Bull. Korean Math. Soc. **39** (2002), No. 4, pp. 705–714

ERROR ESTIMATES FOR OPTION PRICES IN JUMP-DIFFUSION MODELS

IN-SUK WEE

ABSTRACT. We consider a jump-diffusion model generated by a Lévy process for an asset price. We present an error estimate for the option prices between the jump-diffusion model and the Black-Scholes model when the former converges weakly to the latter.

1. Introduction

We consider a jump-diffusion process S_t^δ for an asset price satisfying

$$dS_t^\delta = S_{t-}^\delta dY_t^\delta, \tag{1.1}$$

where Y_t^δ is a Lévy process on $(\Omega, \{\mathcal{F}_t^\delta\}, P^\delta)$ whose characteristic function is represented as

$$E\exp(iuY_t^\delta) = \exp(t\psi^\delta(u)),$$

$$\begin{aligned}\psi^\delta(u) = i\beta^\delta u - \frac{(\sigma^\delta)^2}{2}u^2 \\ + \int_{\{|x|<1\}}(e^{iux} - 1 - iux)\nu^\delta(dx) + \int_{\{|x|>1\}}(e^{iux} - 1)\nu^\delta(dx),\end{aligned}$$

$$\int(1 \wedge x^2)\nu^\delta(dx) < \infty.$$

We shall suppose that the noise processes Y_t^δ converge weakly to Brownian motion as δ tends to zero. Roughly speaking, we consider a jump-diffusion model when the intensity of the jumps is very large and the minimum size of the jumps is very small.

For the asset price as a jump-diffusion model such as (1.1), it is well-known that the market is incomplete in general. There are many

Received May 10, 2001.
2000 Mathematics Subject Classification: 60H30, 91B28, 91B70.
Key words and phrases: jump-diffusion model, Black-Scholes model, option price.
The author was supported by grant No. 2000-0-104-002-3 from the Korea Science & Engineering Foundation.

equivalent martingale measures to which we can price a contingent claim allowing no arbitrages. Among them, the minimal martingale measure was introduced by Föllmer and Schweizer [2] for a general semimartingale model and was derived explicitly by Chan [1] for our model. Hong and Wee [5] proved that if Y^δ converges weakly to a Brownian motion as $\delta \to 0$, the option prices for the jump-diffusion models given by the minimal martingale measures converge to the option price for Black-Scholes model via probabilistic method.

In this work, we propose a method for error estimates between two option prices analytically, when the European contingent claim is given by a smooth function of the asset price at the expiration.

2. Main result

We assume that the filtration $\{\mathcal{F}_t^\delta\}$ is the minimal one generated by Y^δ and satisfies the usual conditions, and the sample paths are right-continuous and have left-hand limits. Furthermore, we assume that

(A1)
$$E|Y_1^\delta|^3 < \infty,$$
which is equivalent to
$$\int_{\{|y|\geq 1\}} |y|^3 \nu^\delta(dy) < \infty.$$

Under this assumption, it is possible to decompose Y_t^δ on $(\Omega, \{\mathcal{F}_t^\delta\}, P^\delta)$ as

$$\begin{aligned} Y_t^\delta &= \sigma^\delta B_t^\delta + \int_0^t \int y(\mu^\delta(ds,dy) - \nu^\delta(dy)ds) + tE_{P^\delta}(Y_1^\delta) \\ &\equiv \sigma^\delta B_t^\delta + M_t^\delta + a^\delta t. \end{aligned}$$

Here $\mu^\delta(ds,dy)$ is a Poisson measure with intensity measure $ds \times \nu^\delta(dy)$. B_t^δ is a Brownian motion and they are independent. In order to obtain weak convergence of Y^δ to a Brownian motion, we make the following assumptions : as $\delta \to 0$,

(A2.1) $\int_{\{|y|>\epsilon\}} y^2 \nu^\delta(dy) \to 0$ for any $\epsilon > 0$,

(A2.2) $a^\delta \to a$,

(A2.3) $(\sigma^\delta)^2 + \int y^2 \nu^\delta(dy) \to \sigma^2 > 0$.

If (A2.1), (A2.2), and (A2.3) hold, then the jump-diffusion models $(S^\delta | P^\delta)$ in (1.1) converge weakly to the following Black-Scholes model $(S|P)$ satisfying

$$dS_t = S_t dY_t, \tag{2.1}$$

where

$$Y_t = \sigma B_t + at$$

and $\{B_t\}$ is a Brownian motion on $(\Omega, \{\mathcal{F}_t\}, P)$. See [5] for details.

We now recall how to price European contingent claims $\Gamma^\delta = \varphi(S_T^\delta)$ and $\Gamma = \varphi(S_T)$ on S^δ and S respectively with the maturity T fixed. As we have mentioned before, there are many equivalent martingale measures on $(\Omega^\delta, \{\mathcal{F}_t^\delta\}, P^\delta)$ under which $e^{-rt}S_t^\delta$ in (1.1) becomes a martingale when r is the risk-free interest rate. Among them, we choose the minimal martingale measure to price the contingent claim Γ^δ which we now explain briefly. See Chan [1] and Hong and Wee [5] for details and further references. To define the minimal martingale measure, we need the following conditions;

(A3) The support of ν^δ is contained in $(-1, -\frac{1}{\gamma^\delta})$, and

$$\int (\ln(1+y))^2 \nu^\delta(dy) < \infty,$$

where

$$\gamma^\delta = \frac{r - a^\delta}{(\sigma^\delta)^2 + \int y^2 \nu^\delta(dy)} < 0.$$

Then the minimal martingale measure Q^δ for $(S^\delta | P^\delta)$ is given by

$$\frac{dQ^\delta}{dP^\delta}\Big|_{\mathcal{F}_t^\delta} = Z_t^\delta,$$

where

$$Z_t^\delta = 1 + \int_0^t \gamma^\delta Z_{s-}^\delta (\sigma^\delta dB_s^\delta + dM_s^\delta).$$

Let φ be a twice differentiable function with bounded first and second derivatives. Then the price of $\Gamma^\delta = \varphi(S_T^\delta)$ with respect to Q^δ is given by

$$u^\delta(x,t) = E_{Q^\delta}\left(e^{-r(T-t)}\varphi(S_{t,x}^\delta(T))\right),$$

where $S^\delta_{t,x}(u) = x + \int_t^u S^\delta_{t,x}(\tau-)dY^\delta_\tau$. Then u^δ is a $C^{2,1}$ function defined on $[0,\infty)\times[0,T]$ and satisfies

$$\begin{aligned}\left(\partial_t u^\delta + A^\delta_t u^\delta - ru^\delta\right)(x,t) &= 0, \qquad (x,t)\in(0,\infty)\times(0,T)\\ u^\delta(x,T) &= \varphi(x),\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}A^\delta_t u^\delta(x,t) &= \frac{1}{2}(\sigma^\delta)^2x^2\partial_{xx}u^\delta(x,t) + rx\partial_x u^\delta(x,t)\\ &\quad + \int (u^\delta(x(1+y),t) - u^\delta(x,t) - xy\partial_x u^\delta(x,t))\tilde{\nu}^\delta(dy),\end{aligned}$$

and

$$\tilde{\nu}^\delta(dy) = (1+\gamma^\delta y)\nu^\delta(dy).$$

For the well-known Black-Scholes model $(S|P)$ in (2.1), there exists unique equivalent martingale measure Q and the price of $\Gamma = \varphi(S_T)$ is given by

$$E_Q(e^{-r(T-t)}\varphi(S_T)|\mathcal{F}_t) = u(S_t,t),$$

where u is a $C^{2,1}$ function satisfying

$$\begin{aligned}(\partial_t u + A_t u - ru)(x,t) &= 0, \qquad (x,t)\in(0,\infty)\times(0,T)\\ u(x,T) &= \varphi(x),\\ (A_t u)(x,t) &= \frac{1}{2}\sigma^2x^2\partial_{xx}u(x,t) + rx\partial_x u(x,t).\end{aligned} \tag{2.3}$$

We assume a stronger condition than (A2.1).

(A2.1)′ For any $\epsilon > 0$,

$$\int_{\{|y|>\epsilon\}} y^3\nu^\delta(dy) \to 0 \text{ as } \quad \delta\to 0.$$

It was shown in [5], that if (A1), (A2) and (A3) hold with (A2.1)′ replacing (A2.1), then as $\delta\to 0$

$$u^\delta(x,t)\to u(x,t).$$

In this work, we find the error estimate $u^\delta(t,x) - u(t,x)$ as $\delta\to 0$ which is basically a Taylor series expansion in terms of δ. Note that u^δ and u are the solutions of the Cauchy problems for the parabolic integro-differential equation and for the parabolic differential equation respectively. Although the uniqueness and existence of solution for the Cauchy problem to a parabolic differential equation are well-known in a wide generality, the similar problem for a parabolic integro-differential equation has been solved in a very restricted sense, to the best of our knowledge. For convenience of readers, we state those relevant results concerning our problem in a standard form. Readers may find the proofs in Garroni and Menaldi [3]. Let

$$\begin{aligned} Q_T &= R \times (0,T) \\ Lu(x,t) &= a_2(x,t)\partial_{xx}u(x,t) + a_1(x,t)\partial_x u(x,t) + a_0(x,t)u(x,t) \\ Iu(x,t) &= \int (u(x+j(y),t) - u(x,t) - j(y)\partial_x u(x,t))\pi(dy), \end{aligned}$$

where π is a σ-finite measure satisfying

$$\int \frac{(j(y))^2}{1+j(y)}\pi(dy) < \infty. \tag{2.4}$$

We further consider, for $0 < \alpha < 1$,

$$a_i \in C^{\alpha,\frac{\alpha}{2}}\left(\overline{Q_T}\right), \tag{2.5}$$

and

(2.6) $\quad j$ is Hölder-continuous with exponent α.

Here $C^{k+\alpha,\frac{k+\alpha}{2}}(\overline{Q_T})$ denotes the usual Hölder space, where k is a non-negative integer and $0 < \alpha < 1$.

Proposition 2.1. (*The Cauchy problem for a parabolic differential equation*)
Consider

$$\begin{aligned} \partial_t u(x,t) - Lu(x,t) &= f(x,t), \quad (x,t) \in Q_T \\ u(x,0) &= \varphi(x). \end{aligned} \tag{2.7}$$

Assume that (2.5) *holds and for* $\mu > 0$,
(1) $a_2(x,t) \geq \mu$ *for* $(x,t) \in Q_T$,
(2) f, φ *are continuous such that*

$$|f(x,t)| + |\varphi(x)| \leq C_1 \exp(C_2 x^2),$$

for some $C_1,\ C_2 > 0$,
(3) f *is Hölder continuous in* x *with exponent* α *uniformly for* t.
Then there is unique solution to (2.7) *satisfying*

$$|u(x,t)| \leq K_1 \exp(K_2 x^2),$$

where K_1 *and* K_2 *depend on* C_1, C_2 *and* T. *If in addition,* $f \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q_T})$, *and* $\varphi \in C^{2+\alpha}(\overline{R})$, *then*

$$\| u \|_{2+\alpha,\frac{2+\alpha}{2}} \leq C(\| f \|_{\alpha,\frac{\alpha}{2}} + \| \varphi \|_{2+\alpha}).$$

PROPOSITION 2.2. (*The Cauchy problem for a parabolic integro-differential equation*)
Consider

$$\begin{aligned} (2.8) \qquad \partial_t u(x,t) - Lu(x,t) - Iu(x,t) &= f(x,t), \quad (x,t) \in Q_T \\ u(x,0) &= \varphi(x). \end{aligned}$$

Assume that (2.4), (2.5) *and* (2.6) *hold and for* $\mu > 0$, $a_2(x,t) \geq \mu$ *for* $(x,t) \in Q_T$. *Then for* $\varphi \in C^{2+\alpha}(\overline{R})$, *and* $f \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q_T})$, (2.8) *has unique solution* $u \in C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q_T})$ *such that*

$$\| u \|_{2+\alpha,\frac{2+\alpha}{2}} \leq C(\| f \|_{\alpha,\frac{\alpha}{2}} + \| \varphi \|_{2+\alpha}).$$

In both propositions, it is essential to have uniform ellipticity and boundedness of the coefficients of the differential operator in (2.7) and (2.8). To apply these results to our problem, we perform a change of the variables. Let

$$\begin{aligned} v(x,t) &= u(e^x, T-t) \\ v^\delta(x,t) &= u^\delta(e^x, T-t) \\ \tilde{\varphi}(x) &= \varphi(e^x). \end{aligned}$$

Then (2.2) and (2.3) are replaced by the following systems of equations, respectively;

$$(2.9) \quad \begin{cases} \partial_t v^\delta(x,t) - B_t^\delta v^\delta(x,t) + rv^\delta(x,t) = 0, \quad (x,t) \in Q_T \\ v^\delta(x,0) = \tilde{\varphi}(x) \end{cases}$$

$$(2.10) \quad \begin{cases} \partial_t v(x,t) - B_t v(x,t) + rv(x,t) = 0, \quad (x,t) \in Q_T \\ v(x,0) = \tilde{\varphi}(x), \end{cases}$$

where

$$\begin{aligned} B_t^\delta v(x,t) &= \frac{1}{2}(\sigma^\delta)^2 \partial_{xx} v(x,t) + \{r - \frac{1}{2}(\sigma^\delta)^2 \\ &\quad + \int (\ln(1+y) - y)\tilde{\nu}^\delta(dy)\}\partial_x v(x,t) \\ &\quad + \int (v(x+\ln(1+y),t) - v(x,t) \\ &\quad - \ln(1+y)\partial_x v(x,t))\tilde{\nu}^\delta(dy), \\ B_t v(x,t) &= \frac{1}{2}\sigma^2 \partial_{xx} v(x,t) + (r - \frac{1}{2}\sigma^2)\partial_x v(x,t). \end{aligned}$$

By Propositions 2.1 and 2.2, there exist unique solutions to (2.9) and (2.10) respectively if $\tilde{\varphi} \in C^{2+\alpha}(\overline{R})$. Throughout the section, we denote by v^δ and v the unique solutions of (2.9) and (2.10) respectively. Now we present the main result.

THEOREM 2.1. *Assume that (A1), (A2) and (A3) hold, with (A2.1)′ replacing (A2.1), $\tilde{\varphi} \in C^{2+\alpha}(\overline{R})$ and there exist bounded operators, for $1 \le l \le k$,*

$$B_t^{(l)} : C_{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q_T}) \longrightarrow C_{\alpha,\frac{\alpha}{2}}(\overline{Q_T}),$$

such that as $\delta \to 0$,

$$\frac{B_t^\delta - B_t - \delta B_t^{(1)} - \cdots - \delta^{k-1} B_t^{(k-1)}}{\delta^k} \longrightarrow B_t^{(k)}$$

in the operator norm. Then there exist $V_0^{(l)} \in C_{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q_T}), 1 \le l \le k$, such that

$$\begin{aligned}(\partial_t - B_t + r)V_0^{(l)} &= B_t^{(1)}V_0^{(l-1)} + B_t^{(2)}V_0^{(l-2)} \\ &\quad + \cdots + B_t^{(l-1)}V_0^{(1)} + B_t^{(l)}v,\end{aligned} \tag{2.11}$$

and

$$\frac{v^\delta - v - \delta V_0^{(1)} - \cdots - \delta^{l-1}V_0^{(l-1)}}{\delta^l} \longrightarrow V_0^{(l)} \tag{2.12}$$

in $C_{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q_T})$ as $\delta \to 0$.

Proof. We prove by induction on k. Let

$$V_\delta^{(1)} = \frac{v^\delta - v}{\delta}.$$

Note that $V_\delta^{(1)}$ satisfies

$$\begin{cases} (\partial_t - B_t^\delta + r)V_\delta^{(1)} = \dfrac{B_t^\delta - B_t}{\delta} v & \text{on } Q_T, \\ V_\delta^{(1)}(x,0) = 0. \end{cases} \tag{2.13}$$

$$\begin{cases} (\partial_t - B_t + r)V_0^{(1)} = B_t^{(1)} v & \text{on } Q_T, \\ V_0^{(1)}(x,0) = 0. \end{cases} \tag{2.14}$$

Combining (2.13) and (2.14), we may rewrite as follows;

$$\begin{cases} (\partial_t - B_t + r)(V_\delta^{(1)} - V_0^{(1)}) \\ = (B_t^\delta - B_t)(V_\delta^{(1)} - V_0^{(1)}) + (B_t^\delta - B_t)V_0^{(1)} \\ \qquad + \left(\dfrac{B_t^\delta - B_t}{\delta} - B_t^{(1)}\right) v(x,t) & \text{on } Q_T \\ V_\delta^{(1)}(x,0) - V_0^{(1)}(x,0) = 0. \end{cases}$$

By Proposition 2.2,

$$\begin{aligned} \| V_\delta^{(1)} - V_0^{(1)} \|_{2+\alpha, \frac{2+\alpha}{2}} \ &\le \ C \, \| (B_t^\delta - B_t)(V_\delta^{(1)} - V_0^{(1)}) \|_{\alpha, \frac{\alpha}{2}} \\ &\quad + C \, \| (B_t^\delta - B_t)V_0^{(1)} \|_{\alpha, \frac{\alpha}{2}} \\ &\quad + C \, \| (\frac{B_t^\delta - B_t}{\delta} - B_t^{(1)})v \|_{\alpha, \frac{\alpha}{2}} . \end{aligned}$$

Examining the proof of Proposition 2.2 carefully, it is not hard to see that C is independent of δ under our conditions. This proves the assertion for $k = 1$. Following the induction hypothesis, we suppose that there exist $V_0^{(l)}, 1 \le l \le k-1$, satisfying (2.11) and (2.12). Let

$$V_\delta^{(l)} = \frac{V_\delta^{(l-1)} - V_0^{(l-1)}}{\delta} \text{ for } 2 \le l \le k-1.$$

Note that for $1 \le l \le k-1$,

$$V_\delta^{(l)} = \frac{v^\delta - v - \delta V_0^{(1)} - \cdots - \delta^{l-1} V_0^{(l-1)}}{\delta^l},$$

and

$$\begin{aligned} \left(\partial_t - B_t^\delta + r\right) V_\delta^{(l)} \ &= \ \frac{(B_t^\delta - B_t)V_0^{(l-1)}}{\delta} + \frac{\left(B_t^\delta - B_t - \delta B_t^{(1)}\right) V_0^{(l-2)}}{\delta^2} \\ &\quad + \cdots \\ &\quad + \frac{\left(B_t^\delta - B_t - \delta B_t^{(1)} - \cdots - \delta^{l-2} B_t^{(l-2)}\right) V_0^{(1)}}{\delta^{l-1}} \\ &\quad + \frac{\left(B_t^\delta - B_t - \delta B_t^{(1)} - \cdots - \delta^{l-1} B_t^{(l-1)}\right) v}{\delta^l}. \end{aligned} \tag{2.15}$$

Define

$$\begin{aligned} V_\delta^{(k)} \ &= \ \frac{V_\delta^{(k-1)} - V_0^{(k-1)}}{\delta} \\ &= \ \frac{v^\delta - v - \delta V_0^{(1)} - \cdots - \delta^{k-1} V_0^{(k-1)}}{\delta^k}. \end{aligned}$$

Let $W_\delta^{(k)}$ be the expression on the right hand side of (2.15) with k replacing l. We note that $V_\delta^{(k)}$ is the unique solution satisfying

$$\begin{cases} (\partial_t - B_t^\delta + r)V_\delta^{(k)} = W_\delta^{(k)} \quad \text{on } Q_T, \\ V_\delta^{(k)}(x,0) = 0, \end{cases}$$

which is given by Proposition 2.2. By Proposition 2.1, there exists unique $V_0^{(k)}$ such that

$$\begin{cases} (\partial_t - B_t + r)V_0^{(k)} = B_t^{(1)}V_0^{(k-1)} + B_t^{(2)}V_0^{(k-2)} \\ \qquad + \cdots + B_t^{(k-1)}V_0^{(1)} + B_t^{(k)}v \qquad\qquad \text{on } Q_T \\ V_0^{(k)}(x,0) = 0. \end{cases}$$

We observe that on Q_T

$$\begin{aligned} &(\partial_t - B_t + r)\left(V_\delta^{(k)} - V_0^{(k)}\right) \\ = \quad & (B_t^\delta - B_t)V_\delta^{(k)} + (B_t^\delta - B_t)(V_\delta^{(k)} - V_0^{(k)}) \\ & + \delta^{-1}\left(B_t^\delta - B_t - \delta B_t^{(1)}\right)V_0^{(k-1)} + \cdots \\ & + \delta^{-k}\left(B_t^\delta - B_t - \delta B_t^{(1)} - \cdots - \delta^{k-1}B_t^{(k-1)} - \delta^k B_t^{(k)}\right)v. \end{aligned}$$

By Proposition 2.1, we have

$$\begin{aligned} & \| V_\delta^{(k)} - V_0^{(k)} \|_{2+\alpha, \frac{2+\alpha}{2}} \\ \leq \quad & C \| (B_t^\delta - B_t)V_0^{(k)} \|_{\alpha, \frac{\alpha}{2}} + C \| (B_t^\delta - B_t)(V_\delta^{(k)} - V_0^{(k)}) \|_{\alpha, \frac{\alpha}{2}} \\ & + C \| \delta^{-1}\left(B_t^\delta - B_t - \delta B_t^{(1)}\right)V_0^{(k-1)} \|_{\alpha, \frac{\alpha}{2}} + \cdots \\ & + C \| \delta^{-k}\left(B_t^\delta - B_t - \delta B_t^{(1)} - \cdots - \delta^{k-1}B_t^{(k-1)} - \delta^k B_t^{(k)}\right) v \|_{\alpha, \frac{\alpha}{2}} \end{aligned}$$

where C is again independent of δ. This completes the proof. □

REMARK. Assume that as $\delta \to 0$, for any $\epsilon > 0$

$$\int_{\{|y|>\epsilon\}} \max\{y^2, (\ln(1+\sigma y))^2\} \frac{\tilde{\nu}^\delta(dy)}{\delta} \longrightarrow 0,$$

$$\sup_\delta \int_{\{|y|<\epsilon\}} y^2 \frac{\tilde{\nu}^\delta(dy)}{\delta} < \infty,$$

and

$$\frac{(\sigma^\delta)^2 - \sigma^2 + \int y^2 \tilde{\nu}^\delta(dy)}{\delta} \longrightarrow c_1.$$

Then

$$\frac{B_t^\delta - B_t}{\delta} \longrightarrow B_t^{(1)}$$

in the operator norm where

$$B_t^{(1)} v = \frac{1}{2} c_1 \sigma^2 (\partial_{xx} v - \partial_x v).$$

If the conditions in Theorem 2.1 hold, then we have

$$\frac{v^\delta - v}{\delta} \to V_0^{(1)} \text{ in } C_{2+\alpha, \frac{2+\alpha}{2}}(\overline{Q_T}),$$

where

$$\begin{cases} (\partial_t - B_t + r) V_0^{(1)} = B_t^{(1)} v & \text{on } Q_T, \\ V_0^{(1)}(x, 0) = 0. \end{cases}$$

References

[1] T. Chan, *Pricing contingent claims on stocks driven by Lèvy processes*, Ann. Appl. Probab. **9** (1999), 504–528.

[2] H. Föllmer and M. Schweizer, *Hedging of contingent claims under incomplete information*, In : Applied Stochastic Analysis (M. H. A. Davis and R. J. Elliott. eds.), Gordon and Breach, New York, (1991), 389–414.

[3] M. G. Garroni and J. L. Menaldi, *Green functions for second order parabolic integro-differential problems*, Longman Scientific and Technical, England, 1992.

[4] I. I. Gihman and A. V. Skorohod, *Stochastic differential equations*, Springer-Verlag, New York, 1972.

[5] D. W. Hong and I. S. Wee, *Convergence of jump-diffusion models to the Black-Scholes model* (To appear in Stochastic Analysis and Applications).

Department of Mathematics, Korea University, Seoul 136-701, Korea
E-mail: insukwee@semi.korea.ac.kr