ON DERIVATIONS IN BANACH ALGEBRAS

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ABSTRACT. Our main goal is to show that if there exist Jordan derivations D and G on a noncommutative $(n + 1)!$-torsion free prime ring R such that

$$D(x)x^n - x^n G(x) \in C(R)$$

for all $x \in R$, then we have $D = 0$ and $G = 0$. We also prove that if there exists a derivation D on a noncommutative 2-torsion free prime ring R such that the mapping $x \mapsto [aD(x), x]$ is commuting on R, then we have either $a = 0$ or $D = 0$.

1. Introduction

In this paper R will represent an associative ring with center $C(R)$, and A will represent an algebra over a complex field $\mathbb{C}$. The (*Jacobson*) *radical* of A will be denoted by $rad(A)$. The commutator $xy - yx$ will be denoted by $[x, y]$. We write $[x, y]$ for $xy - yx$, and use the identities $[xy, z] = [x, z]y + x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$. Recall that R is *prime* if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is *semi-prime* if $aRa = (0)$ implies $a = 0$. An additive mapping D from R to R is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. And also, an additive mapping D from R to R is called a *Jordan derivation* if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$. An additive mapping F from R to R is said to be a *commuting* (resp. *centralizing*) if $[F(x), x] = 0$ (resp. $[F(x), x] \in C(R)$) holds for all $x \in R$.

The Singer-Wermer theorem, which is a classical theorem of Banach algebra theory, states that every continuous derivation on a commutative

Banach algebra maps into its radical [12], and Thomas proved that the Singer-Wermer theorem remains true without assuming the continuity of the derivation [13] (This generalization is called the Singer-Wermer conjecture). On the other hand, Posner [9] obtained two fundamental results in 1957:

(i) the first result (so-called Posner's first theorem) asserts that if D and G are derivations on a 2-torsion free prime ring such that the product DG is also a derivation, then either $D = 0$ or $G = 0$.

(ii) the second result (so-called Posner's second theorem) states that if D is a centralizing derivation on a noncommutative prime ring, then $D = 0$.

It is the purpose of this paper to present some results which is inspired by the above Posner's theorems, and to obtain noncommutative versions of the Singer-Wermer theorem by using his results. Furthermore, we will present a result which can be considered as a contribution to the theory of commuting mappings in prime rings.

2. Derivations on prime rings

We precede the proof of our main result by the following theorem.

THEOREM 2.1. *Let n be a fixed positive integer. Let R be a non-commutative $(n + 1)!$-torsion free prime ring. Suppose that there exist Jordan derivations D and $G : R \to R$ such that*

$$D(x)x^n - x^n G(x) = 0$$

for all $x \in R$. Then we have $D = 0$ and $G = 0$.

Proof. By [1, Theorem 1], D and G are derivations on R.
Suppose that

$$(1) \qquad\qquad D(x)x^n - x^n G(x) = 0$$

for all $x \in R$. Replacing x by $x + ky$ in (1), we have

$$(2) \qquad kQ_1(x,y) + k^2 Q_2(x,y) + \cdots + k^n Q_n(x,y) = 0$$

$k \in \mathbb{Z}$, $x, y \in R$, where $Q_i(x,y)$ denotes the sum of these terms in which y appears as a term in the product i times. Then by [5, Lemma 1], we

have

(3)
$$Q_1(x,y) = D(x)x^{n-1}y + D(x)x^{n-2}yx + D(x)x^{n-3}yx^2 + \cdots$$
$$+ D(x)yx^{n-1} + D(y)x^n - \{x^{n-1}yG(x) + x^{n-2}yxG(x)$$
$$+ x^{n-3}yx^2G(x) + \cdots + yx^{n-1}G(x) + x^nG(y)\} = 0$$

for all $x, y \in R$. Putting xy instead of y in (3), we have

$$D(x)x\{x^{n-1}y + x^{n-2}yx + x^{n-3}yx^2 + \cdots + yx^{n-1}\}$$
$$+ D(x)yx^n + xD(y)x^n$$
$$= x\{x^{n-1}yG(x) + x^{n-2}yxG(x) + x^{n-3}yx^2G(x) + \cdots + yx^{n-1}G(x)$$
$$+ x^nG(y)\} + x^nG(x)y$$
$$= x\{D(x)x^{n-1}y + D(x)x^{n-2}yx + D(x)x^{n-3}yx^2 + \cdots + D(x)yx^{n-1}$$
$$+ D(y)x^n\} + x^nG(x)y$$

for all $x, y \in R$. It follows from (1) that

(4)
$$[D(x), x]x^{n-1}y + [D(x), x]x^{n-2}yx + [D(x), x]x^{n-3}yx^2$$
$$+ \cdots + [D(x), x]yx^{n-1} + D(x)[y, x^n] = 0$$

for all $x, y \in R$. Let $y = x$ in (4). Since R is $(n + 1)!$-torsion free, we have
$$[D(x), x]x^n = 0$$

for all $x \in R$. By [8, Lemma 2.6], we get $D = 0$. Thus (1) becomes $x^nG(x) = 0$ for all $x \in R$, and so $x^n[G(x), x] = 0$ for all $x \in R$. Consequently $G = 0$ by the same reasoning. The proof of the theorem is complete. □

COROLLARY 2.2. *Let n be a fixed positive integer. Let R be a non -commutative $(n + 1)!$-torsion free prime ring. Suppose that there exists a Jordan derivation $D : R \to R$ such that*

$$[D(x), x^n] = 0$$

for all $x \in R$. Then we have $D = 0$.

Now our main result, motivated by Posner's theorems, is as follows:

THEOREM 2.3. *Let n be a fixed positive integer. Let R be a non-commutative $(n+1)!$-torsion free prime ring. Suppose that there exist Jordan derivations D and $G : R \to R$ such that*

$$D(x)x^n - x^n G(x) \in C(R)$$

for all $x \in R$. Then we have $D = 0$ and $G = 0$.

Proof. By [1, Theorem 1], D and G are derivations on R.
Suppose that

(5) $$[D(x)x^n - x^n G(x), z] = 0$$

for all $x, z \in R$. Replacing x by $x + ky$ in (5), we obtain

(6) $$kQ_1(x, y, z) + k^2 Q_2(x, y, z) + \cdots + k^n Q_n(x, y, z) = 0$$

$k \in \mathbb{Z}$, $x, y, z \in R$, where $Q_i(x, y, z)$ denotes the sum of these terms in which y appears as a term in the product i times. Then by [5, Lemma 1], we have

$$\begin{aligned}
Q_1(x, y, z) = {}& [D(x)x^{n-1}y + D(x)x^{n-2}yx + D(x)x^{n-3}yx^2 + \cdots \\
& + D(x)yx^{n-1} + D(y)x^n - x^{n-1}yG(x) - x^{n-2}yxG(x) \\
& - x^{n-3}yx^2 G(x) - \cdots - yx^{n-1}G(x) - x^n G(y), z] = 0
\end{aligned}$$

for all $x, y, z \in R$. It follows that

(7)
$$\begin{aligned}
& D(x)x^{n-1}y + D(x)x^{n-2}yx + D(x)x^{n-3}yx^2 + \cdots \\
& + D(x)yx^{n-1} + D(y)x^n - x^{n-1}yG(x) - x^{n-2}yxG(x) \\
& - x^{n-3}yx^2 G(x) - \cdots - yx^{n-1}G(x) - x^n G(y) \in C(R)
\end{aligned}$$

for all $x, y \in R$. First, assume that there exists a nonzero element c in $C(R)$. Taking $y = c$ in (7), we get

(8) $$c(nD(x)x^{n-1} - nx^{n-1}G(x)) + D(c)x^n - x^n G(c) \in C(R)$$

for all $x \in R$. Let $y = c^2$ in (7). Then we have

(9) $$c^2(nD(x)x^{n-1} - nx^{n-1}G(x)) + 2cD(c)x^n - 2cx^n G(c) \in C(R)$$

for all $x \in R$. Left multiplication of (8) by c gives

$$(10) \qquad c^2(nD(x)x^{n-1} - nx^{n-1}G(x)) + cD(c)x^n - cx^nG(c) \in C(R)$$

for all $x \in R$. Subtracting (10) from (9), we obtain

$$c(D(c)x^n - x^nG(c)) \in C(R)$$

for all $x \in R$. Since $c \in C(R)$ is nonzero and R is prime, we have

$$(11) \qquad D(c)x^n - x^nG(c) \in C(R)$$

for all $x \in R$. Hence we have by (8) and (11)

$$c(nD(x)x^{n-1} - nx^{n-1}G(x)) \in C(R)$$

for all $x \in R$. The fact that $c \in C(R)$ is nonzero and R is prime implies that

$$n(D(x)x^{n-1} - x^{n-1}G(x)) \in C(R)$$

for all $x \in R$. Since R is $(n+1)!$-torsion free,

$$D(x)x^{n-1} - x^{n-1}G(x) \in C(R)$$

for all $x \in R$. Continuing this process, we conclude that

$$(12) \qquad D(x) - G(x) \in C(R)$$

for all $x \in R$. In view of [3], we are forced to conclude that

$$(13) \qquad D(x) - G(x) = 0$$

for all $x \in R$. Thus in case when $C(R) \neq \{0\}$ we obtain by assumption and (13) that

$$[D(x), x^n] \in C(R)$$

for all $x \in R$. By [6, Corollary], we arrive at $D = G = 0$.

In case when $C(R) = \{0\}$, by assumption we get

$$D(x)x^n - x^nG(x) = 0$$

for all $x \in R$. By Theorem 2.1, $D = G = 0$. The proof of the theorem is complete. $\qquad\square$

COROLLARY 2.4. *Let n be a fixed positive integer. Let R be a non-commutative $(n+1)!$-torsion free prime ring. Suppose that there exists a Jordan derivation $D : R \to R$ such that*

$$[D(x), x^n] \in C(R)$$

for all $x \in R$. Then we have $D = 0$.

The motivation for the following result is due to Posner's second theorem.

THEOREM 2.5. *Let R be a noncommutative 2-torsion free prime ring. Suppose that there exists a derivation $D : R \to R$ such that the mapping $x \mapsto [aD(x), x]$ is commuting on R. Then we have either $a = 0$ or $D = 0$.*

Proof. Let a be a nonzero element in R. By [2, Theorem 1], the mapping $x \mapsto aD(x)$ is commuting on R. Thus we have

$$(14) \qquad\qquad [aD(x), x] = 0$$

for all $x \in R$. The linearization of (14) leads to

$$(15) \qquad\qquad [aD(x), y] + [aD(y), x] = 0,$$

for all $x, y \in R$, which gives the relation

$$(16) \qquad a[D(x), y] + [a, y]D(x) + a[D(y), x] + [a, x]D(y) = 0$$

for all $x, y \in R$. Replacing y by yx in (15) we have by (14)

$$(17) \qquad \begin{aligned} 0 = {} & a[D(x), y]x + [a, y]D(x)x + ay[D(x), x] + a[y, x]D(x) \\ & + [a, x]yD(x) + a[D(y), x]x + [a, x]D(y)x \end{aligned}$$

for all $x, y \in R$. Right multiplication of (16) by x gives

$$(18) \qquad a[D(x), y]x + [a, y]D(x)x + a[D(y), x]x + [a, x]D(y)x = 0$$

for all $x, y \in R$. Subtracting (18) from (17), we get

$$(19) \qquad ay[D(x), x] + a[y, x]D(x) + [a, x]yD(x) = 0$$

for all $x, y \in R$. By substituting ay for y in (19), we have

$$(20) \qquad 0 = a^2 y[D(x), x] + a^2[y, x]D(x)$$
$$+ a[a, x]yD(x) + [a, x]ayD(x)$$

for all $x, y \in R$. Left multiplication of (19) by a gives

$$(21) \qquad a^2 y[D(x), x] + a^2[y, x]D(x) + a[a, x]yD(x) = 0$$

for all $x, y \in R$. Subtracting (21) from (20), we arrive at

$$[a, x]ayD(x) = 0$$

for all $x, y \in R$. Since R is prime, we see that for any $x \in R$, either $D(x) = 0$ or $[a, x]a = 0$. That is, R is the union of its additive subgroups $A = \{x \in R : D(x) = 0\}$ and $B = \{x \in R : [a, x]a = 0\}$. Since a group cannot be the union of two proper subgroups, we see that either $A = R$ or $B = R$.

If $A = R$, then $D = 0$. If $B = R$, then it follows that

$$[a, x]a = 0$$

for all $x \in R$. Let us replace x by xy in this relation. Then we obtain that

$$[a, x]ya = 0$$

for all $x, y \in R$. The fact that $a \in R$ is nonzero and R is prime implies that a lies in $C(R)$, and hence by this and (14), the relation (19) reduces to

$$a[y, x]D(x) = 0$$

for all $x, y \in R$, from which it follows that

$$az[y, x]D(x) = 0$$

for all $x, y, z \in R$. Now by primeness of R, we see that

$$[y, x]D(x) = 0$$

for all $x, y \in R$. We obtain, by replacing y by yw, that

$$[y, x]wD(x) = 0$$

for all $x, y, w \in R$. Again using the fact that a group cannot be the union of two proper subgroups, it follows that $D = 0$ since R is non-commutative. Hence we see that, in any case, $D = 0$. The proof of the theorem is complete. $\qquad \square$

3. Jordan derivations on a noncommutative Banach algebras

In this section we obtain some results on noncommutative Banach algebras by using the preceding algebraic results.

THEOREM 2.6. *Let n be a fixed positive integer. Let A be a noncommutative Banach algebra. Suppose that there exist continuous Jordan derivations D and $G : A \to A$ such that*

$$D(x)x^n - x^n G(x) \in rad(A)$$

for all $x \in A$. Then we have $D(A) \subseteq rad(A)$ and $G(A) \subseteq rad(A)$.

Proof. Let P be any primitive ideal of A. Since D and G are continuous, by [10, Lemma 3.2], we have $D(P) \subseteq P$ and $G(P) \subseteq P$. Then we can define Jordan derivations D_P and G_P on the Banach algebra A/P by

$$D_P(x + P) = D(x) + P, \ G_P(x + P) = G(x) + P$$

for all $x \in A$. Note that every derivation on a semisimple Banach algebra is continuous [7, Remark 4.3]. First, in case when A/P is commutative, combining this result with the Singer-Wermer theorem gives $D_P = 0$ and $G_P = 0$ since A/P is semisimple. We intend to show that $D_P = 0$ and $G_P = 0$ in the case when A/P is noncommutative. Since the assumption of the theorem $D(x)x^n - x^n G(x) \in rad(A)$ gives $D_P(x+P)(x+P)^n - (x+P)^n G_P(x+P) = P$ and A/P is prime, we arrive at $D_P = 0$ and $G_P = 0$ by Theorem 2.3. Hence we see that $D(A) \subseteq P$ and $G(A) \subseteq P$. Since the intersection of all primitive ideals is the radical, we have $D(A) \subseteq rad(A)$ and $G(A) \subseteq rad(A)$.   □

COROLLARY 2.7. *Let n be a fixed positive integer. Let A be a noncommutative semisimple Banach algebra. Suppose that there exist Jordan derivations D and $G : A \to A$ such that*

$$D(x)x^n - x^n G(x) \in rad(A)$$

for all $x \in A$. Then we have $D = 0$ and $G = 0$.

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