ON WEYL SPECTRA OF ALGEBRAICALLY TOTALLY-PARANORMAL OPERATORS

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Dedicated to Professor Yong Tae Kim on his 65th birthday

Abstract. In this paper we show that Weyl’s theorem holds for \( f(T) \) when an Hilbert space operator \( T \) is “algebraically totally-paranormal” and \( f \) is any analytic function on an open neighborhood of the spectrum of \( T \).

1. Introduction

Throughout this paper let \( \mathcal{L}(\mathcal{H}) \) denote the algebra of bounded linear operators acting on an infinite dimensional Hilbert space \( \mathcal{H} \). If \( T \in \mathcal{L}(\mathcal{H}) \) write \( N(T) \) and \( R(T) \) for the null space and range of \( T \); \( \sigma(T) \) for the spectrum of \( T \); \( \pi_0(T) \) for the set of eigenvalues of \( T \); \( \pi_{00}(T) \) for the isolated points of \( \sigma(T) \) which are eigenvalues of finite multiplicity. Recall ([5], [7]) that an operator \( T \in \mathcal{L}(\mathcal{H}) \) is called Fredholm if it has closed range with finite dimensional null space and its range of finite co-dimension. The index of a Fredholm operator \( T \in \mathcal{L}(\mathcal{H}) \) is given by

\[
\text{ind}(T) = \dim N(T) - \dim R(T) \uparrow \quad (= \dim N(T) - \dim N(T^*))
\]

An operator \( T \in \mathcal{L}(\mathcal{H}) \) is called Weyl if it is Fredholm of index zero. An operator \( T \in \mathcal{L}(\mathcal{H}) \) is called Browder if it is Fredholm “of finite ascent and descent”: equivalently, if \( T \) is Fredholm and \( T - \lambda I \) is invertible for sufficiently small \( \lambda \neq 0 \) in \( \mathbb{C} \). The essential spectrum \( \sigma_e(T) \), the

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Weyl spectrum $\omega(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in \mathcal{L}(\mathcal{H})$ are defined by

$$
\sigma_c(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm} \},
$$

$$
\omega(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl} \},
$$

$$
\sigma_b(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder} \},
$$

$$
\sigma_c(T) \subseteq \omega(T) \subseteq \sigma_b(T) = \sigma_c(T) \cup \text{acc} \sigma(T),
$$

where we write $\text{acc} \mathbf{K}$ for the accumulation points of $\mathbf{K} \subseteq \mathbb{C}$. Following Coburn ([1]) we say that Weyl’s theorem holds for $T \in \mathcal{L}(\mathcal{H})$ if there is equality

$$
\sigma(T) \setminus \omega(T) = \pi_{00}(T).
$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of $T$. Recall ([8]) that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *totally-paranormal* if

$$
||(T - \lambda)x||^2 \leq ||(T - \lambda)^2 x|| ||x|| \text{ for all } x \in \mathcal{H} \text{ and } \lambda \in \mathbb{C}.
$$

We shall say that the operator $T \in \mathcal{L}(\mathcal{H})$ is *algebraically totally-paranormal* if there exists a nonconstant complex polynomial $p$ such that $p(T)$ is totally-paranormal. Evidently,

$$
\{ \text{hyponormal operators} \} \subseteq \{ \text{totally-paranormal operators} \}
$$

and

$$
\{ \text{algebraically hyponormal operators} \} \subseteq \{ \text{algebraically totally-paranormal operators} \}.
$$

From well-known facts (cf. [8]) of totally-paranormal operators we easily see that

(a) If $T \in \mathcal{L}(\mathcal{H})$ is algebraically totally-paranormal, then so is $T - \lambda I$ for each $\lambda \in \mathbb{C}$.

(b) If $T \in \mathcal{L}(\mathcal{H})$ is algebraically totally-paranormal and $\mathcal{M} \subseteq \mathcal{H}$ is invariant under $T$, then $T|\mathcal{M}$ is algebraically totally-paranormal.

(c) Unitary equivalence preserves algebraical totally-paranormality.

In [4] Han and Lee showed that Weyl’s theorem holds for $f(T)$ when $T$ is an algebraically hyponormal operator and $f$ is an analytic function on an open neighborhood of $\sigma(T)$.

In this paper we extend this result to algebraically totally-paranormal operators: our proof however differs from the correspondence in [4], in that we employ techniques from local spectral theory.

The following is our main result.
THEOREM. If $T \in \mathcal{L}(\mathcal{H})$ is algebraically totally-paranormal, then for every $f \in H(\sigma(T))$, Weyl's theorem holds for $f(T)$, where $H(\sigma(T))$ denotes the set of analytic functions on an open neighborhood of $\sigma(T)$.

2. Proofs

The following two lemmas give important and essential facts for algebraically totally-paranormal operators but its proofs are routine and similar to that of Han and Lee ([4]). Thus we shall just state them without proofs.

The following result is an extension of [4, Lemma 1] to algebraically totally-paranormal operators.

**Lemma 1.** Suppose $T \in \mathcal{L}(\mathcal{H})$.
(i) If $T$ is algebraically totally-paranormal and quasinilpotent, then $T$ is nilpotent.
(ii) If $T$ is algebraically totally-paranormal, then $T$ is isoloid.
(iii) If $T$ is algebraically totally-paranormal, then $T$ has finite ascent.

The following result is an extension of [4, Theorem 3] to algebraically totally-paranormal operators.

**Lemma 2.** If $T \in \mathcal{L}(\mathcal{H})$ is algebraically totally-paranormal, then

$$\omega(f(T)) = f(\omega(T)) \quad \text{for every } f \in H(\sigma(T)).$$

To state next lemma we need some notions from local spectral theory. We say that $T \in \mathcal{L}(\mathcal{H})$ has the single valued extension property (SVEP) if there is implication, for arbitrary open sets $U \subseteq \mathbb{C}$ and holomorphic functions $f : U \to \mathcal{H}$,

$$(T - zI)f(z) = 0 \text{ on } U \implies f(z) = 0 \text{ on } U.$$

If this holds for a neighborhood $U$ of $\lambda \in \mathbb{C}$ we say that $T$ has the SVEP at $\lambda$.

We introduce two important subsets of $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$ and $F$ is a closed set in $\mathbb{C}$, we define

$$\mathcal{H}_T(F) = \{x \in \mathcal{H} : \text{there exists an analytic } \mathcal{H}\text{-valued function } f : \mathbb{C} \setminus F \to \mathcal{H} \text{ such that } (T - \lambda)f(\lambda) = x\}.$$
Then $H_T(F)$ is said to be the spectral manifold of $T$. If $T$ has the SVEP, then the above definition is identical with $H_T(F) = \{ x \in H : \sigma_T(x) \subseteq F \}$, where $\sigma_T(x)$ is the local spectrum of $T$ at $x$. (see [2], [3], [8], [9] for details)

Let $H_o(T) = \{ x \in H : \| T^n x \| \rightarrow 0 \}$. If $H_o(T) = H$, then $T$ is a quasinilpotent operator on $H$ ([2, p.28. Lemma]). Now we are ready for the following result.

**Lemma 3.** Weyl's theorem holds for every algebraically totally-paranormal operator.

**Proof.** Suppose $p(T)$ is totally-paranormal for some nonconstant polynomial $p$. We first prove that $\pi_{00}(T) \subseteq \sigma(T) \setminus \omega(T)$. Without loss of generality, it suffices to show that

$$0 \in \pi_{00}(T) \implies T \text{ is Weyl but not invertible.}$$

Suppose $0 \in \pi_{00}(T)$. Since $0 \in \text{iso}(T)$, we can consider the Riesz spectral projection $P_0$ with respect to $0$ ([7, Theorem 49.1; Proposition 49.1]) such that

$$R(P_0) = H_o(T), \quad (T)|_{N(P_0)} \text{ is invertible, and } H = R(P_0) \oplus N(P_0).$$

It is well known ([8, Proposition 1.8]) that if $T$ has finite ascent, then it has the SVEP at $0$. It is well known ([8, Corollary 2.4]) that if $T$ has the SVEP at $0$, then

$$H_T(\{0\}) = H_o(T).$$

Thus we have

$$R(P_0) = H_o(T) = H_T(\{0\}).$$

By hypothesis $R(T)$ is closed and $0 \in \pi_0(T)$, and so $T$ is semi-Fredholm. Then since $H_T(\{0\})$ is closed, we have by [9, Theorem 2]

$$R(P_0) = H_T(\{0\}) \text{ is finite dimensional.}$$

Thus the restrictions of $T$ to reducing subsets $R(P_0)$ and $N(P_0)$ are finite dimensional and invertible operators, respectively. So we can see that $T$ is Weyl but not invertible. Hence we have that $\pi_{00}(T) \subseteq \sigma(T) \setminus \omega(T)$.

For the reverse inclusion, suppose $0 \in \sigma(T) \setminus \omega(T)$. Thus $T$ is Weyl. Since $T$ has a finite ascent, $T$ has also a finite descent by [10, Theorem 1(4)]. So $T$ is Weyl of finite ascent and descent, and then it is Browder. Therefore $0 \in \pi_{00}(T)$. This completes the proof. $\square$
Now we conclude with the proof of Theorem.

Proof of Theorem. Remembering [12, Lemma] that if $T$ is isoloid, then

$$f(\sigma(T) \setminus \pi_{00}(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T))$$

for every $f \in H(\sigma(T))$; it follows from Lemma 1 (ii), Lemma 2 and Lemma 3 that

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = f(\omega(T)) = \omega(f(T)),$$

which implies that Weyl’s theorem holds for $f(T)$. □

References