Bull. Korean Math. Soc. **39** (2002), No. 4, pp. 681–685

ON A CHARACTERIZATION OF ROUND SPHERES

Leyla Onat

Abstract. It is shown that, an immersion of n-dimensional compact manifold without boundary into $(n+1)$-dimensional Euclidean space, hyperbolic space or the open half spheres, is a totally umbilic immersion if for some r, $r = 2, 3, \cdots, n$ the r-th mean curvature H_r does not vanish and there are nonnegative constants $C_1, C_2, \cdots, C_r$ such that

$$H_r = \sum_{i=1}^{r-1} C_i H_i.$$

1. Introduction

Let $\mathbb{Q}^{n+1}$ be the Euclidean space $\mathbb{R}^{n+1}$, the hyperbolic space $\mathbb{H}^{n+1}$, or the open half sphere $\mathbb{S}^{n+1}_{+}$. Let $x : M \to \mathbb{Q}^{n+1}$ be an isometric immersion, where M is a compact, n-dimensional manifold without boundary. Let H_r denote the r- th mean curvature of M, that is

$$H_r = \frac{1}{\begin{pmatrix} r \\ n \end{pmatrix}} \sum_{i_1 < i_2 \cdots < i_r} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_r},$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ the principal curvatures of M, H_o is defined as 1. Obviously H_1 is the usual mean curvature and H_n is the Gauss Kronecker curvature. In [3], it was shown that if the $(r-1)$-th mean curvature of M does not vanish and the ratio $\frac{H_r}{H_{r-1}}$ is constant for $r = 2, 3, \cdots, n$, then $x(M)$ is a geodesic hypersphere. In [4], the same characterization is obtained in terms of H_1 and H_n. Koh and Lee [5] obtained that if one of the ratio of two mean curvature functions on an isometrically immersed closed hypersurfaces in $\mathbb{Q}^{n+1}$ is constant, then the hypersurfaces is a round sphere.

Received May 8, 2002.
2000 Mathematics Subject Classification: 53C40, 53C42, 53A15.
Key words and phrases: higher order mean curvature, umbilical points, Minkowski formula.

In the sequel we use Minkowski integral formulas [1] which are the main formulas for the proofs of [3], [4] and [5]. In this work $\langle , \rangle$ denotes the usual Euclidean inner product on $\mathbb{R}^{n+1}$ (on $\mathbb{R}^{n+2}$) when $\mathbb{Q}^{n+1}$ is $\mathbb{R}^{n+1}$ (respectively $\mathbb{S}_{+}^{n+1}$) and $\langle , \rangle$ denotes the Lorentzian inner product on $\mathbb{R}^{n+2}$ when $\mathbb{Q}^{n+1}$ is $\mathbb{H}^{n+1}$. The inner product $\langle x, \mu \rangle$ denotes the support function of M, where μ is the normal direction and x is the position vector field on M.

LEMMA 1. [1] *For $k = 1, 2, \cdots, n$ the following identities hold*

1. *When $\mathbb{Q}^{n+1}$ is $\mathbb{R}^{n+1}$,*

$$\int_M \left(H_{k-1} + H_k \langle x, \mu \rangle \right) dM = 0 \tag{1.1}$$

2. *When $\mathbb{Q}^{n+1}$ is $\mathbb{H}^{n+1}$,*

$$\int_M \left(H_{k-1} \langle x, p \rangle + H_k \langle \mu, p \rangle \right) dM = 0 \tag{1.2}$$

for any $p \in \mathbb{R}^{n+2}$,

3. *When $\mathbb{Q}^{n+1}$ is $\mathbb{S}_{+}^{n+1}$*

$$\int_M \left(H_{k-1} \langle x, p \rangle - H_k \langle \mu, p \rangle \right) dM = 0 \tag{1.3}$$

for any $p \in \mathbb{R}^{n+2}$.

For any elementary symmetric polynomial H_k, $k = 1, 2, \cdots, n$ the following algebraic inequalities is well-known [7].

$$H_{k-1} H_{k+1} - H_k^2 \leq 0,$$

where the equality holds if and only if all the principal curvatures are equal (i.e., $x(M)$ is totally umbilical).

2. Main theorem

THEOREM 1. *Let $\mathbb{Q}^{n+1}$ be one of $\mathbb{R}^{n+1}$, $\mathbb{H}^{n+1}$ or $\mathbb{S}_{+}^{n+1}$ and $x : M \to \mathbb{Q}^{n+1}$ be an isometric immersion of a compact, n-dimensional manifold M without boundary. Assume $x(M)$ is convex. If H_r does not vanish and there are nonnegative constants $C_1, C_2, \cdots, C_{r-1}$ such that*

$$H_r = \sum_{i=1}^{r-1} C_i H_i$$

then $x(M)$ is a geodesic hypersphere.

This theorem is also a generalization of [3].

REMARK 1. An analogy of this theorem in the affine differential geometry of $\mathbb{R}^{n+1}$ was proved in [6] .

REMARK 2. When M is an n-dimensional oriented closed submanifold of Euclidean m-space E^m, another version of this theorem was proved in [2].

Proof. **Case 1.** When $\mathbb{Q}^{n+1}$is Euclidean space $\mathbb{R}^{n+1}$, all the principal curvatures and all curvature functions are positive on M since M is convex. Thus

$$H_i H_{r-1} - H_r H_{i-1} \geq 0 \tag{2.1}$$

for $i < r$ where the equality holds if and only if all the principal curvatures are equal. The assumptions and the inequality (2.1) imply

$$1 = \sum_{i=1}^{r-1} C_i \frac{H_i}{Hr} \geq \sum_{i=1}^{r-1} C_i \frac{H_{i-1}}{H_{r-1}} \tag{2.2}$$

or

$$H_{r-1} - \sum_{i=1}^{r-1} C_i H_{i-1} \geq 0.$$

We use the integral formulas (1.1) for $k = 1, 2, \cdots, r$ then we have

$$0 \leq \int_M (H_{r-1} - \sum_{i=1}^{r-1} C_i H_{i-1}) dM = \int_M (-H_r + \sum_{i=1}^{r-1} C_i H_i) \langle x, \mu \rangle \, dM = 0.$$

Now we have from (2.2) the following equality

$$H_{r-1} = \sum_{i=1}^{r-1} C_i H_{i-1}.$$

This equality and the inequality (2.2) imply that the equality holds in (2.1), which implies that all the principal curvatures are equal. Thus every point is umbilical, that is, $x(M)$ is a geodesic hypersphere.
Case 2. From Lemma 1 and the assumption of the theorem, we have

$$\int_M H_{r-1} \langle x, p \rangle \, dM = - \int_M H_r \langle \mu, p \rangle \, dM = - \int_M \sum_{i=1}^{r-1} C_i H_i \langle \mu, p \rangle \, dM$$

$$= \int_M C_1 \langle x, p \rangle \, dM + \int_M C_2 H_1 \langle x, p \rangle \, dM + \cdots + \int_M C_{r-1} H_{r-2} \langle x, p \rangle \, dM.$$

It follows that

$$\int_M (H_{r-1} - \sum_{i=1}^{r-1} C_i H_{i-1}) \langle x, p \rangle \, dM = 0.$$

Now if take $p = (1, 0, \cdots, 0) \in \mathbb{R}^{n+2}$, then the sign of $\langle x, p \rangle$ does not change on M. As a consequence of the equality above we have

$$H_{r-1} = \sum_{i=1}^{r-1} C_i H_{i-1}.$$

This equality shows that $x(M)$ is a geodesic hypersphere as in Case 1.
Case 3. From Lemma 1 and the assumption of the theorem, we have

$$\begin{aligned}\int_M H_{r-1} \langle x, p \rangle \, dM &= \int_M H_r \langle \mu, p \rangle \, dM \\ &= \int_M \sum_{i=1}^{r-1} C_i H_i \langle \mu, p \rangle \, dM \\ &= \int_M \sum_{i=1}^{r-1} C_i H_{i-1} \langle x, p \rangle \, dM.\end{aligned}$$

Thus the following equality holds for every p on M

$$\int_M (H_{r-1} - \sum_{i=1}^{r-1} C_i H_{i-1}) \langle x, p \rangle \, dM = 0.$$

Since M lies in the open half sphere, one can find a vector p so that $\langle x, p \rangle$ is positive on M, then it follows that,

$$H_{r-1} - \sum_{i=1}^{r-1} C_i H_{i-1} = 0.$$

This equality shows that $x(M)$ is a geodesic hypersphere as in Case 1. □

References

[1] I. Bivens, *Integral formulas and hyperspheres in a simply connected space form*, Proc. Amer. Math. Soc. **88** (1983), no. 1, 113–118.
[2] B. Y. Chen and K. Yano, *Some integral formulas for submanifolds and their applications*, J. Differential Geometry **5** (1971), 467–477.
[3] S. E. Koh, *A Characterization of round spheres*, Proc. Amer. Math. Soc. **126** (1998), no. 12, 3657–3660.

[4] S. W. Lee and S. E. Koh, *Another characterization of round spheres*, Bull. Korean Math. Soc. **36** (1999), no. 4, 701–706.

[5] S. E. Koh and S-W Lee, *Addendum to the paper: Sphere theorem by means of the ratio of mean curvature functions*, Glasgow Math. J. **43** (2001), 275–276.

[6] A. M. Li, U. Simon, G. Zhao, *Global affine differential geometry of hypersurfaces*, De Gruyter Berlin, 1993.

[7] K. Nomizu, T. Sasaki, *Affine Differential Geometry*, Cambridge Univ. Press, 1994.

Department of Mathematics, Mersin University, Mersin, Turkey
E-mail: leylao@mersin.edu.tr