ON A CHARACTERIZATION OF ROUND SPHERES

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ABSTRACT. It is shown that, an immersion of n-dimensional compact manifold without boundary into \((n + 1)\)-dimensional Euclidean space, hyperbolic space or the open half spheres, is a totally umbilic immersion if for some \(r, r = 2, 3, \cdots, n\) the \(r\)-th mean curvature \(H_r\) does not vanish and there are nonnegative constants \(C_1, C_2, \cdots, C_r\) such that

\[
H_r = \sum_{i=1}^{r-1} C_i H_i.
\]

1. Introduction

Let \(Q^{n+1}\) be the Euclidean space \(\mathbb{R}^{n+1}\), the hyperbolic space \(\mathbb{H}^{n+1}\), or the open half sphere \(S^{n+1}_+\). Let \(x : M \rightarrow Q^{n+1}\) be an isometric immersion, where \(M\) is a compact, \(n\)-dimensional manifold without boundary. Let \(H_r\) denote the \(r\)-th mean curvature of \(M\), that is

\[
H_r = \frac{1}{\binom{n}{r}} \sum_{i_1 < i_2 < \cdots < i_r} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_r},
\]

where \(\lambda_1, \lambda_2, \cdots, \lambda_n\) the principal curvatures of \(M\), \(H_0\) is defined as 1. Obviously \(H_1\) is the usual mean curvature and \(H_n\) is the Gauss Kronecker curvature. In [3], it was shown that if the \((r - 1)\)-th mean curvature of \(M\) does not vanish and the ratio \(\frac{H_r}{H_{r-1}}\) is constant for \(r = 2, 3, \cdots, n\), then \(x(M)\) is a geodesic hypersphere. In [4], the same characterization is obtained in terms of \(H_1\) and \(H_n\). Koh and Lee [5] obtained that if one of the ratio of two mean curvature functions on an isometrically immersed closed hypersurfaces in \(Q^{n+1}\) is constant, then the hypersurfaces is a round sphere.

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In the sequel we use Minkowski integral formulas [1] which are the main formulas for the proofs of [3], [4] and [5]. In this work \( (, ) \) denotes the usual Euclidean inner product on \( \mathbb{R}^{n+1} \) (on \( \mathbb{R}^{n+2} \) when \( Q^{n+1} \) is \( \mathbb{R}^{n+1} \) (respectively \( S^{n+1}_+ \)) and \( (, ) \) denotes the Lorentzian inner product on \( \mathbb{R}^{n+2} \) when \( Q^{n+1} \) is \( H^{n+1} \). The inner product \( \langle x, \mu \rangle \) denotes the support function of \( M \), where \( \mu \) is the normal direction and \( x \) is the position vector field on \( M \).

**Lemma 1.** [1] For \( k = 1, 2, \cdots, n \) the following identities hold

1. When \( Q^{n+1} \) is \( \mathbb{R}^{n+1} \),

\[
\int_M (H_{k-1} + H_k \langle x, \mu \rangle) \, dM = 0
\]

2. When \( Q^{n+1} \) is \( H^{n+1} \),

\[
\int_M (H_{k-1} \langle x, p \rangle + H_k \langle \mu, p \rangle) \, dM = 0
\]

for any \( p \in \mathbb{R}^{n+2} \),

3. When \( Q^{n+1} \) is \( S^{n+1}_+ \),

\[
\int_M (H_{k-1} \langle x, p \rangle - H_k \langle \mu, p \rangle) \, dM = 0
\]

for any \( p \in \mathbb{R}^{n+2} \).

For any elementary symmetric polynomial \( H_k \), \( k = 1, 2, \cdots, n \) the following algebraic inequalities is well-known [7],

\[
H_{k-1} H_{k+1} - H_k^2 \leq 0,
\]

where the equality holds if and only if all the principal curvatures are equal (i.e., \( x(M) \) is totally umbilical).

2. **Main theorem**

**Theorem 1.** Let \( Q^{n+1} \) be one of \( \mathbb{R}^{n+1}, H^{n+1} \) or \( S^{n+1}_+ \) and \( x : M \to Q^{n+1} \) be an isometric immersion of a compact, \( n \)-dimensional manifold \( M \) without boundary. Assume \( x(M) \) is convex. If \( H_r \) does not vanish and there are nonnegative constants \( C_1, C_2, \cdots, C_{r-1} \) such that

\[
H_r = \sum_{i=1}^{r-1} C_i H_i
\]

then \( x(M) \) is a geodesic hypersphere.
This theorem is also a generalization of [3].

REMARK 1. An analogy of this theorem in the affine differential geometry of $\mathbb{R}^{n+1}$ was proved in [6].

REMARK 2. When $M$ is an $n$-dimensional oriented closed submanifold of Euclidean $m$-space $E^m$, another version of this theorem was proved in [2].

Proof. Case 1. When $\mathbb{Q}^{n+1}$ is Euclidean space $\mathbb{R}^{n+1}$, all the principal curvatures and all curvature functions are positive on $M$ since $M$ is convex. Thus

\begin{equation}
H_i H_{r-1} - H_r H_{i-1} \geq 0
\end{equation}

for $i < r$ where the equality holds if and only if all the principal curvatures are equal. The assumptions and the inequality (2.1) imply

\begin{equation}
1 = \sum_{i=1}^{r-1} C_i \frac{H_i}{H_r} \geq \sum_{i=1}^{r-1} C_i \frac{H_{i-1}}{H_{r-1}}
\end{equation}

or

\begin{equation}
H_{r-1} - \sum_{i=1}^{r-1} C_i H_{i-1} \geq 0.
\end{equation}

We use the integral formulas (1.1) for $k = 1, 2, \cdots, r$ then we have

\[0 \leq \int_{M} (H_{r-1} - \sum_{i=1}^{r-1} C_i H_{i-1}) dM = \int_{M} (-H_r + \sum_{i=1}^{r-1} C_i H_i) \langle x, \mu \rangle dM = 0.\]

Now we have from (2.2) the following equality

\[H_{r-1} = \sum_{i=1}^{r-1} C_i H_{i-1}.
\]

This equality and the inequality (2.2) imply that the equality holds in (2.1), which implies that all the principal curvatures are equal. Thus every point is umbilical, that is, $x(M)$ is a geodesic hypersphere.

Case 2. From Lemma 1 and the assumption of the theorem, we have

\[\int_{M} H_{r-1} \langle x, p \rangle dM = -\int_{M} H_r \langle x, p \rangle dM = -\int_{M} \sum_{i=1}^{r-1} C_i H_i \langle x, p \rangle dM
\]

\[= \int_{M} C_1 \langle x, p \rangle dM + \int_{M} C_2 H_1 \langle x, p \rangle dM + \cdots + \int_{M} C_{r-1} H_{r-2} \langle x, p \rangle dM.
\]
It follows that
\[
\int_M (H_{r-1} - \sum_{i=1}^{r-1} C_i H_{i-1}) \langle x, p \rangle \, dM = 0.
\]

Now if take \( p = (1,0, \cdots, 0) \in \mathbb{R}^{n+2} \), then the sign of \( \langle x, p \rangle \) does not change on \( M \). As a consequence of the equality above we have
\[
H_{r-1} = \sum_{i=1}^{r-1} C_i H_{i-1}.
\]

This equality shows that \( x(M) \) is a geodesic hypersphere as in Case 1.

**Case 3.** From Lemma 1 and the assumption of the theorem, we have
\[
\int_M H_{r-1} \langle x, p \rangle \, dM = \int_M H_r \langle \mu, p \rangle \, dM
\]
\[
= \int_M \sum_{i=1}^{r-1} C_i H_i \langle \mu, p \rangle \, dM
\]
\[
= \int_M \sum_{i=1}^{r-1} C_i H_{i-1} \langle x, p \rangle \, dM.
\]

Thus the following equality holds for every \( p \) on \( M \)
\[
\int_M (H_{r-1} - \sum_{i=1}^{r-1} C_i H_{i-1}) \langle x, p \rangle \, dM = 0.
\]

Since \( M \) lies in the open half sphere, one can find a vector \( p \) so that \( \langle x, p \rangle \) is positive on \( M \), then it follows that,
\[
H_{r-1} - \sum_{i=1}^{r-1} C_i H_{i-1} = 0.
\]

This equality shows that \( x(M) \) is a geodesic hypersphere as in Case 1. \( \Box \)

**References**


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