

RELATIONS BETWEEN CERTAIN DOMAINS IN THE COMPLEX PLANE AND POLYNOMIAL APPROXIMATION IN THE DOMAINS

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ABSTRACT. We show that the class of inner chordarc domains is properly contained in the class of exterior quasiconvex domains. We also show that the class of exterior quasiconvex domains is properly contained in the class of John disks. We give the conditions which make the converses of the above results be true.

Next, we show that an exterior quasiconvex domain satisfies certain growth conditions for the exterior Riemann mapping. From the results we show that the domain satisfies the Bernstein inequality and the integrated version of it.

Finally, we assume that f is a function which is continuous in the closure of a domain D and analytic in D . We show connections between the smoothness of f and the rate at which it can be approximated by polynomials on an exterior quasiconvex domain and a Lip_α -extension domain.

1. Introduction

Suppose that D is a domain in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $CD = \overline{\mathbb{C}} \setminus D$ and $D^* = \overline{\mathbb{C}} \setminus \overline{D}$, $\mathbb{B}(z, r) = \{w : |w - z| < r\}$ for $z \in \mathbb{C}$ and $r > 0$ and let $\mathbb{B} = \mathbb{B}(0, 1)$ be the unit disk in \mathbb{C} . Let $\ell(\gamma)$ denote the euclidean length of a curve γ , $\text{dia}(\gamma)$ be a diameter of γ and $\text{dist}(z, \partial D)$ denote the distance from z to ∂D .

DEFINITION 1.1. We say that a Jordan domain D in $\overline{\mathbb{C}}$ is c -chordarc, $1 \leq c < \infty$, if for each pair of points $z_1, z_2 \in \partial D \setminus \{\infty\}$

$$(1.1) \quad \min(\ell(\gamma_1), \ell(\gamma_2)) \leq c|z_1 - z_2|,$$

Received February 25, 2002.

2000 Mathematics Subject Classification: 30C20.

Key words and phrases: inner chordarc domain, quasiconvex, John disk, the Bernstein inequality, polynomial approximation.

This work was supported by Korea Research Foundation Grant (KRF-1996-002-D0554).

where γ_1 and γ_2 are two components of $\partial D \setminus \{z_1, z_2\}$.

DEFINITION 1.2. We say that a domain D in $\overline{\mathbb{C}}$ is a K -quasidisk, $1 \leq K < \infty$, if it is the image of \mathbb{B} under a K -quasiconformal self mapping of $\overline{\mathbb{C}}$.

From the definition, it follows that a quasidisk is a Jordan domain in $\overline{\mathbb{C}}$. Conversely, a Jordan domain in $\overline{\mathbb{C}}$ with smooth boundary is a quasidisk. This class of Jordan domains admits the following simple characterization which is called Ahlfors' three point condition.

LEMMA 1.3 [1]. A Jordan domain D in $\overline{\mathbb{C}}$ is a K -quasidisk if and only if there exists a constant c , $1 \leq c < \infty$, such that for each pair of points $z_1, z_2 \in \partial D \setminus \{\infty\}$,

$$(1.2) \quad \min(\text{dia}(\gamma_1), \text{dia}(\gamma_2)) \leq c|z_1 - z_2|,$$

where γ_1 and γ_2 are two components of $\partial D \setminus \{z_1, z_2\}$. Here K and c depend on each other.

Thus a chordarc domain is a quasidisk, but there exists a quasidisk which is not chordarc (for example, a snowflake domain).

A Jordan domain D in $\overline{\mathbb{C}}$ is a K -quasidisk if and only if there is a constant $c \geq 1$ such that each two points z_1 and z_2 in D can be joined by an arc γ in D such that

$$\ell(\gamma) \leq c|z_1 - z_2|$$

and

$$(1.3) \quad \min(\ell(\gamma_1), \ell(\gamma_2)) \leq c \, \text{dist}(z, \partial D)$$

for all $z \in \gamma$, where γ_1 and γ_2 are the components of $\gamma \setminus \{z\}$, [1], [4]. Here K and c depend on each other.

Next we consider the one-sided versions of (1.1) and (1.2), where $|z_1 - z_2|$ is replaced by one of the interior distances in D given by

$$\lambda_D(z_1, z_2) = \inf \ell(\beta),$$

$$\delta_D(z_1, z_2) = \inf \text{dia}(\beta).$$

Here both infima are taken over all open arcs β in D which join z_1 and z_2 .

DEFINITION 1.4. Suppose that D is a Jordan domain in \mathbb{C} and ∂D is locally rectifiable. We say that D is a *c-inner chordarc domain* if for each pair of points $z_1, z_2 \in \partial D \setminus \{\infty\}$

$$\min(\ell(\gamma_1), \ell(\gamma_2)) \leq c\lambda_D(z_1, z_2),$$

where γ_1 and γ_2 are two components of $\partial D \setminus \{z_1, z_2\}$.

Thus a chordarc domain is an inner chordarc domain, but there is an inner chordarc domain which is not chordarc (for example, $\mathbb{B} \setminus [0, 1]$).

DEFINITION 1.5. A simply connected bounded domain $D \subset \mathbb{C}$ is said to be a *c-John disk* if there exist a point $z_0 \in D$ and a constant $c \geq 1$ such that each point $z_1 \in D$ can be joined to z_0 by an arc γ in D satisfying

$$\ell(\gamma(z_1, z)) \leq c \operatorname{dist}(z, \partial D)$$

for each $z \in \gamma$, where $\gamma(z_1, z)$ is the subarc of γ with endpoints z_1, z . We call z_0 a *John center*, c a *John constant* and γ a *c-John arc*.

There are several equivalent definitions for John disks, [9], [10], [11], [14]. For example, a domain D in \mathbb{C} is a *c-John disk* if and only if there is a constant $c \geq 1$ such that each pair of points $z_1, z_2 \in D$ can be joined by an arc γ in D which satisfies (1.3) [10]. Thus the class of quasidisks is properly contained in the class of John disks. The converse is not true since a John disk need not even be a Jordan domain. From the definition we can see that a domain is a John domain if it is possible to move from one point to another without passing too close to the boundary. Also to compare with an inner chordarc domain we give the following result of [10, Theorem 6.3]: A simply connected bounded domain $D \subset \mathbb{C}$ is a *c-John disk* if

$$\min(\operatorname{dia}(\gamma_1), \operatorname{dia}(\gamma_2)) \leq c\delta_D(z_1, z_2),$$

where γ_1 and γ_2 are two components of $\partial D \setminus \{z_1, z_2\}$. An inner chordarc domain is a John disk [7, Corollary 2.13]. But the converse is not true, since the boundary of John disk need not be locally rectifiable.

DEFINITION 1.6. We say that a set $A \subset \overline{\mathbb{C}}$ is *c-quasiconvex*, $1 \leq c < \infty$, if each pair of points $z_1, z_2 \in A \setminus \{\infty\}$ can be joined by a rectifiable curve γ in A such that

$$\ell(\gamma) \leq c|z_1 - z_2|.$$

We also say that a domain D in $\overline{\mathbb{C}}$ is a c -exterior quasiconvex domain if CD is c -quasiconvex, $1 \leq c < \infty$.

Then CD is 1-quasiconvex if and only if CD is convex.

In Section 2, we show that if D is a c -inner chordarc domain, then D is a c' -exterior quasiconvex domain, $c' = c'(c)$. We also show that if D is a c -exterior quasiconvex domain, then D is a c' -John disk, $c' = c'(c)$. To show that the converses are not true, we construct the counterexamples. Then we give conditions which make the converse of Theorem 2.1 be true.

DEFINITION 1.7. Suppose that D is a domain in \mathbb{C} and f is a complex valued function defined in D . We say that f is in the *Lipschitz class*, $Lip_\alpha(D)$, $0 < \alpha \leq 1$, if there is a constant m such that

$$(1.4) \quad |f(z_1) - f(z_2)| \leq m |z_1 - z_2|^\alpha$$

for all $z_1, z_2 \in D$. We let $\|f\|_\alpha$ denote the infimum of the numbers m for which (1.4) holds.

f is said to belong to the *local Lipschitz class*, $locLip_\alpha(D)$, if there is a constant m such that (1.4) holds whenever z_1, z_2 lie in any open disk which is contained in D . Let $\|f\|_\alpha^{loc}$ denote the infimum of the numbers m such that (1.4) holds in this situation.

A domain D is called a Lip_α -extension domain if there exists a constant a depending on D and α such that $f \in locLip_\alpha(D)$ implies $f \in Lip_\alpha(D)$ with

$$\|f\|_\alpha \leq a \|f\|_\alpha^{loc}.$$

In [6] Gehring and Martio proved that quasidisks are Lip_α -extension domains for all $\alpha \in (0, 1]$. In the same paper examples are given of domains which are Lip_α -extension domains but not quasidisks.

DEFINITION 1.8. Suppose that E is a bounded continuum whose complement in $\overline{\mathbb{C}}$ is connected. If f is continuous on E and analytic on the interior of E , then the *best approximation* $E_n(f)$ of f by polynomials of degree n is given by

$$E_n(f) = \inf\{\|f - p\|_E : p \in P_n\},$$

where

$$\|f\|_E = \max\{|f(z)| : z \in E\}$$

and P_n denotes the class of polynomials p of degree at most n .

In Section 3, we show that exterior quasiconvex domains satisfy certain growth conditions for the exterior Riemann mapping. From the results we show that the domains satisfy the Bernstein inequality and the integrated version of it. Furthermore we show connections between the smoothness of a function f and the rate at which it can be approximated by polynomials on domains introduced above.

2. Relations between inner chordarc domains, exterior quasiconvex domains and John disks

THEOREM 2.1. *Suppose that D is a Jordan domain in \mathbb{C} and that ∂D is locally rectifiable. If D is a c -inner chordarc domain, then D is a c' -exterior quasiconvex domain, where c' , $1 \leq c' < \infty$, is a constant which depends only on c .*

To show Theorem 2.1, we need the following fact.

PROPOSITION 2.2 [7, Theorem 2.7]. *D is a c -inner chordarc domain if and only if there is a constant c' such that for each straight cross cut $\beta = [z_1, z_2]$ of D*

$$\min(\ell(\gamma_1), \ell(\gamma_2)) \leq c' \ell(\beta),$$

where γ_1 and γ_2 are two components of $\partial D \setminus \{z_1, z_2\}$ and c, c' depend only on each other.

Proof of Theorem 2.1. Suppose that D is a c -inner chordarc domain. Fix $z_1, z_2 \in \partial D$ and let $\beta = [z_1, z_2]$ be the line segment joining z_1 and z_2 . If $\beta \subset \partial D$, then

$$(2.1) \quad \ell(\beta) = |z_1 - z_2|.$$

If $\beta \cap D \neq \emptyset$, let β_j , $j = 1, 2, \dots, k$, be the components of $\beta \cap D$ with end points $y_j, y'_j \in \partial D$ and let α_j be the component of $\partial D \setminus \{y_j, y'_j\}$ with shorter length, labeled from z_1 . Since each β_j is a straight cross cut of D , by Proposition 2.2 for some $c' > 1$, depending only on c ,

$$\ell(\alpha_j) \leq c' \ell(\beta_j) = c' |y_j - y'_j|.$$

Set

$$\gamma = [z_1, y_1] \cup \left(\bigcup_{j=1}^k (\alpha_j \cup [y'_j, y_{j+1}]) \right) \cup [y'_k, z_2].$$

Then γ is a curve in CD joining z_1 and z_2 and

(2.2)

$$\ell(\gamma) \leq |z_1 - y_1| + c' \sum_{j=1}^k |y_j - y'_j| + \sum_{j=1}^{k-1} |y'_j - y_{j+1}| + |y'_k - z_2| \leq c' |z_1 - z_2|.$$

Therefore by (2.1) and (2.2) CD is c' -quasiconvex. \square

THEOREM 2.3. Suppose that D is a Jordan domain in \mathbb{C} .

- (1) If D is a c -exterior quasiconvex domain, then D is a c' -John disk.
- (2) If D^* is c -quasiconvex, then D is a c' -John disk.

Here c' depend only on c .

To prove Theorem 2.3 we need a lemma.

LEMMA 2.4. If D is a Jordan domain in $\overline{\mathbb{C}}$, then the following conditions are equivalent, where the constants in each condition need not be the same:

- (1) D is a c -John disk.
- (2) For every $z \in \mathbb{C}$ and $r > 0$, any two points in $CD \cap \overline{\mathbb{B}}(z, r)$ can be joined by a continuum in $CD \cap \overline{\mathbb{B}}(z, cr)$.
- (3) Condition (2) holds for D^* , i.e., each pair of points $z_1, z_2 \in D^*$ can be joined by a continuum $E \subset D^*$ for which

$$\text{dia}(E) \leq c|z_1 - z_2|.$$

Proof. The equivalence of (1) and (2) is proved in [10, Theorem 4.5]. Suppose that (3) holds, then the proof of [16, Proposition 4.1 (2)] gives (2). Suppose next that (2) holds, fix $z \in \mathbb{C}$ and $r > 0$, and choose $z_1, z_2 \in D^* \cap \overline{\mathbb{B}}(z, r)$. Then by hypothesis there exists an arc γ joining z_1 and z_2 in $CD \cap \overline{\mathbb{B}}(z, cr)$. Now since D^* is a Jordan domain, there exists an imbedding $h : CD \rightarrow D^*$ such that $h(z) = z$ for $z \in CD$ with $d(z, \partial D^*) > \epsilon$, where $0 < \epsilon < \min(d(z_j, \partial D), r)$, $j = 1, 2$. Then for each $z \in CD$, $|h(z) - z| \leq \epsilon$ and hence $h(\gamma)$ is an arc joining z_1 and z_2 in $D^* \cap \overline{\mathbb{B}}(z, (c+1)r)$. \square

Proof of Theorem 2.3. Suppose that CD is c -quasiconvex. Then each pair of points $z_1, z_2 \in CD \setminus \{\infty\}$ can be joined by a rectifiable curve γ in CD such that

$$\text{dia}(\gamma) \leq \ell(\gamma) \leq c|z_1 - z_2|.$$

Therefore by the equivalence of (1) and (2) in Lemma 2.4, D is a c' -John disk, $c' = c'(c)$. Next suppose that D^* is c -quasiconvex. Then similarly D^* satisfies the condition of Lemma 2.4 (3), and then D is a c' -John disk, $c' = c'(c)$. \square

Now we construct a counterexample to show that the converse of Theorem 2.1 is not true.

THEOREM 2.5. *There is a domain D in \mathbb{C} which is an exterior quasiconvex domain, but not an inner chordarc domain.*

Proof. Consider the Jordan arcs $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$ as shown in Figure 1 in [14], where

$$|z_i - w_i| = |w_i - \zeta_i| = |\zeta_i - \xi_i| = |w_i - \xi_i| = 2^{2-i},$$

$i = 1, 2, 3, \dots$ and $z_1 = -8$. Then ξ_i converges to 0. Let α be the Jordan arc obtained by letting $\xi_i = z_{i+1}$ for all $i = 1, 2, 3, \dots$. Then

$$\begin{aligned} \ell(\alpha) &= 2 + 2 \cdot 2 + 1 + 2 \cdot 1 \cdot \frac{4}{3} + \frac{1}{2} + 2 \cdot \frac{1}{2} \cdot \left(\frac{4}{3}\right)^2 + \dots \\ &= \left(2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots\right) + \left(4 + \frac{8}{3} + \frac{16}{9} + \dots\right). \end{aligned}$$

Since the second series diverges, α is nonrectifiable. Next let $\alpha' = \{z = x + iy \in \mathbb{C} : x - iy \in \alpha\}$ and let

$$A = \{w \in \mathbb{C} : w = z e^{i(-\frac{\pi}{4})}, z \in \alpha\}$$

$$A' = \{w' \in \mathbb{C} : w' = z e^{i\frac{\pi}{4}}, z \in \alpha'\}.$$

Now suppose that D is a component of $\mathbb{B}(0, 8) \setminus \{A \cup A'\}$ with larger diameter. Then CD is obviously quasiconvex. But ∂D is not regular, thus by the following Theorem 2.8, D is not an inner chordarc domain. \square

To get conditions which make the converse of Theorem 2.1 be true, we introduce the following concept.

DEFINITION 2.6. A curve α in \mathbb{C} is c -regular if for all $z \in \mathbb{C}$ and each $r > 0$

$$m_1(\alpha \cap \mathbb{B}(z, r)) \leq cr,$$

where m_1 denotes the 1-dimensional Hausdorff measure.

A chordarc domain is a quasidisk with regular boundary [13, Proposition 7.7] and the following tells us that an inner chordarc domain is a John disk with regular boundary.

PROPOSITION 2.7 [7, Theorem 2.8]. Suppose that D be a Jordan domain in \mathbb{C} and that ∂D is locally rectifiable. Then the following conditions are equivalent:

- (1) D is a c -inner chordarc domain,
- (2) D is a c -John disk and ∂D is c' -regular,
- (3) D is a c -John disk and for each bounded subarc γ in ∂D ,

$$\ell(\gamma) \leq c' \operatorname{dia}(\gamma).$$

Here the constants c, c' depend only on each other.

Now we can replace the John disk condition in Proposition 2.7 by exterior quasiconvex condition.

THEOREM 2.8. Suppose that D be a Jordan domain in \mathbb{C} and that ∂D is locally rectifiable. Then the following conditions are equivalent:

- (1) D is a c -inner chordarc domain,
- (2) D is a c_1 -exterior quasiconvex domain and ∂D is c_2 -regular,
- (3) D is a c_1 -exterior quasiconvex domain and for each bounded subarc γ in ∂D ,

$$\ell(\gamma) \leq c_2 \operatorname{dia}(\gamma).$$

Here the constants c, c_1, c_2 depend only on each other.

Proof. Suppose that D is a c -inner chordarc domain. Then by Theorem 2.1 D is a c_1 -exterior quasiconvex domain. By the similar arguments to the proof of Proposition 2.7 we obtain that ∂D is c_2 -regular and that for each bounded subarc γ in ∂D ,

$$\ell(\gamma) \leq c_2 \operatorname{dia}(\gamma).$$

Next suppose that D holds (3). Then by Theorem 2.3 (1), D is a c -John disk. Therefore by Proposition 2.7 D is a c -inner chordarc domain. \square

Next we construct a counterexample to show that the converse of Theorem 2.3 (1) is not true.

THEOREM 2.9. There is a domain D in \mathbb{C} which is a John disk but not an exterior quasiconvex domain.

Proof. Let α be the nonrectifiable Jordan arc in the proof of Theorem 2.5. Then we construct another similar nonrectifiable Jordan arc α' below α . For arbitrary $\epsilon > 0$, consider lines β_{11} , β_{12} and β_{13} which are parallel to the segments $[z_1, w_1]$, $[w_1, \zeta_1]$ and $[\zeta_1, \xi_1]$ with distance $\epsilon 2^{-1}$, respectively. Let w_1' denote an intersection point of β_{11} and β_{12} and let ζ_1' be an intersection point of β_{12} and β_{13} . Also consider a line β_{21} which is parallel to a segment $[z_2, w_2]$ with distance $\epsilon 2^{-2}$ and let ξ_1' denote an intersection point of β_{13} and β_{21} . Next let

$$\alpha_1' = [z_1', w_1'] \cup [w_1', \zeta_1'] \cup [\zeta_1', \xi_1'],$$

where z_1' is an intersection point of β_{11} and $\mathbb{B}(0, 8)$. Similarly, for each $i = 2, 3, \dots$ consider lines which are parallel to each line segment of α_i with distance $\epsilon 2^{-i}$ and consider the intersection points of each neighboring lines. Then we obtain α_i' by joining those points by segments and by letting $z_i' = \xi_{i-1}'$ for $i = 2, 3, \dots$. Then we have a nonrectifiable Jordan arc $\alpha' = \bigcup_{i=1}^{\infty} \alpha_i'$ below α . Since ξ_i' converges to 0, α and α' meet at 0. Let D be the component of $\mathbb{B}(0, 8) \setminus \{\alpha \cup \alpha'\}$ which contains 1. Then any curve γ in D^* joining 0 to a point $z \in \mathbb{C} \setminus \mathbb{B}(0, 8)$ is nonrectifiable, since γ contains a subarc whose length converges to $\ell(\alpha)$ as $\epsilon \rightarrow 0$. Thus CD is not quasiconvex. On the other hand, D is a c -John disk. To show this we will regard D as

$$D = D_1 \cup D_2,$$

where D_1 is the subdomain of D whose boundary is

$$\alpha \cup \{z \in \mathbb{C} : |z| = 8, \operatorname{Im} z \geq 0\} \cup \{z \in \mathbb{C} : |z - 4| = 4, \operatorname{Im} z \leq 0\}$$

and D_2 is the subdomain of D whose boundary is

$$\alpha' \cup \{z \in \mathbb{C} : |z| = 8, \operatorname{Im} z \leq 0\} \cup \{z \in \mathbb{C} : |z - 4| = 4, \operatorname{Im} z \geq 0\}.$$

Then since D_1 and D_2 are quasidisks, $D_1 \cap D_2 \neq \emptyset$, CD is connected and we conclude from the following Lemma 2.10 that $D = D_1 \cup D_2$ is a c -John disk. \square

LEMMA 2.10. *Suppose that D_i , $i = 1, 2$ are K_i -quasidisks such that $D_1 \cap D_2 \neq \emptyset$ and $\overline{\mathbb{C}} \setminus (D_1 \cup D_2)$ is connected. Then $D = D_1 \cup D_2$ is a c -John disk.*

Proof. Choose $z_0 \in \overline{D_1 \cap D_2}$ such that

$$\text{dist}(z_0, \partial D) = \sup_{z \in D_1 \cap D_2} \text{dist}(z, \partial D).$$

Suppose first that $z_0 \in D_1 \cap D_2$, fix $z_1 \in D_1$ and let α_1 be the hyperbolic geodesic in D_1 joining z_0 and z_1 . Since D_1 is a K_1 -quasidisk, we have a constant $a_1 = a_1(K_1) > 1$ such that for each $z \in \alpha_1$

$$(2.3) \quad \ell(\alpha_1(z_1, z)) \leq a_1 |z - z_1|$$

and

$$(2.4) \quad \min_{j=0,1} \ell(\alpha_1(z_j, z)) \leq a_1 \text{dist}(z, \partial D_1) \leq a_1 \text{dist}(z, \partial D).$$

Next let

$$b = \frac{\text{dia}(D)}{\text{dist}(z_0, \partial D)} < \infty$$

and let $c_1 = 2a_1^2 b$. Now we will show that

$$(2.5) \quad \ell(\alpha_1(z, z_1)) \leq c_1 \text{dist}(z, \partial D)$$

for all $z \in \alpha_1$ and hence that D is a c_1 -John disk. We consider two cases

Suppose first that

$$|z - z_0| \leq \frac{1}{2} \text{dist}(z_0, \partial D).$$

Then

$$\text{dist}(z, \partial D) \geq \text{dist}(z_0, \partial D) - |z - z_0| \geq \frac{1}{2} \text{dist}(z_0, \partial D)$$

and hence by (2.3)

$$\begin{aligned} \ell(\alpha_1(z_1, z)) &\leq a_1 |z - z_1| \leq a_1 \text{dia}(D) = a_1 b \text{dist}(z_0, \partial D) \\ &\leq 2a_1 b \text{dist}(z, \partial D) \leq c_1 \text{dist}(z, \partial D). \end{aligned}$$

Suppose next that

$$|z - z_0| \geq \frac{1}{2} \text{dist}(z_0, \partial D).$$

If $\ell(\alpha_1(z_0, z)) \leq \ell(\alpha_1(z, z_1))$, then as above and by (2.4)

$$\begin{aligned} \ell(\alpha_1(z, z_1)) &\leq a_1 \text{dia}(D) \leq a_1 b \text{dist}(z_0, \partial D) \leq 2a_1 b |z - z_0| \\ &\leq 2a_1 b \ell(\alpha_1(z, z_0)) \leq 2a_1^2 b \text{dist}(z, \partial D) = c_1 \text{dist}(z, \partial D). \end{aligned}$$

If $\ell(\alpha_1(z_0, z)) \geq \ell(\alpha_1(z, z_1))$, then by (2.4)

$$\ell(\alpha_1(z, z_1)) \leq a_1 \text{dist}(z, \partial D) \leq c_1 \text{dist}(z, \partial D).$$

Next fix $z_2 \in D_2$ and let α_2 be the hyperbolic geodesic joining z_0 and z_2 in D_2 . Then as above we have

$$(2.6) \quad \ell(\alpha_2(z, z_2)) \leq c_2 \text{dist}(z, \partial D)$$

for each $z \in \alpha_2$, where $c_2 = 2a_2^2 b$. Let $c = \max(c_1, c_2)$. Then by (2.5) and (2.6) D is a c -John disk and c depends on K_1, K_2 and a domain D .

Suppose next that $z_0 \in \partial(D_1 \cap D_2) \setminus \partial D$. Since $z_0 \in \partial(D_1 \cap D_2) \setminus \partial D_i$ for $i = 1$ or 2 , $z_0 \in D_i$. Hence by the same argument as above, D is a c -John disk. \square

3. Exterior Riemann mapping, the Bernstein inequality and polynomial approximation in domains

Let E be the closure of a simply connected domain D in \mathbb{C} . It is a well known fact that if $g : \overline{\mathbb{C}} \setminus \overline{\mathbb{B}} \rightarrow \overline{\mathbb{C}} \setminus E$ is a conformal mapping with $g(\infty) = \infty$, then

$$g(w) = a_{-1} w + \sum_{n=0}^{\infty} a_n w^{-n},$$

for $|w| > 1$, [12]. The number $|a_{-1}|$ is called the *transfinite diameter* of E , denoted by $tr(E)$. By performing a preliminary similarity mapping we may assume without loss of generality that $tr(E) = 1$.

Bernstein proved that if E is the closure of a euclidean disk, then it satisfies the Bernstein inequality

$$\sup_E |p'(z)| \leq \frac{n}{tr(E)} \sup_E |p(z)|$$

for all $p \in P_n$, see [2].

For the purpose of this section, we say that D is an open k -quasidisk, $0 \leq k < 1$, if one and hence each conformal mapping $g : \overline{\mathbb{C}} \setminus \overline{\mathbb{B}} \rightarrow \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ can be extended to a K -quasiconformal mapping of $\overline{\mathbb{C}}$, where $K = \frac{1+k}{1-k}$.

A continuum E in \mathbb{C} is said to be a closed k -quasidisk, if $E = \overline{D}$, where D is as above.

In [2] Anderson, Gehring and Hinkkanen extended the Bernstein's result to the case where E is a closed k -quasidisk, $0 \leq k < 1$, as follows: if E is a closed k -quasidisk, then for each $p \in P_n$ it satisfies the Bernstein inequality

$$\sup_E |p'(z)| \leq a \frac{n^b}{tr(E)} \sup_E |p(z)|,$$

and the integrated version of the Bernstein inequality

$$\left| \frac{p(z_1) - p(z_2)}{z_1 - z_2} \right| \leq c_1 \frac{n^b}{tr(E)} \sup_E |p(z)|, \quad z_1, z_2 \in E,$$

where $a = 2^{-k}e$, $c_1 = 2^{-k}e(\frac{\pi}{4} + 1)$ and $1 \leq b = 1 + k < 2$.

In [8], we extend the Bernstein inequality for a closed k -quasidisk, $0 \leq k < 1$, to the case where E is the closure of a John disk.

PROPOSITION 3.1 [8, THEOREM 1.3]. *If E is the closure of a c -John disk with $tr(E) = 1$, then the Bernstein inequality*

$$(3.1) \quad \sup_E |p'(z)| \leq an^b \sup_E |p(z)|$$

holds for each $p \in P_n$, where $a = a(D)$ and $b = b(c)$, $1 \leq b < 2$.

The following is a characterization of any bounded continuum which satisfies (3.1) in terms of the normalized exterior Riemann mapping g .

PROPOSITION 3.2 [8, Corollary 3.22]. *Suppose that E is a bounded continuum in \mathbb{C} with connected complement and $tr(E) = 1$. Then (3.1) holds for some a and $1 \leq b < 2$ if and only if there exist constants $c, 1 \leq c < 2$, and $m > 0$ which satisfy*

$$|g'(w)| \geq m \left(1 - \frac{1}{r^2}\right)^{c-1}$$

for $1 < |w| = r < \sqrt{2} + 1$. Here for the sufficiency $a = a(m, c)$, $b = b(c)$ and for the necessity $c = c(b)$, $m = m(a, b)$.

Combining Proposition 3.1 and Proposition 3.2, we obtain the following:

THEOREM 3.3. If E is the closure of a c -John disk with $tr(E) = 1$, then

$$(3.2) \quad |g'(w)| \geq m \left(1 - \frac{1}{r^2}\right)^k$$

for $1 < |w| = r < \sqrt{2} + 1$. Here $m = m(c, D)$ and $k = k(c)$, $0 < k < 1$.

Next we show that exterior quasiconvex domains satisfy the similar growth conditions for the exterior Riemann mapping g to Theorem 3.3. From the results we show that the domains satisfy the Bernstein inequality and also the integrated version of it.

DEFINITION 3.4. Let Γ be a curve family in a domain D in $\bar{\mathbb{C}}$. A nonnegative Borel function ρ is said to be *admissible* for Γ if

$$\int_{\gamma} \rho ds \geq 1$$

for all locally rectifiable curves $\gamma \in \Gamma$. Then

$$M(\Gamma) = \inf_{\rho} \iint_{\mathbb{C}} \rho^2 dx dy$$

is said to be the *modulus* of Γ , where the infimum is taken over all admissible functions ρ for the curve family Γ .

THEOREM 3.5. If E is the closure of a c -exterior quasiconvex bounded domain D in \mathbb{C} , then E holds (3.2) and

$$(3.3) \quad \left| \frac{g(w_1) - g(w_2)}{w_1 - w_2} \right| \geq m' \left(1 - \frac{1}{r^2}\right)^k$$

for $1 < |w_1| = |w_2| = r < \sqrt{2} + 1$, where $k = k(c)$, $0 \leq k < 1$, m and m' are constants depending on c, D .

Proof. Suppose that E is the closure of a c -exterior quasiconvex bounded domain D in \mathbb{C} . Then by Theorem 2.3 (1) and by Theorem 3.3 E holds (3.2).

To show that E satisfies (3.3), let w_1, w_2 be two points in \mathbb{B}^* with $1 < |w_1| = |w_2| = r < \sqrt{2} + 1$. First we show that if $|\arg w_1 - \arg w_2| = \pi$, then for some $b > 0$,

$$(3.4) \quad |g(w_1) - g(w_2)| \geq b > 0.$$

Then we may assume that $|\arg w_1 - \arg w_2| \leq \theta_0$ for some $0 < \theta_0 < \pi$.

To show (3.4), suppose that (3.4) does not hold. Then for $z_1 = g(w_1)$ and $z_2 = g(w_2)$ in D^* , $|z_1 - z_2|$ can be arbitrary small. Also since D is a c -exterior quasiconvex bounded domain, z_1, z_2 can be joined by a rectifiable curve γ in CD such that

$$\text{dia}(\gamma) \leq \ell(\gamma) \leq c|z_1 - z_2|.$$

Thus $\text{dia}(\gamma)$ can be arbitrary small. Let $\gamma' = g^{-1}(\gamma)$, F' be a circle centered at 0 and containing γ' , and let $F = g(F')$. Let $E = \{z : |z - z_2| = \text{dia}(\gamma) = a\}$ and $E_1 = \{z : |z - z_2| = \text{dist}(z_2, F) = b\}$. Let Γ' be the family of curves joining γ' and F' in the intersection of \mathbb{B}^* and interior of F' . Also let Γ be the family of curves joining γ and F in the intersection of CD and interior of F .

Let Γ_1 be the family of curves joining E and E_1 in $\mathbb{B}(0, b) \setminus \mathbb{B}(0, a)$. Then every curve $\gamma \in \Gamma$ has a subcurve which belongs to Γ_1 . Thus by [15, Theorem 6.4 and Theorem 7.5]

$$M(\Gamma) \leq M(\Gamma_1) = \frac{2\pi}{\log \frac{b}{a}}.$$

Since $\text{dia}(\gamma) = a$ is arbitrary small, we have $M(\Gamma) = 0$. On the other hands, by [15, Theorem 10.12 and Theorem 11.3]

$$\begin{aligned} M(\Gamma') &\geq \frac{2^2}{2\pi} \log \frac{\text{dia}(\gamma')}{\text{dist}(\gamma', F')} \geq \frac{2}{\pi} \log \frac{\pi}{\text{dist}(\gamma', F')} \\ &\geq c_0 > 0. \end{aligned}$$

But by [15, Theorem 8.1], $M(\Gamma) = M(\Gamma')$ and this is a contradiction.

Then we assume that $|\arg w_1 - \arg w_2| \leq \theta_0$ for some $0 < \theta_0 < \pi$. By Theorem 2.3 (1) D is a c' -John disk, $c' = c'(c)$, and thus there is a John center $z_0 \in D$. By performing a preliminary similarity mapping we may assume without loss of generality that $z_0 = 0$ and $\text{dist}(z_0, \partial D) > 1$. Let η be a mapping from \mathbb{B}^* onto \mathbb{B} such that

$$\eta(w) = \frac{1}{w}, \quad w'_j = \eta(w_j), \quad j = 1, 2.$$

Also let h be a self mapping of $\overline{\mathbb{C}}$ such that

$$h(z) = \frac{1}{z}, \quad z'_j = h(z_j), \quad j = 1, 2,$$

for $z_j = g(w_j)$. Next let f be a conformal mapping from $\mathbb{B} = \eta(\mathbb{B}^*)$ onto $h(D^*)$ such that $f(w'_j) = z'_j$, $j = 1, 2$. Let α'' be a hyperbolic geodesic in \mathbb{B} joining w'_1 and w'_2 . Then $\alpha' = f(\alpha'')$ is a hyperbolic geodesic in $h(D^*)$ joining z'_1 and z'_2 . Also we have

$$(3.5) \quad \alpha'' \subset \mathbb{B}(0, \frac{1}{r}).$$

Now let $\alpha = h^{-1}(\alpha')$ and let $\alpha''' = \eta^{-1}(\alpha'')$.

Since D is a c -exterior quasiconvex domain, there is a rectifiable curve γ in CD joining z_1 and z_2 such that

$$(3.6) \quad \ell(\gamma) \leq c|z_1 - z_2|.$$

Let $\gamma' = h(\gamma)$ and let $M = \sup\{|z_0 - z| : z \in \partial D\}$. Then since $\text{dist}(z_0, \partial D) > 1$, we have $M > 1$. Since D is a c' -John disk with $c' = c'(c)$,

$$\alpha \subset \mathbb{B}(z_0, c_1 M)$$

for some constant $c_1 = c_1(c)$, $1 < c_1 < \infty$. Otherwise D has a outward cusp and thus D is not a John disk. Hence for each $z \in \alpha$, we have $1 < |z| < c_1 M$ and thus for each $z' \in \alpha'$, we obtain $\frac{1}{c_1 M} < |z'| < 1$. Therefore for each $w' \in \alpha''$, we have $0 < d \leq |w'| < 1$ for $d = \text{dist}(0, s'')$, where $s'' = f^{-1}(s')$, $s' = \{z \in \mathbb{C} : |z| = \frac{1}{c_1 M}\}$. Then since $1 < |w| < \frac{1}{d}$ for $w \in \alpha'''$, by (3.5) α''' joins w_1 and w_2 in $C\mathbb{B}(0, r)$ without passing through ∞ .

Now since for $1 < |z| < c_1 M$

$$\frac{1}{(c_1 M)^2} < |h'(z)| = \frac{1}{|z|^2} < 1,$$

we have

$$(3.7) \quad \frac{1}{(c_1 M)^2} \ell(\gamma) \leq \ell(\gamma') = \int_{\gamma} |h'(z)| ds \leq \ell(\gamma).$$

Similarly

$$(3.8) \quad \frac{1}{(c_1 M)^2} \ell(\alpha) \leq \ell(\alpha') \leq \ell(\alpha).$$

Also by [5, Theorem 2] and (3.7) we have

$$(3.9) \quad \ell(\alpha') \leq b_1 \ell(\gamma') \leq b_1 \ell(\gamma),$$

where b_1 is a universal constant. Since $\alpha = g(\alpha''')$, by (3.8)

$$(3.10) \quad \ell(\alpha') \geq a \ell(\alpha) = a \ell(g(\alpha''')) = a \int_{\alpha'''} |g'(z)| ds,$$

where $a = \frac{1}{(c_1 M)^2}$. Therefore by the first part of Theorem 3.5, (3.6), (3.9) and (3.10),

$$\begin{aligned} |g(w_1) - g(w_2)| &= |z_1 - z_2| \geq \frac{1}{c} \ell(\gamma) \geq \frac{1}{b_1 c} \ell(\alpha') \\ &\geq \frac{a}{b_1 c} \int_{\alpha'''} |g'(z)| ds \geq \frac{am}{b_1 c} \int_{\alpha'''} \left(1 - \frac{1}{r^2}\right)^k ds \\ &\geq \frac{am}{b_1 c} \ell(\alpha''') \left(1 - \frac{1}{r^2}\right)^k \\ &\geq \frac{am}{b_1 c} |w_1 - w_2| \left(1 - \frac{1}{r^2}\right)^k \end{aligned}$$

for some $k = k(c)$, $0 \leq k < 1$. Hence (3.3) holds with a constant m' depending on c and D . \square

The following Bernstein inequality (3.11) for the closure of an exterior quasiconvex domain is an immediate result from Theorem 2.3 (1) and Proposition 3.1. In [2], Anderson, Gehring and Hinkkanen showed that the integrated version (3.12) of the Bernstein inequality holds for the closure of a k -quasidisk with $0 \leq k < 1$. Now we extend their result to the closure of an exterior quasiconvex domain with $0 \leq k < 1$ by using (3.3) in Theorem 3.5 and by the similar argument to the proof in [2, Theorem 1].

THEOREM 3.6. *Suppose that E is the closure of a c -exterior quasiconvex domain D . Then for each polynomial p in z of degree n it satisfies the Bernstein inequality*

$$(3.11) \quad \sup_E |p'(z)| \leq a \frac{n^{1+k}}{tr(E)} \sup_E |p(z)|,$$

where $a = a(D)$ and $k = k(c)$, $0 \leq k < 1$, as in (3.1). Also it satisfies the integrated version of the Bernstein inequality

$$(3.12) \quad \left| \frac{p(z_1) - p(z_2)}{z_1 - z_2} \right| \leq c_1 \frac{n^{1+k}}{tr(E)} \sup_E |p(z)|,$$

where $c_1 = c_1(c, D)$ and k is same as the above (3.11).

Next we assume that f is a function which is continuous in E and analytic in $D = \text{interior of } E$. If E is a euclidean disk, then there is a connection between the smoothness of a function f and the rate at which it can be approximated by polynomials as follows.

PROPOSITION 3.7 [3, p.147 and p.200]. *If E is a euclidean disk, then for $0 < \alpha < 1$, $E_n(f) = O(n^{-\alpha})$ as $n \rightarrow \infty$ if and only if $f \in Lip_\alpha(E)$.*

In [2], they extended the result to the case where E is a closed k -quasidisk, $0 \leq k < 1$, as follows: if E is a closed k -quasidisk, if $0 < \alpha < 1 + k$ and if

$$E_n(f) = O(n^{-\alpha})$$

as $n \rightarrow \infty$, then $f \in Lip_\beta(E)$, where $\beta = \frac{\alpha}{(1+k)}$.

Now we extend their result to the closure of an exterior quasiconvex domain.

THEOREM 3.8. *Suppose E is the closure of a c -exterior quasiconvex domain. Then there exists a constant k , $0 \leq k < 1$, depending only on c such that if $0 < \alpha < 1 + k$ and if*

$$E_n(f) = O(n^{-\alpha})$$

as $n \rightarrow \infty$, then $f \in Lip_\beta(E)$, where $\beta = \frac{\alpha}{(1+k)}$.

Proof. To prove Theorem 3.8, the most important fact is the inequality (3.12) and then by the same arguments to the proof in [2, Theorem 2] we can prove it. \square

Finally, for Lip_α -extension domains we show a similar result to Theorem 3.8.

THEOREM 3.9. *Suppose E is the closure of a Lip_α -extension domain D , $0 < \alpha < 1$. If*

$$E_n(f) = O(n^{-\alpha})$$

as $n \rightarrow \infty$, then $f \in Lip_\alpha(E)$.

Proof. Suppose that $E_n(f) = O(n^{-\alpha})$ as $n \rightarrow \infty$. Then for any open disk U in D

$$\|f - p\|_{\overline{U}} \leq \|f - p\|_E \leq cn^{-\alpha},$$

where $c > 0$ is a constant. By Proposition 3.7 $f \in Lip_\alpha(\overline{U})$. Since D is a Lip_α -extension domain, $f \in Lip_\alpha(D)$. Now since f is continuous on

∂D , for each pair of points $z_1, z_2 \in \partial D$ and for sequences $\{w_n\}, \{y_n\}$ in D with $\lim_{n \rightarrow \infty} w_n = z_1, \lim_{n \rightarrow \infty} y_n = z_2$ we have

$$\lim_{n \rightarrow \infty} f(w_n) = f(z_1), \quad \lim_{n \rightarrow \infty} f(y_n) = f(z_2).$$

Thus

$$\begin{aligned} |f(z_1) - f(z_2)| &= \lim_{n \rightarrow \infty} |f(w_n) - f(y_n)| \leq \lim_{n \rightarrow \infty} m |w_n - y_n|^\alpha \\ &\leq |z_1 - z_2|^\alpha. \end{aligned}$$

Therefore, $f \in Lip_\alpha(E)$. □

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