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A FREQUENCY–DOMAIN METHOD FOR FINITE ELEMENT SOLUTIONS OF PARABOLIC PROBLEMS

Chang-Ock Lee, Jongwoo Lee, and Dongwoo Sheen

Abstract. We introduce and analyze a frequency-domain method for parabolic partial differential equations. The method is naturally parallelizable. After taking the Fourier transformation of given equations in the space-time domain into the space-frequency domain, we propose to solve an indefinite, complex elliptic problem for each frequency. Fourier inversion will then recover the solution in the space-time domain. Existence and uniqueness as well as error estimates are given. Fourier invertibility is also examined. Numerical experiments are presented.

1. Introduction

Let Ω be an open bounded Lipschitz domain in $\mathbb{R}^N$, $N = 2, 3$, with the $\Gamma = \partial\Omega$. Set $J = [0, \infty)$. We are interested in a parallel numerical method for the following parabolic problem:

$$\beta u_t - \nabla \cdot (\alpha \nabla u) = f, \quad \Omega \times J, \tag{1.1a}$$

$$u = 0, \quad \Gamma \times J, \tag{1.1b}$$

$$u(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{x} \in \Omega, \tag{1.1c}$$

where $\beta \in L^\infty(\Omega)$ and $\alpha \in W^{1,\infty}(\Omega)$ are defined on Ω, satisfying $\beta_* \le \beta \le \beta^*$, $\alpha_* \le \alpha \le \alpha^*$, $|\nabla\alpha| \le \alpha^*$, with positive constants $\beta_*, \beta^*, \alpha_*, \alpha^*$.

The most popular strategy to obtain numerical solutions of (1.1) may be to apply to the Problem (1.1) marching algorithms such as backward-Euler and Crank-Nicolson methods. Such traditional methods have proven to solve many practical problems effectively. These schemes,

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using the solutions at the previous time steps, solve elliptic problems in order to obtain the solutions at the current step. Most parallel algorithms are restricted to solving these elliptic systems in parallel; for example, domain decomposition methods are the most successful such methods. One of disadvantages of such approaches seems to arise from the nontriviality and expenses in reducing communication costs among processors.

This paper investigates an alternative method in order to reduce communication costs significantly by taking the Fourier transformation in the time variable of Problem (1.1).

Recall that the Fourier transform $\widehat{v}(\cdot,\omega)$ of a function $v(\cdot,t)$ in time is defined by

$$\widehat{v}(\cdot,\omega) = \int_{-\infty}^{\infty} v(\cdot,t)e^{-i\omega t}dt$$

while the Fourier inversion formula is given by

$$v(\cdot,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{v}(\cdot,\omega)e^{i\omega t}d\omega.$$

Note that if $v(\boldsymbol{x},t)$ is a real function, its Fourier transform satisfies the conjugate relation $\widehat{v}(\boldsymbol{x},-\omega) = \overline{\widehat{v}(\boldsymbol{x},\omega)}, \quad \omega \in \mathbb{R}$, from which the Fourier inversion formula takes the form

$$v(\boldsymbol{x},t) = \frac{1}{\pi}\mathrm{Re}\int_{0}^{\infty} \widehat{v}(\boldsymbol{x},\omega)e^{i\omega t}\,d\omega. \tag{1.2}$$

By extending f and u by zero to $t < 0$, the space-time formulation of the equations (1.1) is transformed to a space-frequency formulation by the Fourier transformation of (1.1) in time. Then the following set of elliptic problems is obtained: for $\omega \in (0,\infty)$, find $\widehat{u} = \widehat{u}(\cdot,\omega)$ such that

$$i\omega\beta\widehat{u} - \nabla\cdot(\alpha\nabla\widehat{u}) = \widehat{f}, \quad \boldsymbol{x} \in \Omega, \tag{1.3a}$$

$$\widehat{u} = 0, \quad \boldsymbol{x} \in \Gamma. \tag{1.3b}$$

Then $u(\cdot,t)$ is obtained by the formula (1.2).

In practice, the integral in (1.2) is approximated by a suitable quadrature with discrete

$$\{\omega_1,\omega_2,\omega_3,\dots\} \subset (0,\infty),$$

and Problems (1.3) are solved for these ω_j's by finite element or finite difference methods. Of primary interest is that Problem (1.3) for any ω_j is independent of other ω_k's, which theorizes our parallel method to solve the set of elliptic problems in a naturally parallel manner without any data communication.

The above parallel procedure has been already applied to solve wave propagation problems with absorbing boundary conditions [7, 8, 10]. Recently this frequency-domain method was applied to solve a parabolic problem with Robin boundary condition [15], which dealt with parallelization of the method and simulated a model problem using an MIMD machine. See also [14] for an analysis of a linearized Navier-Stokes equations. See also [18] for different parallel algorithmic approaches for parabolic problems by using the Laplace transformation.

Practically the source terms f in (1.1) are given only up to finite time intervals $(0, T)$. In order to apply the frequency domain method, the source terms should be extended to the infinite interval $(0, \infty)$. The current paper presents an analysis for the Problem (1.1) with least squares approximations of source terms so that their Fourier transformations are well-defined.

This paper is organized as follows. In §2, we show that the equation (1.3) has the unique solution $\widehat{u}(\cdot, \omega)$ for $\omega > 0$, and regularity and stability results are proved for such solutions. In §3 we first treat a finite element procedure for (1.3) depending on ω and derive error estimates for the procedure. We then derive a full error estimate for solving (1.1) via the inverse Fourier transformation. We approximate source functions by smoothly decaying functions in §4. Finally in §5 some results from numerical experiments are given.

2. Continuous problems

2.1. Notations and variational formulation

All functions are assumed to have values in the complex field $\mathbb{C}$. But, they are considered in the real field $\mathbb{R}$ for the time-dependent problems. Standard notations for function spaces and their norms will be used in this paper. Let $L^2(\Omega)$ be the space of square integrable functions on Ω. The corresponding inner product and norm will be denoted by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. Let $H^m(\Omega)$, for nonnegative integer m, denote the usual Sobolev spaces with norms $\|\cdot\|_m$, and $H_0^m(\Omega)$ the completion of $C_0^\infty(\Omega)$ in the norm of $H^m(\Omega)$; see [1, 6] for more details of function spaces and related norms.

Define the sesquilinear form $a_\omega(\cdot, \cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ by

$$a_\omega(u, v) = i\omega\,(\beta u, v) + (\alpha \nabla u, \nabla v), \qquad u, v \in H_0^1(\Omega).$$

A *variational formulation* of Problem (1.3) is then to find $\widehat{u}(\cdot,\omega) \in H_0^1(\Omega)$ for given $\widehat{f} \in H^{-1}(\Omega)$ such that

$$a_\omega(\widehat{u}, v) = \langle \widehat{f}, v \rangle, \quad v \in H_0^1(\Omega), \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

2.2. Uniqueness and existence

From the definition of $a_\omega(\cdot,\cdot)$ and Poincaré lemma,

$$|a_\omega(\widehat{u},\widehat{u})| \geq |\mathrm{Re}\ a_\omega(\widehat{u},\widehat{u})| = \|\sqrt{\alpha}\nabla\widehat{u}\|^2 \geq C\|\widehat{u}\|_1^2, \tag{2.2}$$

where $C > 0$ depends on only Ω and α_*. Thus, $a_\omega(\cdot,\cdot)$ is *coercive*. Moreover, from (2.1) with $v = \widehat{u}$ and (2.2) it follows that $\|\widehat{u}\|_1 \leq C\|\widehat{f}\|_{-1}$. Also note that for $\widehat{u}, \widehat{v} \in H_0^1(\Omega)$,

$$\begin{aligned} |a_\omega(\widehat{u},\widehat{v})| &\leq \omega\beta^* \|\widehat{u}\|\,\|\widehat{v}\| + \alpha^*\|\nabla\widehat{u}\|\|\nabla\widehat{v}\| \\ &\leq C(1+\omega)\|\widehat{u}\|_1\|\widehat{v}\|_1, \end{aligned} \tag{2.3}$$

where C depends on only Ω, β_* and α^*. Thus, $a_\omega(\cdot,\cdot)$ is *continuous.* An application of the Lax-Milgram lemma [6, 17] gives the following uniqueness and existence result:

THEOREM 2.1. *Suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^N, N = 2, 3$, with the boundary Γ. Assume that $\widehat{f}(\cdot,\omega) \in H^{-1}(\Omega)$. Then for each ω, the equation (2.1) has a unique solution $\widehat{u}(\cdot,\omega) \in H_0^1(\Omega)$. Moreover,*

$$\|\widehat{u}(\cdot,\omega)\|_1 \leq C\|\widehat{f}(\cdot,\omega)\|_{-1}, \tag{2.4}$$

with C independent of ω.

REMARK 2.1. The estimate (2.2) implies that the coercivity is independent of ω, and (2.3) implies $\|a_\omega\| \leq C(1+\omega)$.

2.3. Stability and regularity

Assume that Ω is a convex polygon in $\mathbb{R}^2$ or a C^2–domain in $\mathbb{R}^N, N = 2, 3$, and that $\widehat{f} \in L^2(\Omega)$. We begin by multiplying (1.3a) by $\overline{\widehat{u}}$, and integrating over Ω. Then an integration by parts yields: $i\omega\,(\beta\widehat{u},\widehat{u}) + (\alpha\nabla\widehat{u},\nabla\widehat{u}) = (\widehat{f},\widehat{u})$, from which the real and imaginary parts give

$$\omega\beta_*\|\widehat{u}\|^2 \leq \omega\left\|\sqrt{\beta}\widehat{u}\right\|^2 = \mathrm{Im}\,(\widehat{f},\widehat{u}) \leq \|\widehat{f}\|\,\|\widehat{u}\|, \tag{2.5a}$$

$$\alpha_*\|\nabla\widehat{u}\|^2 \leq \|\sqrt{\alpha}\nabla\widehat{u}\|^2 = \mathrm{Re}\,(\widehat{f},\widehat{u}) \leq \|\widehat{f}\|\,\|\widehat{u}\|. \tag{2.5b}$$

From (2.5a) it follows that

$$\|\widehat{u}\| \le \frac{1}{\omega\beta_*}\|\widehat{f}\|. \tag{2.6}$$

Combining (2.5b) and (2.6) yields $\|\nabla\widehat{u}\|^2 \le \frac{1}{\alpha_*}\|\widehat{f}\|\,\|\widehat{u}\| \le \frac{1}{\omega\beta_*\alpha_*}\|\widehat{f}\|^2$, and thus, $\|\nabla\widehat{u}\| \le \frac{1}{\sqrt{\omega\beta_*\alpha_*}}\|\widehat{f}\|$. Summarizing the above estimates, one has the following lemma.

LEMMA 2.1. *If $\widehat{u}(\cdot,\omega) \in H_0^1(\Omega)$ is a solution of Problem (2.1), then*

$$\|\widehat{u}(\cdot,\omega)\| \le C\min\Big\{1, \frac{1}{\omega}\Big\}\|\widehat{f}(\cdot,\omega)\|, \tag{2.7a}$$

$$\|\nabla\widehat{u}(\cdot,\omega)\| \le C\min\Big\{1, \frac{1}{\sqrt{\omega}}\Big\}\|\widehat{f}(\cdot,\omega)\|, \tag{2.7b}$$

$$\|\widehat{u}(\cdot,\omega)\|_1 \le C\min\Big\{1, \frac{1}{\sqrt{\omega}}\Big\}\|\widehat{f}(\cdot,\omega)\|. \tag{2.7c}$$

Let us now turn to get an $H^2(\Omega)$-estimate for the solution $\widehat{u}$ of (1.3). Recall the following [12]:

$$\sum_{i,j=1}^{N}\Big\|\frac{\partial^2\widehat{u}}{\partial x_i\partial x_j}\Big\|^2 \le C\|\Delta\widehat{u}\|^2, \quad \widehat{u} \in H^2(\Omega)\cap H_0^1(\Omega). \tag{2.8}$$

We then have the following lemma.

LEMMA 2.2. *Assume that ω is given and $\widehat{f}(\cdot,\omega) \in L^2(\Omega)$. If $\widehat{u}(\cdot,\omega) \in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of (2.1), then there exists a positive constant C such that*

$$\|\widehat{u}(\cdot,\omega)\|_2 \le C\|\widehat{f}(\cdot,\omega)\|,$$

where $C > 0$ is independent of ω.

Proof. Using (2.8), (1.3), (2.7a) and (2.7b), the following estimate is obtained:

$$\begin{aligned} |\widehat{u}|_2 \le C\|\Delta\widehat{u}\| &\le \frac{C}{\alpha_*}\Big\|i\omega\beta\widehat{u} - \nabla\alpha\cdot\nabla\widehat{u} - \widehat{f}\Big\| \\ &\le C\Big\{\omega\beta^*\|\widehat{u}\| + \|\widehat{f}\| + \|\nabla\alpha\|\|\nabla\widehat{u}\|\Big\} \\ &\le C\|\widehat{f}\|. \end{aligned}$$

This completes the proof. □

In particular, the estimate of Lemma 2.2 shows the existence of $\widehat{u} \in H^2(\Omega) \cap H_0^1(\Omega)$ if $\widehat{f} \in L^2(\Omega)$ by the method of Galerkin approximation [16].

We summarize the above results in the following theorem.

THEOREM 2.2. *Suppose Ω is a convex polygon in $\mathbb{R}^2$ or a C^2–domain in $\mathbb{R}^N, N = 2,3$. Then for any $\widehat{f}(\cdot,\omega) \in L^2(\Omega)$, there exists a unique solution $\widehat{u}(\cdot,\omega) \in H_0^1(\Omega) \cap H^2(\Omega)$ with*

$$\|\widehat{u}(\cdot,\omega)\|_2 \leq C\|\widehat{f}(\cdot,\omega)\|.$$

REMARK 2.2. The estimate in Lemma 2.2 means that the elliptic regularity coefficient C for Problem (1.1) corresponding to ω is not singular as ω tends to zero or ∞. This comes from the nature of parabolicity of (1.1), which differs from that of hyperbolicity of wave equations resulting into Helmholtz equations [10].

As immediate results of Theorems 2.1 and 2.2, we have the followings.

COROLLARY 2.1. *If $\|\widehat{f}(\cdot,\omega)\|_{-1}$ is integrable over the frequency domain $\mathbb{R}$ with respect to ω, then there exist Fourier inverses of $\widehat{u}(\cdot,\omega)$ and $\dfrac{\partial \widehat{u}}{\partial x_i}$, $1 \leq i \leq N$.*

COROLLARY 2.2. *If $\|\widehat{f}(\cdot,\omega)\|$ is integrable over the frequency domain $\mathbb{R}$ with respect to ω, then there exist Fourier inverses of $\widehat{u}(\cdot,\omega)$, $\dfrac{\partial \widehat{u}}{\partial x_i}$ and $\dfrac{\partial^2 \widehat{u}}{\partial x_i \partial x_j}$, $1 \leq i,j \leq N$.*

3. Finite element approximation

3.1. Finite element method for a single frequency

Let $h > 0$ be a discretization parameter tending to zero and $V_h \subset H_0^1(\Omega)$ be a finite element space. Then the discrete problem corresponding to (2.1) reads: *Find $\widehat{u} \in V_h$ such that for a given $\widehat{f} \in H^{-1}(\Omega)$,*

$$a_\omega(\widehat{u}, v) = \langle \widehat{f}, v\rangle, \qquad v \in V_h. \tag{3.1}$$

We shall assume that V_h satisfy the following property: There exist a positive constant C and an operator $\pi_h : H^2(\Omega) \to V_h$, independent of h such that

$$\|v - \pi_h v\|_k \leq Ch^{2-k}\|v\|_2, \quad v \in H^2(\Omega), k = 0,1. \tag{3.2}$$

For such finite element spaces, see, for example, [2, 3, 4, 5, 11, 13]. Let $\widehat{u}_h(\cdot,\omega) \in V_h$ be the Galerkin approximation to $\widehat{u}(\cdot,\omega)$ of (2.1). Then $\widehat{u}_h(\cdot,\omega)$ exists uniquely due to Theorem 2.1. Furthermore we have the following error estimates.

THEOREM 3.1. *Suppose that Ω is a convex polygon in $\mathbb{R}^2$ or a C^2–domain in $\mathbb{R}^N, N = 2,3$, and $\widehat{f}(\cdot,\omega) \in L^2(\Omega)$. Then the approximate solution $\widehat{u}_h(\cdot,\omega)$ of* (3.1) *to the solution $\widehat{u}(\cdot,\omega)$ of* (2.1) *for each ω satisfies that*

$$\|\widehat{u}(\cdot,\omega) - \widehat{u}_h(\cdot,\omega)\|_1 \leq C(1+\omega)h\|\widehat{f}(\cdot,\omega)\|, \tag{3.3a}$$

$$\|\widehat{u}(\cdot,\omega) - \widehat{u}_h(\cdot,\omega)\| \leq C(1+\omega)^2h^2\|\widehat{f}(\cdot,\omega)\|. \tag{3.3b}$$

Proof. From (2.1) and (3.1), we have the error equation:

$$a_\omega(\widehat{u} - \widehat{u}_h, v) = 0, \qquad v \in V_h.$$

By coercivity and continuity of a_ω and the above error equation, for arbitrary $\chi \in V_h$,

$$\begin{aligned}\|\widehat{u} - \widehat{u}_h\|_1^2 &\leq Ca_\omega(\widehat{u} - \widehat{u}_h, \widehat{u} - \widehat{u}_h)\\ &= C\left\{a_\omega(\widehat{u} - \widehat{u}_h, \widehat{u} - \chi) + a_\omega(\widehat{u} - \widehat{u}_h, \chi - \widehat{u}_h)\right\}\\ &= Ca_\omega(\widehat{u} - \widehat{u}_h, \widehat{u} - \chi)\\ &\leq C(1+\omega)\|\widehat{u} - \widehat{u}_h\|_1\,\|\widehat{u} - \chi\|_1.\end{aligned}$$

Then by using (3.2) and Theorem 2.2, an appropriate choice of χ yields (3.3a).

For a proof of the second inequality the usual duality argument will be used. Let $z \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of

$$a_\omega(z, v) = (\widehat{u} - \widehat{u}_h, v), \quad v \in H_0^1(\Omega).$$

Then, by Lemma 2.2, we have $\|z\|_2 \leq C\|\widehat{u} - \widehat{u}_h\|$, from which, using the continuity of a_ω, (3.2), and (3.3a), we have

$$\begin{aligned}\|\widehat{u} - \widehat{u}_h\|^2 &= a_\omega(z, \widehat{u} - \widehat{u}_h)\\ &= a_\omega(z - \pi_h z, \widehat{u} - \widehat{u}_h)\\ &\leq C(1+\omega)\|\widehat{u} - \widehat{u}_h\|_1\,\|z - \pi_h z\|_1\\ &\leq C(1+\omega)h\|\widehat{u} - \widehat{u}_h\|_1\,\|z\|_2\\ &\leq C(1+\omega)^2h^2\|\widehat{f}\|\,\|\widehat{u} - \widehat{u}_h\|.\end{aligned}$$

This completes the proof. □

3.2. Full error estimate

Following [8], we will give the full estimate of errors introduced by the truncation and discretization of a quadrature of the inverse Fourier transform, and caused by finite element approximations. The following lemma will be useful.

LEMMA 3.1. *Let $T > 0$ be given and suppose that*

$$\int_0^T t^{2k}\|f(\cdot,t)\|^2\,dt < \infty,$$

$k = 0, 1, 2, \dots, m$ *for some nonnegative integer* m. *Let* u *be the solution of Problem* (1.1). *Then we have the following estimates: for* $k = 0, 1, 2, \dots, m$,

$$\int_0^T \|t^k u(\cdot,t)\|_1^2 dt \le C \sum_{j=0}^{k} \int_0^T t^{2j}\|f(\cdot,t)\|^2\,dt. \tag{3.4}$$

Proof. Multiply (1.1a) by $u(\cdot,t)$ to get, for $\varepsilon > 0$,

$$\frac{d}{dt}\frac{1}{2}\|\beta^{1/2}u(\cdot,t)\|^2 + \|\alpha^{1/2}\nabla u(\cdot,t)\|^2 \le \frac{1}{4\epsilon}\|f(\cdot,t)\|^2 + \epsilon\|u(\cdot,t)\|^2.$$

By integrating this inequality with respect to t over $[0,T]$, we get

$$\frac{1}{2}\|\beta^{1/2}u(\cdot,T)\|^2 + \int_0^T \|\alpha^{1/2}\nabla u(\cdot,t)\|^2 dt \le \frac{1}{4\epsilon}\int_0^T \|f(\cdot,t)\|^2 dt + \epsilon\int_0^T \|u(\cdot,t)\|^2 dt.$$

An application of Poincaré inequality and a choice of a sufficiently small $\epsilon > 0$ lead to

$$\|u(\cdot,T)\|^2 + \int_0^T \|\nabla u(\cdot,t)\|^2 dt \ \le\ C\int_0^T \|f(\cdot,t)\|^2 dt. \tag{3.5}$$

This yields (3.4) for $k = 0$.

Now, observe that $tu(\cdot,t)$ satisfies

$$(\beta t u)_t - \nabla\cdot(\alpha\nabla t u) = tf + \beta u,$$

and repeat the above argument to arrive at

$$\|Tu(\cdot,T)\|^2 + \int_0^T t^2\|\nabla u(\cdot,t)\|^2 dt \le C\int_0^T (1+t^2)\|f(\cdot,t)\|^2 dt,$$

where we used the estimate (3.5). This gives (3.4) for $k = 1$. An inductive argument for $k > 1$ completes the proof. □

We consider restricted sources such that $|\widehat{f}(\cdot,\omega)|$ are square integrable with respect to ω and thus negligible for large $|\omega|$. We then choose a sufficiently large $\omega^* > 0$ so that both $\widehat{u}(\cdot,\omega)$ and $\widehat{f}(\cdot,\omega)$ are negligible for $|\omega| > \omega^*$. Also recall that the computation of $\widehat{u}(\cdot,\omega)$ for $\omega < 0$ is not necessary. Let N_ω be a positive integer and define the discretization parameter $\Delta\omega$ of the frequency domain by the formula $\Delta\omega = \omega^*/N_\omega$, and introduce the mesh points $\omega_{j-\frac{1}{2}} = (j-\frac{1}{2})\Delta\omega, j = 1,\ldots,N_\omega$ on the interval $(0,\omega^*)$. Due to (1.2), the time-domain solution u of (1.1) will then be approximated by

$$u^h_{\omega^*,\Delta\omega}(\boldsymbol{x},t) = \frac{1}{\pi}\sum_{j=1}^{N_\omega} \widehat{u}_h(\boldsymbol{x},\omega_{j-1/2})e^{it\omega_{j-1/2}}\Delta\omega.$$

We now turn to estimate the convergence of $u^h_{\omega^*,\Delta\omega}(\cdot,t)$ to $u(\cdot,t)$ for a fixed time $t>0$. Setting

$$u_{\omega^*}(\boldsymbol{x},t) = \frac{1}{\pi}\int_0^{\omega^*} \widehat{u}(\boldsymbol{x},\omega)e^{i\omega t}d\omega$$

and

$$u_{\omega^*,\Delta\omega}(\boldsymbol{x},t) = \frac{1}{\pi}\sum_{j=1}^{N_\omega} \widehat{u}(\boldsymbol{x},\omega_{j-1/2})e^{it\omega_{j-1/2}}\Delta\omega,$$

we have

$$\begin{aligned} u(\boldsymbol{x},t) - u^h_{\omega^*,\Delta\omega}(\boldsymbol{x},t) &= (u(\boldsymbol{x},t) - u_{\omega^*}(\boldsymbol{x},t)) + (u_{\omega^*}(\boldsymbol{x},t) - u_{\omega^*,\Delta\omega}(\boldsymbol{x},t)) \\ &\quad + \left(u_{\omega^*,\Delta\omega}(\boldsymbol{x},t) - u^h_{\omega^*,\Delta\omega}(\boldsymbol{x},t)\right) \\ &\equiv E_1(\boldsymbol{x},t) + E_2(\boldsymbol{x},t) + E_3(\boldsymbol{x},t). \end{aligned} \tag{3.6}$$

First, by Minkowski's inequality for integral and from the inequality (2.7b) in Lemma 2:2 we get

$$\|E_1(\cdot,t)\|_1 \le C\int_{\omega>\omega^*} \|\nabla\widehat{u}(\cdot,\omega)\|d\omega \le C\int_{\omega>\omega^*} \left\|\frac{1}{\sqrt{\omega}}\widehat{f}(\cdot,\omega)\right\|d\omega. \tag{3.7}$$

Thus

$$\|E_1(\cdot,t)\|_1 \to 0 \qquad \text{as } \omega^* \to \infty.$$

We also have

$$
\begin{aligned}
\|E_2(\cdot,t)\|_1^2 &\le \frac{C}{\pi^2}\int_\Omega \left|\int_0^{\omega^*} \nabla\widehat{u}(\boldsymbol{x},t)e^{i\omega t}d\omega - \sum_{j=1}^{N_\omega} \nabla\widehat{u}(\boldsymbol{x},\omega_{j-1/2})e^{it\omega_{j-1/2}}\Delta\omega\right|^2 dx \\
&\le C(\Delta\omega)^4 \int_\Omega \left\|\frac{\partial^2\left(\nabla\widehat{u}(\boldsymbol{x},\omega)e^{i\omega t}\right)}{\partial\omega^2}\right\|^2_{L^2(0,\omega^*)} dx \\
&\le C(\Delta\omega)^4 \int_\Omega \left\|-\widehat{t^2\nabla u}(\boldsymbol{x},\cdot) + 2t\widehat{t\nabla u}(\boldsymbol{x},\cdot) - t^2\nabla\widehat{u}(\boldsymbol{x},\cdot)\right\|^2_{L^2(0,\omega^*)} dx \\
&\le C(\Delta\omega)^4 \int_\Omega \Big\{\|\widehat{t^2\nabla u}(\boldsymbol{x},\cdot)\|^2_{L^2(0,\infty)} \\
&\quad + t^2\|\widehat{t\nabla u}(\boldsymbol{x},\cdot)\|^2_{L^2(0,\infty)} + t^4\|\nabla\widehat{u}(\boldsymbol{x},\cdot)\|^2_{L^2(0,\infty)}\Big\} dx \\
&\le C(\Delta\omega)^4 \Big\{\|t^2 u\|^2_{L^2((0,\infty);H^1(\Omega))} \\
&\quad + t^2\|tu\|^2_{L^2((0,\infty);H^1(\Omega))} + t^4\|u\|^2_{L^2((0,\infty);H^1(\Omega))}\Big\}.
\end{aligned} \tag{3.8}
$$

Assume further that, for $k = 0,1,2,$

$$\int_0^\infty s^{2k}\|f(\cdot,s)\|^2\,ds < \infty.$$

Then, using Lemma 3.1, we see that

$$\|E_2(\cdot,t)\|_1 \le C(\Delta\omega)^2 \sum_{k=0}^{2} t^{2-k} \sum_{j=0}^{k} \left[\int_0^\infty s^{2j}\|f(\cdot,s)\|^2 ds\right]^{1/2} \to 0$$

as $\Delta\omega \to 0$.

Finally, from Theorem 3.1 and Theorem 2.2, we have

$$
\begin{aligned}
\|E_3(\cdot,t)\|_1 &\le C\left\|\frac{1}{\pi}\sum_{j=1}^{N_\omega}\left(\nabla\widehat{u}_h(\cdot,\omega_{j-1/2}) - \nabla\widehat{u}(\cdot,\omega_{j-1/2})\right)e^{it\omega_{j-1/2}}\Delta\omega\right\| \\
&\le C\Delta\omega\sum_{j=1}^{N_\omega}\left\|\nabla\widehat{u}_h(\cdot,\omega_{j-1/2}) - \nabla\widehat{u}(\cdot,\omega_{j-1/2})\right\| \\
&\le Ch\Delta\omega\sum_{j=1}^{N_\omega}(1+\omega_{j-1/2})\|\widehat{f}(\cdot,\omega_{j-1/2})\| \\
&\le Ch\left\|(1+\omega)\,\widehat{f}(\cdot,\omega)\right\|_{L^2_\omega((0,\infty);L^2(\Omega))}.
\end{aligned} \tag{3.9}
$$

Thus, if we assume that

$$\left\|(1+\omega)\widehat{f}(\cdot,\omega)\right\|_{L^2_\omega((0,\infty);L^2(\Omega))} \equiv \left[\int_0^\infty \|(1+\omega)\widehat{f}(\cdot,\omega)\|^2 d\omega\right]^{1/2} < \infty$$

we have that

$$\|E_3(\cdot,t)\|_1 \to 0 \qquad \text{as } h \to 0.$$

Combining the estimates (3.7), (3.8), and (3.9), and using Lemma 3.1, we have the full error estimate.

THEOREM 3.2. *Suppose that Ω is as in Theorem 3.1. Assume that for $k = 0, 1, 2$*

$$\int_0^\infty s^{2k}\|f(\cdot, s)\|^2\, ds < \infty,$$

and

$$\left\|(1+\omega)\,\widehat{f}(\cdot,\omega)\right\|_{L^2_\omega((0,\infty);L^2(\Omega))} < \infty.$$

Then $u^h_{\omega^*,\Delta\omega}(\cdot, t)$ converges to $u(\cdot, t)$ for a fixed time t; moreover, for $t > 0$,

$$\begin{aligned}\|u(\cdot,t) - u^h_{\omega^*,\Delta\omega}(\cdot,t)\|_1 \le C_1 \int_{\omega>\omega^*} &\left\|\frac{1}{\sqrt{\omega}}\widehat{f}(\cdot,\omega)\right\| d\omega \\ &+ C_2(\Delta\omega)^2 \sum_{k=0}^{2} t^{2-k} \sum_{j=0}^{k} \left[\int_0^\infty s^{2j}\|f(\cdot,s)\|^2 ds\right]^{1/2} \\ &+ C_3 h \left\|(1+\omega)\,\widehat{f}(\cdot,\omega)\right\|_{L^2_\omega((0,\infty);L^2(\Omega))},\end{aligned}$$

with $C_j, j = 1, 2, 3$, dependent only on the domain Ω and the coefficients β and α.

Similarly, if we estimate $\|E_1(\cdot,t)\|, \|E_2(\cdot,t)\|$, and $\|E_3(\cdot,t)\|$ in (3.6), using (2.7a) instead of (2.7b) for the first error term $\|E_1(\cdot,t)\|$, and using (3.3b) instead of (3.3a) for the third error term $\|E_3(\cdot,t)\|$, we obtain the following theorem.

THEOREM 3.3. *Suppose that Ω is as in Theorem 3.1. Assume that for $k = 0, 1, 2$*

$$\int_0^\infty s^{2k}\|f(\cdot, s)\|^2\, ds < \infty,$$

and

$$\left\|(1+\omega)^2\,\widehat{f}(\cdot,\omega)\right\|_{L^2_\omega((0,\infty);L^2(\Omega))} < \infty.$$

Then $u^h_{\omega^*,\Delta\omega}(\cdot,t)$ *converges to* $u(\cdot,t)$ *for a fixed time* t*; moreover, for* $t>0$,

$$\begin{aligned}\|u(\cdot,t)-u^h_{\omega^*,\Delta\omega}(\cdot,t)\| \le C_1 &\int_{\omega>\omega^*}\left\|\frac{1}{\omega}\widehat{f}(\cdot,\omega)\right\| d\omega \\ &+C_2(\Delta\omega)^2\sum_{k=0}^{2} t^{2-k}\sum_{j=0}^{k}\left[\int_0^\infty s^{2j}\|f(\cdot,s)\|^2 ds\right]^{1/2} \\ &+C_3h^2\left\|(1+\omega)^2\,\widehat{f}(\cdot,\omega)\right\|_{L^2_\omega((0,\infty);L^2(\Omega))},\end{aligned}$$

with $C_j, j=1,2,3$, *dependent only on the domain* Ω *and the coefficients* β and α.

4. Source functions and least square approximation

Assume first that the source $f(\boldsymbol{x},t)$ is given in the form $f(\boldsymbol{x},t)=\phi(\boldsymbol{x})g(t)$. In practice $g(t)$ is not known for all $t\in(0,\infty)$. We also assume that $g(t)\in L^2(0,T)$ for some $T<\infty$. We thus have to extend $g(t)$ to $(0,\infty)$ so that the Fourier transformation of the extended source function $g(t)$ can be calculated effectively and efficiently.

The following is a simple application of Weierstrass Theorem.

LEMMA 4.1. *Let* c *be a positive number. Then the following set* $\mathcal{S}$ *is dense in* $L^2(0,T)$.

$$\mathcal{S}=\operatorname{Span}\left\{t^k e^{-ct}\,|\,k=0,1,\ldots,\quad t\in(0,T)\right\}.$$

Let $g_n(t)\in\mathcal{S}$ be a least square approximation to $g(t)$ in $(0,T]$ such that $\|g(t)-g_n(t)\|_{L^2(0,T)}<\varepsilon$, where $\varepsilon>0$ is a prescribed tolerance. Write such a g_n in the form

$$g_n(t)=\sum_{j=0}^{n} a_j t^j e^{-ct},$$

and extend g_n to (T,∞). We now approximate the solution to our Problem (1.1) with $f_n(\boldsymbol{x},t)=\phi(\boldsymbol{x})g_n(t)$ for an appropriate $n\ge 0$ instead of $f(\boldsymbol{x},t)$.

Let $u^n(\boldsymbol{x},t)$ be the solution to (1.1) with $f=f_n(\boldsymbol{x},t)$. Due to the relation (1.2) the time-domain solution $u^n(\boldsymbol{x},t)$ will then be approximated by

$$u^{n,h}_{\omega^*,\Delta\omega}(\boldsymbol{x},t)=\frac{1}{\pi}\sum_{j=1}^{N_\omega}\widehat{u}^n_h(\boldsymbol{x},\omega_{j-1/2})e^{it\omega_{j-1/2}}\Delta\omega.$$

We now turn to estimate the L^2-norm convergence of $u^{n,h}_{\omega^*,\Delta\omega}(\cdot,t)$ to $u(\cdot,t)$ for a fixed time $t>0$. Setting

$$u^n_{\omega^*}(\boldsymbol{x},t)=\frac{1}{\pi}\int_0^{\omega^*}\widehat{u^n}(\boldsymbol{x},\omega)e^{i\omega t}d\omega$$

and

$$u^n_{\omega^*,\Delta\omega}(\boldsymbol{x},t)=\frac{1}{\pi}\sum_{j=1}^{N_\omega}\widehat{u^n}(\boldsymbol{x},\omega_{j-1/2})e^{it\omega_{j-1/2}}\Delta\omega,$$

we have

$$\begin{aligned} u(\boldsymbol{x},t)-u^{n,h}_{\omega^*,\Delta\omega}(\boldsymbol{x},t) &= (u(\boldsymbol{x},t)-u^n(\boldsymbol{x},t))+(u^n(\boldsymbol{x},t)-u^n_{\omega^*}(\boldsymbol{x},t))\\ &\quad+\left(u^n_{\omega^*}(\boldsymbol{x},t)-u^n_{\omega^*,\Delta\omega}(\boldsymbol{x},t)\right)\\ &\quad+\left(u^n_{\omega^*,\Delta\omega}(\boldsymbol{x},t)-u^{n,h}_{\omega^*,\Delta\omega}(\boldsymbol{x},t)\right). \end{aligned}$$

Under the same assumption as in Theorem 3.3, using the inequality (3.5) and following the argument in Section 3.2, we have that for a fixed t $(0<t\le T)$,

$$\begin{aligned} \|u(\cdot,t)-u^{n,h}_{\omega^*,\Delta\omega}(\cdot,t)\| &\le C_0\|f(\cdot,s)-f_n(\cdot,s)\|_{L^2_s((0,t);L^2(\Omega))}\\ &\quad+C_1\int_{\omega>\omega^*}\left\|\frac{1}{\omega}\widehat{f_n}(\cdot,\omega)\right\|d\omega\\ &\quad+C_2(\Delta\omega)^2\sum_{k=0}^{2}t^{2-k}\sum_{j=0}^{k}\left[\int_0^\infty s^{2j}\|f_n(\cdot,s)\|^2ds\right]^{1/2}\\ &\quad+C_3h^2\left\|(1+\omega)^2\widehat{f_n}(\cdot,\omega)\right\|_{L^2_\omega((0,\infty);L^2(\Omega))}, \end{aligned} \tag{4.1}$$

with $C_j, j=0,1,2,3$, dependent only on the domain Ω and the coefficients β and α.

Note that due to Theorem 3.3, $\widehat{g_n}(\omega)$ should decay faster than $1/\omega^3$ as $\omega\to\infty$ and that

$$\int_0^\infty t^ke^{-ct}e^{-i\omega t}dt=\frac{k!}{(c+i\omega)^{k+1}},\ k=0,1,\dots,\infty. \tag{4.2}$$

So we take $n\ge 3$. If the source function needs to satisfy that $g(0)=0$, we can take $g_n(t)$ in the form

$$g_n(t)=\sum_{j=3}^{n}a_jt^je^{-ct},\quad t\in(0,\infty).$$

For a source function given by $f(\boldsymbol{x},t) = \phi(\boldsymbol{x})g(t)$, we may choose $n \geq 3$ in the inequality (4.1) so that

$$
\begin{aligned}
\|f(\cdot,s) - f_n(\cdot,s)\|_{L^2_s((0,t);L^2(\Omega))} &= \|\phi(\boldsymbol{x})g(s) - \phi(\boldsymbol{x})g_n(s)\|_{L^2_s((0,t);L^2(\Omega))} \\
&= \|\phi\|_{L^2(\Omega)}\,\|g(s) - g_n(s)\|_{L^2(0,t)} \\
&\leq \|\phi\|_{L^2(\Omega)}\,\|g(s) - g_n(s)\|_{L^2(0,T)} \\
&< \varepsilon \quad \text{for some prescribed } \varepsilon\,.
\end{aligned}
$$

Since

$$
\widehat{g_n}(\omega) = \sum_{j=3}^{n} \frac{a_j\, j!}{(c+i\omega)^{j+1}} \tag{4.3}
$$

we may control the second term of the right hand side of the inequality (4.1) using the following.

$$
\begin{aligned}
\int_{\omega>\omega^*} \left\| \frac{1}{\omega}\widehat{f_n}(\cdot,\omega) \right\| d\omega &\leq \|\phi\|_{L^2(\Omega)} \int_{\omega>\omega^*} \left| \frac{1}{\omega}\widehat{g_n}(\omega) \right| d\omega \\
&\leq \|\phi\|_{L^2(\Omega)} \sum_{j=3}^{n} \int_{\omega>\omega^*} \frac{|a_j| j!}{\omega(c^2+\omega^2)^{(j+1)/2}} d\omega \\
&< \|\phi\|_{L^2(\Omega)} \sum_{j=3}^{n} \frac{|a_j| j!}{(j+1)(\omega^*)^{j+1}}. \qquad (4.4)
\end{aligned}
$$

5. Numerical experiments

Numerical experiments were performed for Problem (1.1) with $\beta = \alpha = 1$. Let $\Omega = (0,1)^2$ in $\mathbb{R}^2$ and $J = (0,T)$, $T = 1$. The source $f(\boldsymbol{x},t) = \phi(\boldsymbol{x})g(t)$ was chosen so that the analytic solution u to Problem (1.1) is $u(\boldsymbol{x},t) = \sin(2\pi x)\sin(3\pi y)h(t), \boldsymbol{x} = (x,y) \in \Omega, t \in J$, where $h(t) = \frac{t^2}{1+10t^2}$.

First we approximate $g(t)$ by $g_n(t) = \sum_{j=3}^{n} a_j t^j e^{-ct}$. However we can not take arbitrarily large n since the induced linear system may be ill-conditioned.

We choose $c = 10$ so that $g_n(t)$ does not grow too large in finite time. To pick a reasonable n, we compared two values

$$
\|g(t) - g_n(t)\| \quad \text{and} \quad \|g(t) - \mathcal{T}(g,n,\omega^*,\Delta\omega)\|\ , \tag{5.1}
$$

where

$$
\mathcal{T}(g,n,\omega^*,\Delta\omega) = \sum_{j=1}^{N_\omega} \frac{1}{\omega_{j-1/2}} \widehat{g_n}(\omega_{j-1/2}) e^{it\omega_{j-1/2}} \Delta\omega \approx \int_0^{\omega^*} \frac{1}{\omega}\widehat{g_n}(\omega) e^{i\omega t}\, d\omega\,,
$$

# of basis	$\|g(t) - g_n(t)\|$	$\|g(t) - \mathcal{T}(g, n, \omega^*, \Delta\omega)\|$			ω^*
		$\Delta\omega = 2$	$\Delta\omega = 1$	$\Delta\omega = 1/2$	
9	1.150E-3	2.075E-3	1.147E-3	1.147E-3	120
10	1.005E-3	1.743E-3	1.002E-3	1.002E-3	125
11	7.949E-4	9.903E-3	7.908E-4	7.908E-4	132
12	6.584E-4	6.229E-2	6.540E-4	6.540E-4	138
13	6.593E-4	6.510E-2	6.549E-4	6.549E-4	138

TABLE 1. Relative $\|g(t) - g_n(t)\|$, relative $\|g(t) - \mathcal{T}(g, n, \omega^*, \Delta\omega)\|$, and ω^*.

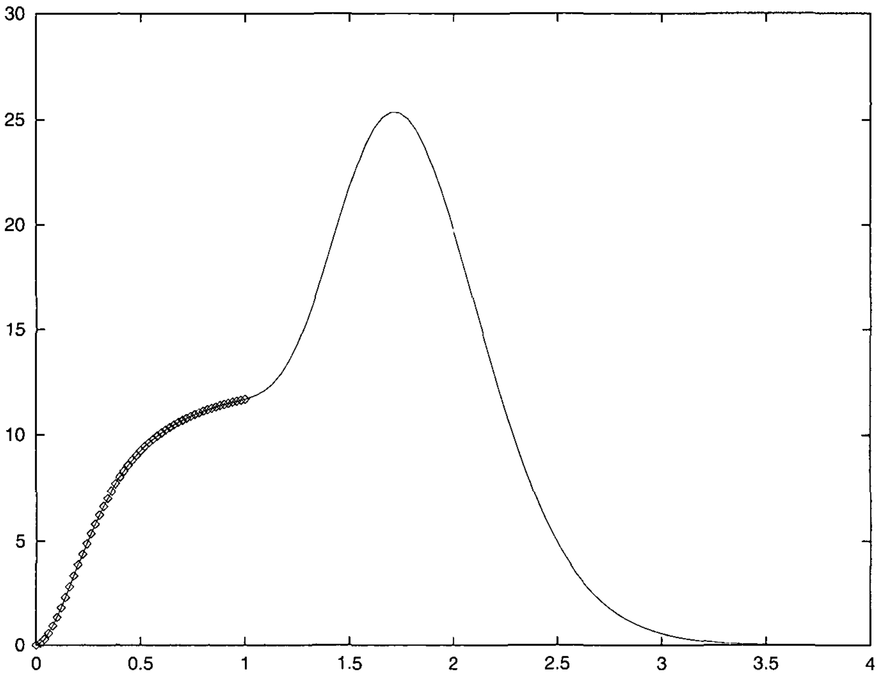

FIGURE 1. Time Shape of Source Function $g(t)$ and $g_n(t)$: $\diamond$'s represent the original source $g(t)$ and the solid line represents the graph of a least square approximation extended beyond $[0, 1]$.

ω^* is an integer satisfying $\sum_{j=3}^{n} \frac{|a_j| j!}{(j+1)(\omega^*)^{j+1}} < 10^{-4}$ in (4.4), and $N_\omega = \omega^*/\Delta\omega$. Table 1 shows the values of (5.1) with various n and $\Delta\omega$. It seems that eleven basis functions, i.e., $\sum_{j=3}^{13} a_j t^j e^{-10t}$ and $\Delta\omega = 1$

$N_x = N_y = 50$	$N_x = N_y = 100$	$N_x = N_y = 200$
2.317E-3 (2.086E-3)	6.727E-4 (3.902E-4)	4.393E-4 (3.209E-4)

TABLE 2. Relative $L^2(J; L^2(\Omega))$-errors and relative $L^2(\Omega)$-errors at time $t = 1$ for the solution by Frequency–Domain Method. The numbers in parentheses denote the relative $L^2(\Omega)$-errors at time $t = 1$.

time	$N_x = N_y = 50$	$N_x = N_y = 100$	$N_x = N_y = 200$
0.1	5.034E-3	5.700E-3	5.887E-3
0.2	1.334E-3	1.072E-3	1.299E-3
0.3	2.797E-3	1.215E-3	8.560E-4
0.4	1.931E-3	3.809E-4	4.082E-4
0.5	2.610E-3	8.842E-4	4.769E-4
0.6	2.197E-3	4.493E-4	1.399E-4
0.7	2.416E-2	6.401E-4	1.984E-4
0.8	2.450E-3	6.696E-4	2.293E-4
0.9	2.269E-3	4.893E-4	1.056E-4
1.0	2.086E-3	3.902E-4	3.209E-4

TABLE 3. Comparison of Relative $L^2(\Omega)$-Errors at time $t = 0.1, 0.2, \ldots, 1$ for the solutions by Frequency–Domain Method.

are reasonable choices. In Figure 1, $\diamond$'s represent the original source function $g(t)$, and the solid line represents the graph of a least square approximation extended smoothly beyond $[0, 1]$.

We solved Problem (3.1) for $\omega_{j-1/2}, j = 1, \ldots, N_\omega$. In the calculation of finite element solutions $N_x \times N_y$ uniform triangular meshes were taken for the triangulation of Ω, and C^0 piecewise linear finite elements were used. The resulting algebraic problems were solved by using a Gaussian elimination type solver, Yale Sparse Matrix Package [9].

Table 2 shows the relative $L^2(J; L^2(\Omega))$-errors and the relative $L^2(\Omega)$-errors at time $t = 1$. We observe that the errors decay, but not drastically, as the number, say $N_x (= N_y)$, of points in the space Ω increases because it reduces only the last part of (4.1).

We also present the relative $L^2(\Omega)$-errors for each time step $t = 0.1, 0.2, \ldots, 1$ for the cases $N_x = N_y = 50, 100, 200$ in Table 3. It is worthwhile to observe that the errors are up and down but do not increase as time grows. This feature should be emphasized compared to error behaviors for traditional solvers; as time increases, errors generated by traditional solvers in the space-time formulation, such as Crank-Nicolson and backward-Euler methods, usually grow.

Among all the features mentioned above, the most favorable advantage for our scheme lies in the natural parallelization when there are given massively parallel processors.

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CHANG-OCK LEE, DIVISION OF APPLIED MATHEMATICS, KAIST, TAEJON 305-701, KOREA
E-mail: colee@amath.kaist.ac.kr

JONGWOO LEE, DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139-701, KOREA
E-mail: jlee@mail.kwangwoon.ac.kr

DONGWOO SHEEN, DEPARTMENT OF MATHEMATICS, SEOUL NATIONAL UNIVERSITY, SEOUL 151-747, KOREA
E-mail: sheen@math.snu.ac.kr