies Lemma 1: $1.3333j$, the imaginary pole of Eq. (25), is an eigenvalue of H. From the proof of Lemma 2, we can show that $K(r)$ defined by Eq. (20) is a nonzero solution to Eq. (19), where $K(0)$ is given by

$$K(0) = \begin{bmatrix} 0.6539 & 0.1196 \\ 0.5922 & 0.2135 \end{bmatrix}.$$

5. Conclusion

In this paper, periodic properties of a Lyapunov functional are shown by investigating a solution of a Lyapunov equation. This result clarifies the relationship between the frequency domain stability conditions and the time domain conditions. As a byproduct of the result, an easy way to compute pure imaginary poles of a state delay system is also obtained.

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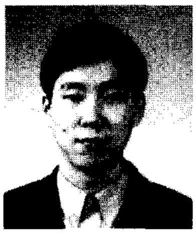

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