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AVERAGING PROPERTY IN BANACH SPACES

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ABSTRACT. In this paper, we study averaging properties in Banach space. We prove that the convex block Banach-Saks property is equivalent to the reflexivity in a Banach space. And we show that a weakly compact operator is a convex block Banach-Saks operator.

A Banach space X is said to have the Banach-Saks property if every bounded sequence in X admits a subsequence whose arithmetic means converge in norm. In 1938, S. Kakutani [3] showed that if X is uniformly convex then X has the Banach-Saks property. In 1963, T. Nishiura and D. Waterman [4] proved that if a Banach space X has the Banach-Saks property then X is reflexive.

The natural questions are the followings : For a Banach space X with the Banach-Saks property, is it uniformly convex? And does every reflexive Banach space have the Banach-Saks property? In 1972, A. Baernstein [1] gave an example of a reflexive Banach space which does not have the Banach-Saks property. In 1978, C. J. Seifert [5] showed that the dual of Baernstein space which is not uniformly convex has the Banach-Saks property.

We introduce the following averaging property.

DEFINITION 1. A Banach space X is said to have the convex block Banach-Saks property if every bounded sequence in X admits a convex block sequence whose arithmetic means converge in norm.

It is clear that the Banach-Saks property implies the convex block Banach-Saks property. Since the Banach-Saks property implies the reflexivity in a Banach space, it is an apparent question whether a Banach space with the convex block Banach-Saks property is reflexive.

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THEOREM 2. *A Banach space X has the convex block Banach-Saks property if and only if X is reflexive.*

To prove Theorem 2, we consider the followings.

LEMMA 3. *Let (x_n) be a weakly convergent sequence. Then there exists a convex block sequence (y_n) of (x_n) such that (y_n) is norm convergent.*

Proof. Suppose (x_n) is weakly convergent to x. Then we have

$$x \in \overline{co}^{w}\{x_n\}_{n\geq s} = \overline{co}^{\|\cdot\|}\{x_n\}_{n\geq s}, \qquad \text{for all } s \in \mathbb{N}.$$

Then for $s = 1$, there exists a convex combination $y_1 = \sum_{j=1}^{p_1} a_j x_j$ in $co\{x_n\}_{n\geq 1}$ such that $\|y_1 - x\| < \frac{1}{2}$. Since

$$x \in \overline{co}^{w}\{x_n\}_{n\geq p_1+1} = \overline{co}^{\|\cdot\|}\{x_n\}_{n\geq p_1+1},$$

there exists a convex combination $y_2 = \sum_{j=p_1+1}^{p_2} a_j x_j$ in $co\{x_n\}_{n\geq p_1+1}$ such that $\|y_2 - x\| < \frac{1}{2^2}$. Continue this process, we get a convex block sequence (y_n) of (x_n) with $\|y_n - x\| < \frac{1}{2^n}$. This means that $y_n \to x$ in norm. □

DEFINITION 4. A real infinite matrix $A = (\alpha_{ij})$ is called an R-matrix if and only if

(1) $\sum_{j=1}^{\infty} \alpha_{ij} \not\to 0$ if $i \to \infty$

(2) $\lim_{i\to\infty} \alpha_{ij} = 0$ for all $j \in \mathbb{N}$.

An R-matrix $A = (\alpha_{ij})$ is called positive if none of its entries is negative.

The next theorem can be found in [2].

THEOREM 5. *Let K be a weakly closed bounded convex subset of a Banach space X. Then the following are equivalent :*

(1) *K is weakly compact ;*

(2) *given* $(x_n) \subseteq K$ *there is a positive* R*-matrix* $A = (\alpha_{ij})$ *such that* $\left\{\sum_{j=1}^{\infty} \alpha_{ij} x_j\right\}_{i=1}^{\infty}$ *converges in norm ;*

(3) *given* $(x_n) \subseteq K$ *there is a positive* R*-matrix* $A = (\alpha_{ij})$ *such that* $\left\{\sum_{j=1}^{\infty} \alpha_{ij} x_j\right\}_{i=1}^{\infty}$ *converges weakly.*

Proof of Theorem 2. Suppose that a Banach space X has the convex block Banach-Saks property. Let $\{x_n\}$ be a sequence in the unit ball B_X of X. Then there exists a convex block sequence $\{y_n\}$ of $\{x_n\}$ such that $\left(\frac{1}{n}\sum\limits_{i=1}^{n} y_i\right)$ converges in norm where $y_n = \sum\limits_{p_{n-1}+1}^{p_n} a_j x_j$, $0 \leq a_j \leq 1$ and $\sum\limits_{j=p_{n-1}+1}^{p_n} a_j = 1$, for increasing sequence $\{p_n\}$ of nonnegative integers. Define a real infinite matrix $A = (\alpha_{ij})$ with

$$\alpha_{ij} = \begin{cases} \dfrac{1}{i} a_j, & 1 \leq j \leq p_i \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{j=1}^{\infty} \alpha_{ij} = \sum_{j=1}^{p_i} \frac{1}{i} a_j = \frac{1}{i}\left[\sum_{j=1}^{p_1} a_j + \cdots + \sum_{j=p_{i-1}+1}^{p_i} a_j\right] = 1$$

and

$$0 \leq \lim_{i\to\infty} \alpha_{ij} \leq \lim_{i\to\infty} \frac{1}{i} = 0,$$

since $0 \leq a_j \leq 1$. Then $A = (\alpha_{ij})$ is a positive R-matrix. Since

$$\begin{aligned} \sum_{j=1}^{\infty} \alpha_{ij} x_j &= \frac{1}{i}\sum_{j=1}^{p_i} a_j x_j \\ &= \frac{1}{i}\left[\sum_{j=1}^{p_1} a_j x_j + \cdots + \sum_{j=p_{i-1}+1}^{p_i} a_j x_j\right] = \frac{1}{i}\sum_{j=1}^{i} y_j, \end{aligned}$$

$\left(\sum\limits_{j=1}^{\infty} \alpha_{ij} x_j\right)_{i=1}^{\infty}$ converges in norm. By Theorem 5, B_X is weakly compact and X is reflexive.

Suppose X is reflexive. Let (x_n) be a bounded sequence. Then (x_n) has a weakly convergent subsequence (x_{n_k}). By Lemma 3, there exists a convex block sequence (y_n) of (x_{n_k}) such that (y_n) is norm convergent. Then the arithmetic means of (y_n) converge in norm. This completes our proof. □

DEFINITION 6. A bounded operator $T : X \to Y$ is a convex Banach-Saks operator if whenever (x_n) is a bounded sequence in X, (Tx_n) has a convex block sequence whose arithmetic means are norm convergent in Y.

Certainly if either X or Y has the convex block Banach-Saks property then a bounded operator $T : X \to Y$ is a convex block Banach-Saks operator. An operator $T : X \to Y$ is weakly compact if T takes bounded sets to relatively weakly compact sets. It is clear that if either X or Y is reflexive, a bounded operator $T : X \to Y$ is weakly compact. Considering Theorem 2, we may conjecture that if a bounded operator T is weakly compact, then T is a convex block Banach-Saks operator.

THEOREM 7. *$T : X \to Y$ is weakly compact if and only if $T : X \to Y$ is a convex block Banach-Saks operator.*

Proof. Suppose $T : X \to Y$ is weakly compact. Let (x_n) be a sequence in B_X. Then there exists subsequence (Tx_{n_k}) of (Tx_n) which is weakly convergent. By Lemma 3, there exist a convex block sequence (y_n) of (Tx_{n_k}) which is norm-convergent in Y and the arithmetic means of (y_n) are norm convergent in Y.

Suppose $T : X \to Y$ is a convex block Banach-Saks operator. Let (x_n) be a bounded sequence in X. Then there exists a convex block sequence $\{y_n\}$ of $\{Tx_n\}$ such that $\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)$ converges in norm, where $y_n = \sum_{p_{n-1}+1}^{p_n} a_j T x_j$, $0 \leq a_j \leq 1$ and $\sum_{j=p_{n-1}+1}^{p_n} a_j = 1$, for increasing sequence $\{p_n\}$ of nonnegative integers. Define a real infinite matrix $A = (\alpha_{ij})$ with

$$\alpha_{ij} = \begin{cases} \dfrac{1}{i} a_j, & 1 \leq j \leq p_i \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{j=1}^{\infty} \alpha_{ij} = \sum_{j=1}^{p_i} \frac{1}{i} a_j = \frac{1}{i} \left[\sum_{j=1}^{p_1} a_j + \cdots + \sum_{j=p_{i-1}+1}^{p_i} a_j \right] = 1$$

and

$$0 \le \lim_{i\to\infty} \alpha_{ij} \le \lim_{i\to\infty} \frac{1}{i} = 0,$$

since $0 \le a_j \le 1$. Then $A = (\alpha_{ij})$ is a positive R-matrix. Since

$$\begin{aligned}\sum_{j=1}^{\infty} \alpha_{ij} T x_j &= \frac{1}{i} \sum_{j=1}^{p_i} a_j T x_j \\ &= \frac{1}{i} \left[\sum_{j=1}^{p_1} a_j T x_j + \cdots + \sum_{j=p_{i-1}+1}^{p_i} a_j T x_j \right] = \frac{1}{i} \sum_{j=1}^{i} y_j,\end{aligned}$$

$\left(\sum_{j=1}^{\infty} \alpha_{ij} T x_j \right)_{i=1}^{\infty}$ converges in norm. By Theorem 5, TB_X is weakly compact and $T : X \to Y$ is weakly compact. This completes our proof.□

References

[1] A. Baernstein, *On reflexivity and summability*, Studia Math. **42** (1972), 91–94.

[2] A. Brunel, H. Fong and L. Sucheston, *An ergodic superproperty of Banach spaces defined by a class of matrices*, Proc. Amer. Math. Soc. **49** (1975), 373–378.

[3] S. Kakutani, *Weak convergence in uniformly convex spaces*, Tôhoku Math. J. **45** (1938), 188–193.

[4] T. Nishiura and D. Waterman,, *Reflexivity and summability*, Studia Math. **23** (1963), 53–57.

[5] C. J. Seifert, *The dual of Baernstein's space and the Banach-Saks property*, Bull. Acad. Polon. Sci. **26** (1978), 237–239.

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