Journal of the Korean Statistical Society (2002), **31**: *4, pp 533–543*

A Change-point Estimator with Unsymmetric Fourier Series[†]

Jaehee Kim[1]

ABSTRACT

In this paper we propose a change-point estimator with left and right regressions using the sample Fourier coefficients on the orthonormal bases. The window size is different according to the data in the left side and in the right side at each point. The asymptotic properties of the proposed change-point estimator are established. The limiting distribution and the consistency of the estimator are derived.

Keywords. Change-point model, sample Fourier coefficients, stochastic process, consistent.
AMS 2000 subject classifications. Primary 62G08; Secondary 62G20.

1. Introduction

Consider a sequence $\{y_i\}$ of observations with finite fourth moment. The model considered is

$$Y_i = f(x_i) + \epsilon_i \tag{1.1}$$

where the errors are independent $N(0,1)$ and $x_i = i/n$, $i = 1, 2, \ldots, n$. Also f is right continuous and left continuous except at an unknown change-point or discontinuity point $\tau \in (0,1)$. For definiteness, we suppose that τ is an event time, *i.e.*, $\tau = x_i$ for some i. Left and right local regressions with the sample Fourier coefficients are estimated at each design point.

Hinkley (1970) proposed the maximum likelihood estimator of the change-point estimator with the normal errors in the mean shift model. McDonald and Owen (1986) and Hall and Titterington (1992) investigated change-point estimation based on three smoothed estimates of the function. Loader (1996) proposed

Received March 2002; accepted September 2002.
[†]This work was supported by grant No. R04-2001-00135 from the Basic Research Program of the Korea Science & Engineering Foundation.
[1]Department of Statistics, Duksung Women's University, Seoul 132-714, Korea

an estimate of the location of the discontinuity based on one-side nonparametric regression estimates of the mean function using the kernel function. Lombard (1988) introduced the sample Fourier series coefficients for testing for the existence of change. Kim and Hart (1998) proposed a test for change using the sample Fourier coefficients with the dependent data.

In Section 2 we suggest the change-point estimator with Fourier coefficient series estimation and the asymptotic properties are shown. The proofs are in Section 3. The simulation results are given in Section 4 and concluding remarks are in Section 5.

2. Left and Right Function Estimation Using the Sample Fourier Coefficients in the Unsymmetric Window

We assume that the underlying function f has the following Fourier series representation with the cosine system $\{\cos \pi jx\}$ and $x \in (0,1)$:

$$f(x) = \phi_0 + 2\sum_{j=1}^{\infty} \phi_j \cos \pi jx, \tag{2.1}$$

where $\phi_j = \int_0^1 f(x) \cos \pi jx dx$, $j = 0, 1, \ldots$, which are the Fourier coefficients of f. The sample Fourier coefficients using a cosine system are defined by

$$\hat{\phi}_j = \frac{1}{n}\sum_{i=1}^{n} y_i \cos \pi jx_i, \qquad j = 0, 1, \ldots, n-1, \tag{2.2}$$

where $x_i = i/n$. Consider the left Fourier series estimator with the data in the left and the right Fourier series estimator with the data in the right at each point. Then the left and right window sizes vary according to the location of the point. That is, unsymmetric window is used at each point and the Fourier estimation is done with the different number of data, with whole data in the left and whole data in the right. Let the left window size be h and the right window size be g ($g + h = 1$) and the estimators can be obtained as follows:

$$\hat{f}_{K+}(x) = \hat{\phi}_0 + 2\sum_{j=1}^{K} \hat{\phi}_j \cos \pi jt_1, \tag{2.3}$$

$$\hat{f}_{K-}(x) = \hat{\psi}_0 + 2\sum_{j=1}^{K} \hat{\psi}_j \cos \pi jp_e \tag{2.4}$$

with $t_u = (u-0.5)/ng$, $p_u = (u-0.5)/nh$, $p_e = p_{nh}$ in $x \in (\delta, 1-\delta)$ and $n\delta \le m = nx \le n - n\delta$,
where

$$\hat{\phi}_0 = \frac{1}{ng}\sum_{u=1}^{ng} y_{m+u},$$
$$\hat{\phi}_j = \frac{1}{ng}\sum_{u=1}^{ng} y_{m+u}\cos\pi j t_u,$$
$$\hat{\psi}_0 = \frac{1}{nh}\sum_{u=1}^{nh} y_u,$$
$$\hat{\psi}_j = \frac{1}{nh}\sum_{u=1}^{nh} y_u \cos\pi j p_u.$$

Define the size of change is measured by

$$\Delta = \Delta_\tau = f(\tau+) - f(\tau-).$$

The estimator of Δ_x at the point x can be defined as

$$\hat{\Delta}_x = \hat{f}_{K+}(x) - \hat{f}_{K-}(x), \qquad \delta \le x \le 1-\delta. \tag{2.5}$$

The estimate $\hat{\tau}$ of τ is the value of x which maximizes $\hat{\Delta}_x^2$ over $\delta \le x \le 1-\delta$. That is,

$$\hat{\tau} = \arg\max_{\delta\le x\le 1-\delta} \hat{\Delta}_x^2. \tag{2.6}$$

One could also consider the maximizer of $|\hat{\Delta}_x|$ or the maximizer of $\hat{\Delta}_x$ if $\Delta > 0$. The choice of the number of the sample Fourier coefficients K and the bandwidth h depend on the selection criteria. We do not discuss the selection criteria in this paper.

Theorem 2.1. *Let $\ldots, \epsilon_{-1}, \epsilon_0, \epsilon_1, \ldots$ be independent $N(0,1)$ random variables. Then we have the following limiting process as*

$$\lim_{n\to\infty} P\left(n(\hat{\tau}-\tau) = s\right) = P\left(S_\Delta = s\right), \tag{2.7}$$

where S_Δ is the location of the maximum of the process

$$Z_i = \begin{cases} -i\Delta^2 + 2\Delta\sum_{u=1}^{i}\epsilon_u, & i > 0,\\ 0, & i = 0,\\ -|i|\Delta^2 + 2\Delta\sum_{u=i}^{-1}\epsilon_u, & i < 0.\end{cases} \tag{2.8}$$

The local regression estimates will require a larger n, nh and ng for the estimation and for the asymptotics to be applicable.

Theorem 2.2. *$\hat{\tau}$ is a consistent estimator of τ, i.e., for $i_0 = o(\min\{nh, ng\})$ and $i_0 > 0$,*

$$P\left(|\hat{\tau} - \tau| \geq \frac{i_0}{n}\right) \xrightarrow{p} 0. \tag{2.9}$$

3. Proofs

Lemma 1.

$$\sup_{t \geq \tau + \delta} |E\hat{\Delta}_t| = O\left(\frac{\delta}{h}\right)$$

and

$$\text{Var}\left[\hat{\Delta}_t\right] = O\left(\max\left\{\frac{1}{nh}, \frac{1}{ng}\right\}\right).$$

Proof of Lemma 1. For $t \geq \tau + \delta$, by simple calculation

$$\begin{aligned} E[\hat{\Delta}_t] &= E\left[\hat{f}_+(t) - \hat{f}_-(t)\right] \\ &= O\left(\frac{K\delta}{h}\right) \to 0 \ \text{ as } \frac{\delta}{h} \to 0. \end{aligned}$$

For the variance, at $t = m + i$,

$$\begin{aligned} \text{Var}\left[\hat{f}_{K+}(t)\right] &= \text{Var}\left[\hat{\phi}_0 + 2\sum_{j=1}^{K} \hat{\phi}_j \cos \pi j t_1\right] \\ &= \text{Var}\left[\frac{1}{ng}\sum_{u=1}^{ng} \epsilon_{m+i-1+u}\left(1 + 2\sum_{j=1}^{K}\sum_{u=1}^{ng} \cos \pi j t_u \cdot \cos \pi j t_1\right)\right] \\ &= O\left(\frac{1}{ng}\right) + o(1), \end{aligned}$$

$$\begin{aligned} \text{Var}\left[\hat{f}_{K-}(t)\right] &= \text{Var}\left[\hat{\psi}_0 + 2\sum_{j=1}^{K} \hat{\psi}_j \cos \pi j t\right] \\ &= \text{Var}\left[\frac{1}{nh}\sum_{u=1}^{nh} \epsilon_u\left(1 + 2\sum_{j=1}^{K}\sum_{u=1}^{nh} \cos \pi j p_u \cdot \cos \pi j t_e\right)\right] \\ &= O\left(\frac{1}{nh}\right) + o(1). \end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{Var}\left[\hat{\Delta}_t\right] &= \operatorname{Var}\left[\hat{f}_{K+}(t)\right] + \operatorname{Var}\left[\hat{f}_{K-}(t)\right] \\ &= O\left(\max\left\{\frac{1}{nh}, \frac{1}{ng}\right\}\right).\end{aligned}$$

□

Proof of Theorem 2.1. We consider $nh(\hat{\Delta}_{\tau+i/n} - \hat{\Delta}_\tau)$ in one level change model in which the amount of change is Δ.

Let $m = n\tau$, consider the mean level change model

$$y_i = \begin{cases} \mu + \epsilon_i, & i = 1, \dots, \tau, \\ \mu + \Delta + \epsilon_i, & i = \tau + 1, \dots, n, \end{cases}$$

in which the difference about the change-point is Δ. Let $q = 1 - \tau$.

The difference of the right estimators is

$$\hat{f}_{K+}(\tau + i/n) - \hat{f}_{K+}(\tau) = \hat{\phi}_0 - \hat{\phi}_{0\tau} + 2\sum_{j=1}^{K}(\hat{\phi}_j - \hat{\phi}_{j\tau}) \cos \pi j t_1.$$

We have

$$\begin{aligned}\hat{\phi}_0 - \hat{\phi}_{0\tau} &= \frac{1}{ng}\sum_{u=1}^{ng} \epsilon_{m+i+u} - \frac{1}{nq}\sum_{u=1}^{nq} \epsilon_{m+u} \\ &= \left(\frac{i}{ng \cdot nq}\right)\sum_{u=1}^{ng} \epsilon_{m+i+u} - \frac{1}{nq}\sum_{u=1}^{i} \epsilon_{m+u}\end{aligned}$$

and for $t_u = (u - 0.5)/ng$, $s_u = (u - 0.5)/nq$,

$$\begin{aligned}\hat{\phi}_j - \hat{\phi}_{j\tau} &= \frac{1}{ng}\sum_{u=1}^{ng} \epsilon_{m+i+u} \cos \pi j t_u - \frac{1}{nq}\sum_{u=1}^{nq} \epsilon_{m+u} \cos \pi j s_u \\ &= \sum_{u=1}^{ng} \epsilon_{m+i+u}\left(\frac{\cos \pi j t_u}{ng} - \frac{\cos \pi j s_u}{nq}\right) - \frac{1}{nq}\sum_{u=1}^{i} \epsilon_{m+u} \cos \pi j s_u.\end{aligned}$$

Since as $ng \to \infty$, $nq \to \infty$,

$$\begin{aligned}\frac{\cos \pi j t_u}{ng} - \frac{\cos \pi j s_u}{nq} &= \frac{\cos \pi j t_u}{nq} - \frac{\cos \pi j s_u}{nq} + \frac{i}{ng \cdot nq} \cos \pi j t_u \\ &= \frac{\pi j}{ng \cdot nq} \sin \pi j t_u^* + o(1).\end{aligned}$$

And then for $t_u^* \in (t_u, s_u)$,

$$\begin{aligned}
&\hat{f}_{K+}(\tau + i/n) - \hat{f}_{K+}(\tau) \\
&= \sum_{u=1}^{ng} \epsilon_{m+i+u} \left(\frac{i}{ng \cdot nq} + 2\sum_{j=1}^{K} \frac{\pi j}{ng \cdot nq} \sin \pi j t_u^* \cos \pi j t_1 \right) \\
&\quad - \frac{1}{nq} \sum_{u=1}^{i} \epsilon_{m+u} \left(1 + 2\sum_{j=1}^{K} \cos \pi j s_u \cos \pi j t_1 \right) + o(1).
\end{aligned}$$

For $p_e = (nh - 0.5)/nh$, the end point of the basis, the difference of the left estimators is

$$\hat{f}_{K-}(\tau + i/n) - \hat{f}_{K-}(\tau) = \hat{\psi}_0 - \hat{\psi}_{0\tau} + 2\sum_{j=1}^{K} (\hat{\psi}_j - \hat{\psi}_{j\tau}) \cos \pi j p_e.$$

We have

$$\hat{\psi}_0 - \hat{\psi}_{0\tau} = \frac{i\Delta}{nh} - \frac{i}{nh \cdot n\tau} \sum_{u=1}^{m} \epsilon_u + \frac{1}{nh} \sum_{u=1}^{i} \epsilon_{m+u}$$

and for $p_u = (u - 0.5)/nh$, $u = 1, \ldots, nh$, $a_u = (u - 0.5)/n\tau$, $u = 1, \ldots, n\tau$,

$$\begin{aligned}
\hat{\psi}_j - \hat{\psi}_{j\tau} &= \frac{\Delta}{nh} \sum_{u=1}^{i} \cos \pi j p_u + \frac{1}{n\tau} \sum_{u=1}^{m} \epsilon_u \left(\frac{\cos \pi j p_u}{nh} - \frac{\cos \pi j a_u}{n\tau} \right) \\
&\quad + \frac{1}{nh} \sum_{u=1}^{i} \epsilon_{m+u} \cos \pi j p_u.
\end{aligned}$$

And then for $p_u^* \in (p_u, a_u)$,

$$\begin{aligned}
\hat{f}_{K-}(\tau + i/n) - \hat{f}_{K-}(\tau) &= \frac{i\Delta}{nh} \left(1 + 2\sum_{j=1}^{K} \sum_{u=1}^{i} \cos \pi j p_u \cos \pi j p_e \right) \\
&\quad - \frac{1}{nh} \sum_{u=1}^{m} \epsilon_u \left(\frac{i}{n\tau} + 2\sum_{j=1}^{K} \frac{\pi j}{n\tau} \sin \pi j p_u^* \cos \pi j p_e \right) \\
&\quad + \frac{1}{nh} \sum_{u=1}^{i} \epsilon_{m+u} \left(1 + 2\sum_{j=1}^{K} \cos \pi j p_u \cos \pi j p_e \right) + o(1)
\end{aligned}$$

where for $|i-m| < i_0$, $i < i_0$, $o(1)$ holds uniformly. Therefore

$$nh(\hat{\Delta}_{\tau+i/n} - \hat{\Delta}_\tau)$$
$$= -i\Delta\left(1 + 2\sum_{j=1}^{K}\sum_{u=1}^{i}\cos\pi jp_u\cos\pi jp_e\right)$$
$$+\sum_{u=1}^{m}\epsilon_u\left(\frac{i}{n\tau} + 2\sum_{j=1}^{K}\frac{\pi j}{n\tau}\sin\pi jp_u^*\cos\pi jp_e\right)$$
$$-\sum_{u=1}^{i}\epsilon_u\left\{\frac{h}{q}\left(1 + 2\sum_{j=1}^{K}\cos\pi js_u\cos\pi jt_1\right) + \left(1 + 2\sum_{j=1}^{K}\cos\pi jp_u\cos\pi jp_e\right)\right\}$$
$$+\sum_{u=m+i+u}^{n}\epsilon_u\left\{\frac{h}{g}\left(\frac{i}{nq} + 2\sum_{j=1}^{K}\frac{\pi j}{nq}\sin\pi jt_u^*\cos\pi jt_1\right)\right\} + o(1)$$

Therefore we have as $n\tau \to \infty$, $h/q \to 1$ and $h/g \to 1$, $nq \to \infty$

$$\frac{nh(\hat{\Delta}_{\tau+i/n} - \hat{\Delta}_\tau)}{(1+2K^*)} = -i\Delta + 2\sum_{u=1}^{i}\epsilon_{m+u} + o_p(1) + o(1)$$

where

$$K^* = \sum_{j=1}^{K}\sum_{u=1}^{i}\cos\pi jp_u\cos\pi jt_e.$$

Since $\hat{\Delta}_{\tau+i/n} + \hat{\Delta}_\tau \to 2\Delta$ for $|i-m| \leq i_0$. Then we achieve

$$\frac{nh(\hat{\Delta}^2_{\tau+i/n} - \hat{\Delta}^2_\tau)}{2(1+2K^*)} = -i\Delta^2 + 2\Delta\sum_{u=1}^{i}\epsilon_u + o_p(1) + o(1).$$

By the same method,

$$\frac{nh(\hat{\Delta}^2_{\tau-i/n} - \hat{\Delta}^2_\tau)}{2(1+2K^*)} = -i\Delta^2 + 2\Delta\sum_{u=-i}^{1}\epsilon_u + o_p(1) + o(1).$$

□

Proof of Theorem 2.2. We assume $\Delta > 0$; the case $\Delta < 0$ is similar.

$$P\left(|\hat{\tau} - \tau| \geq \frac{i_0}{n}\right) = 1 - P\left(\tau - \frac{i_0}{n} \leq \hat{\tau} \leq \tau + \frac{i_0}{n}\right)$$
$$= P\left(\tau + \frac{i_0}{n} \leq \hat{\tau} \leq \tau + q\right) + P\left(\tau - q \leq \hat{\tau} \leq \tau - \frac{i_0}{n}\right).$$

We only consider that

$$
\begin{aligned}
&P\left(\tau+\frac{i_0}{n}\le\hat\tau\le\tau+q\right)\\
&=P\left(\hat\tau=\tau+\frac{i_0}{n}\ \text{ or }\ \hat\tau=\tau+\frac{i_0+1}{n},\ldots,\ \text{ or }\ \hat\tau=\tau+q\right)\\
&\le\sum_{i=i_0}^{nq}P\left(\hat\Delta^2_{\tau+i/n}\ge\hat\Delta^2_\tau\right)\\
&\le\sum_{i=i_0}^{nq}P\left(\hat\Delta_{\tau+i/n}-\hat\Delta_\tau>0\right)+\sum_{i=i_0}^{nq}P\left(\hat\Delta_{\tau+i/n}+\hat\Delta_\tau<0\right).
\end{aligned}
$$

Then we have $P\left(\hat\Delta_{\tau+i/n}+\hat\Delta_\tau<0\right)\to 0$ by the assumption and

$$
\begin{aligned}
\sum_{i=i_0}^{nq}P\left(\hat\Delta_{\tau+i/n}-\hat\Delta_\tau>0\right) &= \sum_{i=i_0}^{nq}P\left(nh(\hat\Delta_{\tau+i/n}-\hat\Delta_\tau)>0\right)\\
&=\sum_{i=i_0}^{nq}P\left(\frac{nh(\hat\Delta_{\tau+i/n}-\hat\Delta_\tau)-\Delta^*}{\sqrt{M}}>\frac{-\Delta^*}{\sqrt{M}}\right)\\
&=\sum_{i=i_0}^{nq}\left\{1-\Phi(C_1\sqrt{i})\right\}\\
&\le\sum_{i=i_0}^{nq}\frac{1}{C_1\sqrt{i}}\exp\left(-\frac{C_1^2 i}{2}\right)\\
&\to 0
\end{aligned}
$$

using $\phi(a)/a\ge 1-\Phi(a)$ for $a>0$, $ng\to\infty$ and $i_0/K^2\to\infty$, where C_1 is a generic constant,

$$\Delta^*=E\left[nh(\hat\Delta_{\tau+i/n}-\hat\Delta_\tau)\right]=O(Ki)$$

and

$$M=\text{Var}\left[nh(\hat\Delta_{\tau+i/n}-\hat\Delta_\tau)\right]=O(K^2 i).$$

□

4. Simulation

A simulation study was done to investigate the behavior of the proposed change-point estimator. The data are generated from the various change-point model with *iid* normal errors with mean 0 and variance 1:

(1) mean level change model with $\mu = 0$, $\Delta = 2, 3$

$$y_i = \begin{cases} \mu + \epsilon_i, & i = 1, \ldots, \tau, \\ \mu + \Delta + \epsilon_i, & i = \tau + 1, \ldots, n; \end{cases}$$

(2) linear function change model with $\Delta = 2, 3$

$$y_i = \begin{cases} 5x + \epsilon_i, & i = 1, \ldots, \tau, \\ 5x + \Delta + \epsilon_i, & i = \tau + 1, \ldots, n; \end{cases}$$

(3) smooth function change model with $\Delta = 2, 3$

$$y_i = \begin{cases} \cos(3\pi x) + \epsilon_i, & i = 1, \ldots, \tau, \\ \cos(3\pi x) + \Delta + \epsilon_i, & i = \tau + 1, \ldots, n; \end{cases}$$

(4) smooth function change model with $\Delta = 3, 7$

$$y_i = \begin{cases} \cos(5\pi x) + \epsilon_i, & i = 1, \ldots, \tau, \\ 3 + 2 \cdot \cos(\Delta\pi x) + \epsilon_i, & i = \tau + 1, \ldots, n. \end{cases}$$

We compute the mean, MSE(mean square error) and the proportion that the estimate is within the true change-point ± 1 based on the sample size $n = 100$ in 1,000 repetitions. The bandwidth $h = 0.2$ and the order of Fourier estimation $K = 1, 2, 3, 4, 5$ are used.

In each unsymmetric window in the left and in the right, those two linear models

$$L : y = \beta_0 + \beta_1 x + \epsilon, \qquad Q : y = \beta_0 + \beta_1 x + \beta_2 x^2 + \epsilon$$

are fitted and left and right estimators are compared to estimate the change-point. The change-point estimators are the points at which the maximum difference of the left and the right estimators occur.

Table 1 shows the results in various change models that the proposed estimator depends on K for a fixed h and the proposed change-point estimator works better than those with the linear and the quadratic regressions. Especially in the model(4) we find that $K = 3$ gives the best estimation. In practice, we need to choose K and h appropriately. Because one can reflect the smoothness of the underlying function in the estimation procedure, the proposed change-point estimator detects the break point better than the simple regression estimators.

TABLE 1 *Mean, MSE and proportion of the change-point estimators with L, Q and* $K = 1, 2, 3, 4, 5$ *based on* $n = 100$ *in* $1,000$ *repetitions*

	(1)	$\Delta = 2$	$h = 0.2$	(1)	$\Delta = 3$	$h = 0.2$
K	*mean*	*MSE*	*proportion*	*mean*	*MSE*	*proportion*
L	49.99	4.593	0.834	49.97	0.315	0.968
Q	62.25	181.679	0.000	61.69	147.106	0.000
K= 1	49.98	3.996	0.838	49.96	0.318	0.967
2	49.66	21.345	0.814	49.96	0.339	0.963
3	49.65	60.173	0.750	49.85	4.817	0.960
4	49.84	90.453	0.697	49.83	8.080	0.949
5	50.43	135.571	0.624	50.08	18.030	0.928
	(2)	$\Delta = 2$	$h = 0.2$	(2)	$\Delta = 3$	$h = 0.2$
K	*mean*	*MSE*	*proportion*	*mean*	*MSE*	*proportion*
L	50.07	0.385	0.965	50.00	0.415	0.958
Q	63.25	190.234	0.000	61.71	149.14	0.000
K= 1	50.07	0.381	0.963	50.02	0.392	0.960
2	50.05	0.443	0.960	50.01	0.407	0.959
3	50.05	0.680	0.955	49.88	3.644	0.952
4	50.00	3.654	0.954	49.93	7.089	0.949
5	50.13	16.027	0.932	50.03	20.534	0.927
	(3)	$\Delta = 2$	$h = 0.2$	(3)	$\Delta = 3$	$h = 0.2$
K	*mean*	*MSE*	*proportion*	*mean*	*MSE*	*proportion*
L	49.95	1.115	0.882	50.01	0.232	0.976
Q	63.85	197.783	0.000	63.29	180.91	0.000
K=1	49.97	1.082	0.883	50.01	0.233	0.976
2	49.96	19.546	0.847	49.99	1.004	0.969
3	49.80	37.132	0.802	50.01	2.042	0.962
4	49.87	43.405	0.768	49.99	3.446	0.956
5	49.95	55.616	0.719	49.97	77.677	0.767
	(4)	$\Delta = 3$	$h = 0.2$	(4)	$\Delta = 7$	$h = 0.2$
K	*mean*	*MSE*	*proportion*	*mean*	*MSE*	*proportion*
L	50.46	10.808	0.980	50.32	6.720	0.984
Q	58.07	65.797	0.000	57.99	64.567	0.000
K=1	50.64	16.606	0.978	50.56	14.264	0.978
2	54.35	71.964	0.739	54.28	70.836	0.743
3	50.30	1.017	0.959	50.29	0.863	0.956
4	51.84	33.539	0.763	51.95	31.823	0.766
5	50.77	8.699	0.882	50.99	15.049	0.877

5. Concluding Remarks

The objective of this research is to find a consistent change-point or discontinuity estimator which can detect the break-point in the regression funciton. The left and right estimators with the sample Fourier coefficients in the unsymmetric window were used for the proposed change-point estimator. The selection of the bandwidth and the truncated number of the Fourier series affects the estimation. The appropriate selection will increase the ability of detection. The proposed estimator with the sample Fourier series has better performance especially when the underlying function is more wiggly. And the proposed method can be applied to any situation where the underlying function is various with the smoothness. The choice problem of the bandwidth and the window will be left for another research.

REFERENCES

Hall, R. and Titterington, D. M. (1992). "Edge-preserving and peak-preserving smoothing", *Technometrics*, **34**, 429–440.

Hinkley, D. V. (1970). "Inference about the change-point in a sequence of random variables", *Biometrika*, **57**, 1–17.

Kim, J. H. and Hart, J. D. (1998). "Tests for change in a mean function when the data are dependent", *Journal of Time Series Analysis*, **19**, 399–424.

Loader, C. R. (1996). "Change point estimation using nonparametric regression", *The Annals of Statistics*, **24**, 1667–1678.

Lombard, F. (1988). "Detecting change points by Fourier analysis", *Technometrics*, **30**, 305–310.

McDonald, J. A. and Owen, A. B. (1986). "Smoothing with split linear fits", *Technometrics*, **28**, 195–208.

Tolstov, G. P. (1962). *Fourier Series*, Dover, New York.