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ON k-HYPONORMAL WEIGHTED TRANSLATION SEMIGROUPS

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ABSTRACT. The weighted shifts have been made a central role in the study of k-hyponormality for hyponormal and subnormal operators in discrete notion. In this note we discuss the k-hyponormality of weighted translation semigroups with a symbol function in continuous notion.

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $A, B \in B(\mathcal{H})$, let $[A, B] := AB - BA$. We say that an n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of operators in $B(\mathcal{H})$ is *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (n-copies). For a natural number k and $T \in B(\mathcal{H})$, T is *k-hyponormal* if $(I, T, \cdots, T^k)$ is hyponormal. It is well-known that subnormal $\Rightarrow$ k-hyponormal $\Rightarrow$ hyponormal, for every $k \geq 1$; the study for the gaps among the above classes was discussed in [8], [9], [10] and [11]. In particular, unilateral weighted shifts were considered to study such gab theory (cf. [8], [9], [10], [11] and [12]). The weighted shifts were good roles in the study of bridges between hyponormal and subnormal operators. So it is valuable to study other models in this topic. In this note, we discuss a weighted translation semigroup which is much wider notion than weighted shifts.

Let $\mathbf{R}_+$ be the set of nonnegative real numbers and $L^2 := L^2(\mathbf{R}_+)$ the Hilbert space of square integrable Lebesgue measurable complex valued functions on $\mathbf{R}_+$. Let $B(L^2)$ be the algebra of all bounded linear operators on L^2. A family $\{S_t : t \in \mathbf{R}_+\}$ in $B(L^2)$ is a *semigroup* if $S_0 = I$ and $S_t S_s = S_{t+s}$ for all t and s in $\mathbf{R}_+$. In particular, we consider a weighted translation semigroup $\{S_t\}$ on L^2 defined by $(S_t f)(x) = (\phi(x)/\phi(x-t))f(x-t)$ if $x \geq t$ and 0 otherwise, where ϕ

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is a continuous nonzero complex-valued function on $\mathbf{R}_+$ and is called a symbol. A semigroup $\{S_t\}$ is *strongly continuous* if, for each f in L^2, the mapping $t \to S_t f$ is continuous from $\mathbf{R}_+$ into L^2. It is well-known that $\{S_t\}$ is strongly continuous on $\mathbf{R}_+$ if and only if $\{S_t\}$ is strongly continuous at $t = 0$. Since the weighted translation semigroups with symbols ϕ and $|\phi|$ are unitarily equivalent, we shall assume throughout this paper that all symbols of weighted translation semigroups are positive valued strongly continuous semigroups.

Let $\{S_t\}$ be a weighted translation semigroup with symbol ϕ. For each $t \geq 0$ and each nonnegative integer n, let $X_n^{(t)}$ be the characteristic function of the interval $[nt, (n+1)t]$. Set M_t equal to the closed linear span of $\{\phi \cdot X_n^{(t)} : n \geq 0\}$. Let

$$\lambda_n^{(t)} = \left[\int_{nt}^{(n+1)t} \phi(x)^2 dx \right]^{\frac{1}{2}}.$$

Then for each t, M_t is invariant subspace for S_t, and the restriction $S_t|M_t$ to M_t is a weighted shift with weight sequence $\{\lambda_0^{(t)}, \frac{\lambda_1^{(t)}}{\lambda_0^{(t)}}, \frac{\lambda_2^{(t)}}{\lambda_1^{(t)}}, \ldots\}$. It follows from [1, Theorem 3.2] that $\{S_t\}$ is hyponormal if and only if $\{S_t|M_t\}$ is hyponormal, which is generalized to the following lemma.

LEMMA 1. *Let $\{S_t\}$ be a weighted translation semigroup with symbol ϕ. Then $\{S_t\}$ is k-hyponormal if and only if $\{S_t|M_t\}$ is k-hyponormal, $1 \leq k \leq \infty$.*

Proof. Mimic the proof of [1, Theorem 3.2]. □

The following is the main theorem of this paper.

THEOREM 2. *Let $\{S_t\}$ be a weighted translation semigroup with symbol ϕ. Then the following statements are equivalent:*

(i) *$\{S_t\}$ is k-hyponormal;*

(ii) *the following holds*

$$D_{\phi,k}(x,t) := \begin{pmatrix} \phi^2(x) & \phi^2(x+t) & \cdots & \phi^2(x+kt) \\ \phi^2(x+t) & \phi^2(x+2t) & \cdots & \phi^2(x+(k+1)t) \\ \vdots & \vdots & \ddots & \vdots \\ \phi^2(x+kt) & \phi^2(x+(k+1)t) & \cdots & \phi^2(x+2kt) \end{pmatrix} \geq 0, \text{ for all t} \geq 0.$$

To prove Theorem 2, we need several lemmas. We begin with the following lemma, which means that the Embry type and Bram-Halmos type for subnormality are equivalent in the case of weighted translation semigroups (cf. [13]).

LEMMA 3. *Let $\{S_t\}$ be a weighted translation semigroup with symbol ϕ. Then the following statements are equivalent:*
(i) *$\{S_t\}$ is k-hyponormal;*
(ii) *$\{S_t|M_t\}$ is k-hyponormal;*
(iii) $\sum_{i,j}\left\langle S_t^{i+j}f_i, S_t^{i+j}f_j\right\rangle \geq 0$ *for all* $f_i, f_j \in M_t$, $0 \leq i,j \leq k, t \geq 0$;
(iv) $\sum_{i,j}\left\langle S_t^{i+j}f_i, S_t^{i+j}f_j\right\rangle \geq 0$ *for all* $f_i, f_j \in L^2$, $0 \leq i,j \leq k$.

Proof. (i) $\Leftrightarrow$ (ii): It is Lemma 1.
(ii) $\Leftrightarrow$ (iii): It follows from the fact in [13] that if T is a weighted shift on a Hilbert space $\mathcal{H}$, then T is k-hyponormal if and only if

$$\langle T^{i+j}f_j, T^{i+j}f_i\rangle \geq 0 \quad \text{for all} \quad f_i, f_j \in \mathcal{H},\ 1 \leq i,j \leq k.$$

(iii) $\Leftrightarrow$ (iv): Since $\{\phi\chi_n^{(t)} : nt \geq 0\}$ is dense in $L^2(\mathbf{R}_+)$, it is obvious. □

LEMMA 4. *Let $\{S_t\}$ be a weighted translation semigroup with symbol ϕ. Then*

$$\langle S_t^n f, S_t^n g\rangle = \int_0^\infty \left|\frac{\phi(x+nt)}{\phi(x)}\right|^2 f(x)\overline{g(x)}dx.$$

Proof. Since

$$\left(S_t^n f\right)(x) = \begin{cases} \frac{\phi(x)}{\phi(x-nt)}f(x-nt) & \text{if } x \geq nt, \\ 0 & \text{if } 0 \leq x < nt, \end{cases}$$

we have

$$\begin{aligned} \langle S_t^n f, S_t^n g\rangle &= \int_{nt}^\infty \left|\frac{\phi(x)}{\phi(x-nt)}\right|^2 f(x-nt)\overline{g(x-nt)}dx \\ &= \int_0^\infty \left|\frac{\phi(x+nt)}{\phi(x)}\right|^2 f(x)\overline{g(x)}dx. \end{aligned}$$

□

Recall (cf. [8]) that if

$$T := \begin{pmatrix} I & B^* \\ B & C \end{pmatrix} \in L(\mathcal{H}\oplus\mathcal{H}), \tag{1}$$

then $T > 0 \Longleftrightarrow C - BB^* > 0$.

LEMMA 5. *Assume that* $a_{ij}(x) = \overline{a_{ji}(x)}$, *for any* $0 \le i,j \le k$. *Then the following statements are equivalent*:

(i) *for any* $f_i \in L^2(\mathbf{R}_+)$, *we have*

$$\sum_{i,j=0}^{k} \int_0^\infty a_{ji}(x) f_i(x) \overline{f_j(x)}\, dx \ge 0; \tag{2}$$

(ii) *it holds that*

$$\begin{pmatrix} a_{00}(x) & a_{01}(x) & \cdots & a_{0k}(x) \\ a_{10}(x) & a_{11}(x) & \cdots & a_{1k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{k0}(x) & a_{k1}(x) & \cdots & a_{kk}(x) \end{pmatrix} \ge 0, \quad \text{for } x \text{ a.e. in } \mathbf{R}_+. \tag{3}$$

Proof. Without loss of generality, we may assume that $a_{00}(x) = 1$. We will use the mathematical induction to prove the lemma. For $k = 1$, first observe that

$$\begin{aligned} &\sum_{i,j=0}^{1} \int_0^\infty a_{ji}(x) f_i(x) \overline{f_j(x)}\, dx \\ = &\int_0^\infty (|f_0(x)|^2 + a_{01} f_1(x) \overline{f_0(x)} + a_{10}(x) f_0(x) \overline{f_1(x)} \\ &+ a_{11}(x) |f_1(x)|^2)\, dx \\ = &\int_0^\infty |f_0(x) + a_{01} f_1(x)|^2 dx + \int_0^\infty (a_{11}(x) - |a_{01}(x)|^2) |f_1(x)|^2 dx. \end{aligned}$$

If $a_{11}(x) - |a_{01}(x)|^2 \ge 0$, then

$$\sum_{i,j=0}^{1} \int_0^\infty a_{ji}(x) f_i(x) \overline{f_j(x)}\, dx \ge 0.$$

Conversely, suppose $a_{11}(x) - |a_{01}(x)|^2 < -\varepsilon$ on nonzero measure set E and some $\varepsilon > 0$. Let $f_1(x) = \chi_E$ and $f_0(x) = -a_{01}(x)\chi_E$. Then

$$\begin{aligned} &\sum_{i,j=0}^{1} \int_0^\infty a_{ji}(x) f_i(x) \overline{f_j(x)}\, dx \\ = &\int_0^\infty (a_{11}(x) - |a_{01}(x)|^2) |f_1(x)|^2 dx < -\varepsilon \mu(E)^2 < 0. \end{aligned}$$

Hence, we have that

$$a_{11}(x) - |a_{01}(x)|^2 \geq 0 \Longleftrightarrow \sum_{i,j=0}^{1} \int_0^\infty a_{ji}(x) f_i(x) \overline{f_j(x)}\, dx \geq 0,$$

which proves the case $k = 1$. Assume that (2) and (3) are equivalent for the case $k = n - 1$. And we will consider the case $k = n$. Then

$$\begin{aligned}
& \sum_{i,j=0}^{n} \int_0^\infty a_{ji}(x) f_i(x) \overline{f_j(x)}\, dx \\
= & \int_0^\infty |f_0(x) + a_{01} f_1(x) + \cdots + a_{0n}(x) f_n(x)|^2 dx \\
& + \sum_{i,j=1}^{n} \int_0^\infty a_{ji}(x) f_i(x) \overline{f_j(x)}\, dx - \sum_{i,j=1}^{n} \int_0^\infty a_{i0}(x) a_{0j}(x) f_i(x) \overline{f_j(x)}\, dx \\
= & \int_0^\infty |f_0(x) + a_{01} f_1(x) + \cdots + a_{0n}(x) f_n(x)|^2 dx \\
& + \sum_{i,j=1}^{n} \int_0^\infty (a_{ji}(x) - a_{j0}(x) a_{0i}(x)) f_i(x) \overline{f_j(x)}\, dx \\
= & : A + B.
\end{aligned}$$

We will claim that $B \geq 0$ is equivalent to

$$\sum_{i,j=0}^{n} \int_0^\infty a_{ji}(x) f_i(x) \overline{f_j(x)}\, dx \geq 0 \text{ for any } f_i \in L^2, 0 \leq i \leq n.$$

Suppose there exist functions $f_i \in L^2$, $1 \leq i \leq n$, such that

$$\sum_{i,j=1}^{n} \int_0^\infty (a_{ji}(x) - a_{j0}(x) a_{0i}(x)) f_i(x) \overline{f_j(x)}\, dx < -\varepsilon \text{ for some } \varepsilon > 0.$$

Take

$$f_0(x) = -a_{01}(x) f_1(x) - \cdots - a_{0n}(x) f_n(x).$$

Since $A = 0$, we have

$$\sum_{i,j=0}^{n} \int_0^\infty a_{ji}(x) f_i(x) \overline{f_j(x)}\, dx < -\varepsilon,$$

which proves the claim. Also by mathematical induction, $B \geq 0$ is equivalent to the $n \times n$ matrix

$$\tag{4} \begin{pmatrix} a_{11} - a_{10}a_{01} & a_{12} - a_{10}a_{02} & \cdots & a_{1n} - a_{10}a_{0n} \\ a_{21} - a_{20}a_{01} & a_{22} - a_{20}a_{02} & \cdots & a_{2n} - a_{20}a_{0n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} - a_{n0}a_{01} & a_{n2} - a_{n0}a_{02} & \cdots & a_{nn} - a_{n0}a_{0n} \end{pmatrix} \geq 0.$$

Moreover, by (1), the positivity of (4) is equivalent to the following $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} 1 & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{pmatrix} \geq 0.$$

So the proof is complete. $\square$

Proof of Theorem 2. According to Lemmas 3, 4, and 5, we have that $\{S_t\}$ is k-hyponormal if and only if

$$\begin{pmatrix} 1 & \frac{\phi^2(x+t)}{\phi^2(x)} & \cdots & \frac{\phi^2(x+kt)}{\phi^2(x)} \\ \frac{\phi^2(x+t)}{\phi^2(x)} & \frac{\phi^2(x+2t)}{\phi^2(x)} & \cdots & \frac{\phi^2(x+(k+1)t)}{\phi^2(x)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\phi^2(x+kt)}{\phi^2(x)} & \frac{\phi^2(x+(k+1)t)}{\phi^2(x)} & \cdots & \frac{\phi^2(x+2kt)}{\phi^2(x)} \end{pmatrix} \geq 0,$$

which proves Theorem 2. $\square$

COROLLARY 6 ([2]). *Let $\{S_t\}$ be a weighted translation semigroup with symbol ϕ. Then $\{S_t\}$ is hyponormal if and only if $\phi(x)\phi(x+2t) \geq \phi^2(x+t)$ for any $x, t \in \mathbf{R}_+$, which is equivalent to $\ln \phi$ is convex.*

COROLLARY 7. *Let $\{S_t\}$ be a weighted translation semigroup with symbol ϕ. Then $\phi(x)\phi(x+2t) = \phi^2(x+t)$ for any $x, t \in \mathbf{R}_+$ if and only if $\ln \phi$ is a line.*

Proof. Since $\phi(x)\phi(x+2t) = \phi^2(x+t)$ is equivalent to $\frac{1}{2}(\ln \phi(x) + \ln \phi(x+2t)) = \ln \phi(x+t)$, the proof follows easily. $\square$

COROLLARY 8. *Let $\{S_t\}$ be a weighted translation semigroup with symbol ϕ. If $\det D_{\phi,1}(x,t) = 0$, i.e., $\ln \phi$ is a line, then $\det D_{\phi,n}(x,t) = 0$, for all $n > 2$.*

Proof. Let $\ln \phi(x) = ax + b$. Then $\phi(x) = e^{ax}e^b$. Hence

$$D_{\phi,k}(x,t) = \begin{pmatrix} e^{2ax}e^{2b} & e^{2a(x+t)}e^{2b} & \cdots & e^{2a(x+kt)}e^{2b} \\ e^{2a(x+t)}e^{2b} & e^{2a(x+2t)}e^{2b} & \cdots & e^{2a(x+(k+1)t)}e^{2b} \\ \vdots & \vdots & \ddots & \vdots \\ e^{2a(x+kt)}e^{2b} & e^{2a(x+(k+1)t)}e^{2b} & \cdots & e^{2a(x+2kt)}e^{2b} \end{pmatrix}$$

$$(1) \qquad = e^{2b+2ax} \begin{pmatrix} 1 & e^{2at} & \cdots & e^{2kat} \\ e^{2at} & e^{4at} & \cdots & e^{2(k+1)at} \\ \vdots & \vdots & \ddots & \vdots \\ e^{2kat} & e^{2(k+1)at} & \cdots & e^{4akt} \end{pmatrix},$$

and so each columns in the above matrix are linearly dependent. Thus $\det D_{\phi,k}(x,t) = 0$ for $k \geq 2$. □

REMARK 9. Let $\phi(x) = \frac{1}{x+1}$ and let $D_{\phi,k}$ be the matrix in Theorem 2 (ii). Since the corresponding semigroup $\{S_t\}$ is subnormal (cf. [2]), obviously $D_{\phi,k}(x,t) \geq 0$ for any $x, t \in \mathbf{R}^+$. In fact, by direct computations we have $\det D_{\phi,2}(x,t) > 0$, $\det D_{\phi,3}(x,t) > 0$, $\det D_{\phi,4}(x,t) > 0$, etc. Let $\phi(x) = xe^{-x}$. Then by a direct computation $\det D_{\phi,2}(x,t) = 0, \det D_{\phi,3}(x,t) = 0$, etc., even $\ln \phi(x)$ is not a line.

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