The Korean Communications
in Statistics Vol. 9, No. 3, 2002
pp. 753-763

Comparison of Change-point Estimators in Hazard Rate Models[1)]

Jaehee Kim[2)]

Abstract

When there is one change-point in the hazard rate model, a change-point estimator with the partial score process is suggested and compared with the previously developed estimators. The limiting distribution of the partial score process we used is a function of the Brownian bridge. Simulation study gives the comparison of change-point estimators.

Keywords : Change-point, Hazard rate, Maximum likelihood. Score-statistic, Ornstein-Uhlenbeck process, Nelson-Aalen cumulative hazard estimator

1. Introduction

In life testing, medical follow up and other studies, the observation of the occurrence of the event of interest (called a failure) may have some pattern to be modelled and investigated. Inference about rates has been studied in survival analysis, demography, reliability, and many other fields.

Suppose patients in a clinical trial receive a treatment at time 0. The survival times may represent the time until unwanted side effects occur, in which case we would expect a high initial hazard rate and a lower hazard rate after the treatment has been in place for some time.

Let $T_1, \ldots, T_n$ denote independently distributed lifetimes of n subjects, but that early failures appear to occur at one rate and late failures appear to occur at another rate. The nonnegative continuous type random variable T has the density function f, the distribution function F. The hazard rate h satisfy:

$$h(t) = \begin{cases} \lambda_0, & 0 \le t \le \tau \\ \lambda_1, & t > \tau \end{cases} \tag{1}$$

---

1) This research was supported by R04-2001-000-00135-0 KOSEF.
2) Associate Professor, Dept. of Statistics, Duksung Women's University, Ssangmun-Dong Tobong-Gu Seoul S.Korea
E-mail : jaehee@center.duksung.ac.kr

in which the hazard rates λ_0, λ_1 and the change-point τ are all unknown with the parameter space $\{(\lambda_0, \lambda_1, \tau): \lambda_0 \geq 0, \lambda_1 \geq 0, \tau \geq 0\}$. The function $h(t)$ is also called conditional failure rate, mortality, intensity, age-specific death rate, instantaneous death rate or force of mortality. Then we can obtain the density of T as

$$f(t) = \begin{cases} \lambda_0 \exp(-\lambda_0 t) & t \leq \tau \\ \lambda_1 \exp(-\lambda_0 \tau - \lambda_1 (t-\tau)) & t > \tau. \end{cases} \tag{2}$$

Questions of interest are testing the null hypothesis of no change, and if we conclude that there is a change, inference about τ and the size of the change should be made. In this paper we are interested in comparison of change-point estimators with the proposed one. In section 2, the previously suggested change-point estimators are reviewed including the maximum likelihood estimator proposed by Matthews and Farewell (1982), Yao (1986), the modified likelihood estimator suggested by Henderson (1990) and an alternative likelihood estimator given by Nguyen, Rogers and Walker (1984), the score process estimator by Matthews, Farewell and Pyke (1985), the Poisson process approach by Akman and Raftery (1986). Recently Jung and Han (1998) considered the change-point estimation with the likelihood for the Weibull distribution and suggested the bootstrap technique to get the confidence set. Section 3 contains a proposed estimator with the partial score statistic and its limiting distribution and section 4 shows the result of comparison via simulation studies. And finally section 5 presents concluding remarks.

2. A Review of Change-point Estimators

Matthews and Farewell (1982) considered the model using the data obtained in the treatment of leukaemia patients and the log likelihood of observations $T_1, \ldots, T_n$

$$L_1(\tau) = r \log \lambda_0 + (n-r) \log \lambda_1 - \lambda_0 \sum_{i=1}^{r} t_i - \lambda_1 \sum_{i=r+1}^{n} t_i + (n-r)\tau(\lambda_1 - \lambda_0)$$

where $N(\tau) = r$ and the order statistics are denoted by $\{t_i\}$. For fixed τ, the maximum likelihood estimators of λ_0 and λ_1 can be found by differentiating $L_1(\tau)$ giving

$$\hat{\lambda}_0 = \frac{r}{\sum_{i=1}^{r} t_i + (n-r)\tau} \quad \text{and} \quad \hat{\lambda}_1 = \frac{n-r}{\sum_{i=r+1}^{n} t_i - (n-r)\tau}. \tag{3}$$

They suggested the maximum likelihood change-point estimator as

$$\hat{\tau}_L = \arg\max_{\tau_l \leq \tau \leq \tau_u} L_1(\tau). \tag{4}$$

Yao (1986) suggested the restriction $\hat{\tau} < t_n$ since $L(t, \hat{\lambda}_0, \hat{\lambda}_1) \to \infty$ as $t \to t_n$ and showed that

$\hat{\tau} - \tau = O_p(n^{-1})$ with the natural constraint $\hat{\tau} \leq t_{n-1}$. Yao (1986) considered the likelihood function as

$$l(\tau) = -(sup_{\lambda_0, \lambda_1} L_1(\tau) + n)/n$$

Yao (1986) suggested the change-point estimator which minimizes $l(\tau)$ over $\tau \le t_{n-1}$.

When the classical maximum likelihood method and the method of moments cannot be used due to the irregularity of the models, Nguyen, Rogers and Walker (1984) derived an alternative likelihood estimator. The mean and the variance of the right-hand portion of the sample denoted by

$$M(\tau) = \sum_{i=r+1}^{n} t_i / (n-r), \quad S(\tau)^2 = \frac{\sum_{i=r+1}^{n} t_i^2}{(n-r)} - \{M(\tau)\}^2.$$

They noted that

$$E_n = \log\left\{\frac{n}{(n-r)}\right\}\{M(\tau) - S(\tau)\}^{-1} \text{ converges to } \lambda_0$$

and

$$W_n = \frac{r - (n-r)\log\left\{\frac{n}{(n-r)}\right\}}{nZ_n - (n-r)M(\tau)} \text{ converges to } \lambda_0.$$

Since the limits of the denominators in E_n and W_n are not zero, the limit of the numerator of their difference is zero. This difference can be

$$X_n(\tau) = \frac{S(\tau)}{n}\left[(n-r)\log\left\{\frac{n}{(n-r)} - r\right\}\right] + \frac{r}{n}M(\tau) - Z_n \log\left\{\frac{n}{n-r}\right\}$$

converges to 0. By the convergence property and equating the same expected values, Nguyen, Rogers and Walker (1984) derived the change-point estimator as the difference of two statistics whose expected values are zero. These two statistics include the classical maximum likelihood estimators derived by differentiation. The change-point estimator was suggested as

$$\hat{\tau}_N = \inf\{\ t \in [\tau_l, \tau_u]:\ X_n(t) \text{ is close to } 0\}. \qquad (5)$$

Henderson (1990) considered for a constant hazard against the alternative of a step change at an unknown time point. Henderson (1990) wrote L_k^+ and L_k^- for twice the log likelihood ratio evaluated at $\tau = t_k^+$ and $\tau = t_k^-$ respectively,

$$L_k^+ = 2\left\{k\log\left(\frac{k}{U_k}\right) + (n-k-1)\log\left(\frac{n-k-1}{1-U_k}\right) - (n-1)\log(n-1)\right\},$$

$$L_k^- = 2\left\{(k-1)\log\left(\frac{k}{U_k}\right) + (n-k)\log\left(\frac{n-k}{1-U_k}\right) - (n-1)\log(n-1)\right\}$$

where

$$U_k = \left\{\sum_{i=1}^{k} t_i + (n-k)t_k\right\} / \sum_{i=1}^{n} t_i\ .$$

An adjustment to the likelihood ratio procedure is to standardize the L_k^+ and L_k^- terms before maximization over k. Henderson (1990) considered the modification with the standardization as

$$L_2(\tau)=\max\{(L_k^+-\mu_k^+)/\sigma_k^+,(L_k^--\mu_k^-)/\sigma_k^-\}$$

where $\mu_k^+=E(L_k^+)$, $\sigma_k^+=\sqrt{var(L_k^+)}$, $\mu_k^-=E(L_k^-)$ and $\sigma_k^-=\sqrt{var(L_k^-)}$. Also he considered the alternative possibility with the greater credence to the more central values as

$$L_3(\tau)=L_1(\tau)+\log\{4k(n-k)/n^2\}$$

and combined approach with weighting and standardizing

$$L_4(\tau)=L_k^*+\frac{1}{2}\log\{4k(n-k)/n^2\}$$

where $L_k^*=\max\{(L_k^+-\mu_k^+)/\sigma_k^+,(L_k^{-1}-\mu_k^-)/\sigma_k^-\}$. Henderson (1990) suggested the change-point estimators in the standardized version as

$$\hat{\tau}_{HU}=\inf\{t\in[\tau_l,\tau_u]:\sup\ L_2(\tau)\ \text{ occurs}\} \tag{6}$$

and in the modified version as

$$\hat{\tau}_{H}=\inf\{t\in[\tau_l,\tau_u]:\sup\ L_4(\tau)\ \text{ occurs}\} \tag{7}$$

Matthews, Farewell and Pyke (1985) suggested the test based on the score statistics for the existence of the change-point. And also they suggested the test statistic based on the partial score-statistic process as

$$M_n=\sup_{\tau_l\le\tau\le\tau_u} Z_n(\tau)$$

where $Z_n(\tau)=n^{-1/2}\sum_{i=1}^{N(\tau)}e^{\lambda\tau/2}(\lambda t_i-1)$ is obtained by getting score. Then the process M_n converges in distribution to $\sup_{\tau_l\le\tau\le\tau_u} Z(\tau)$ where Z is the appropriate Ornstein-Uhlenbeck process.

Akman and Raftery (1986) considered the inference based on a Poisson process. The n events on $[0,T]$ have been observed at times $t_1,\dots,t_n$. The rate of occurrence at time t is denoted by $h(t)$. Let $\tau/T=\theta$. They considered the following process

$$A(s;\ c,d)=\{s'(1-s')\}^{1/2}\left\{\frac{\Pi(s)-\Pi(c)}{s'}-\frac{\Pi(d)-\Pi(s)}{1-s'}\right\}$$

where $0\le c<d\le 1$, $s'=(s-c)/(d-c)$, $\Pi(s)=N(sT)$, and $N(t)$ is the number of events that occurred in the time interval $(0,t]$. They suggested the change-point estimator as

$$\hat{\tau}_A=\hat{\theta}T \tag{8}$$

where

$$\hat{\theta}=\inf\{s=t/T:\ A(s;\ 0,1)=\sup\ A(u;\ 0,1)\}.$$

And they proposed the estimators for hazard as

$$\hat{\lambda}_0=\frac{N(\hat{\tau})}{\hat{\tau}},\quad \hat{\lambda}_1=\frac{N(T)-N(\hat{\tau})}{T-\hat{\tau}}$$

which are consistent.

3. A Proposed Change-point Estimator with the Partial Score-statistic

Consider the hazard function (1) with the change-point and decreasing hazard such as

$$h(t)=\begin{cases}\lambda & =\lambda_0, \quad 0\le t\le \tau \\ (1-\xi)\lambda & =\lambda_1, \quad t>\tau\end{cases}$$

in which the two parameters satisfy $0\le\xi<1$ and $\tau\ge 0$.

Matthews, Farewell and Pyke (1985) considered the normalized score-statistics when λ is known and the partial score-statistics in case of unknown λ for testing for the existence of the change-point.

We suggest the change-point estimator with the partial-score statistic as

$$\begin{aligned} Z_n(\tau) &= \frac{\dfrac{\partial \log L}{\partial \xi}}{\dfrac{\partial^2 \log L}{\partial \xi^2}-\dfrac{\partial \log L}{\partial \xi \partial \lambda}\left(\dfrac{\partial^2 \log L}{\partial \lambda^2}\right)^{-1}\dfrac{\partial^2 \log L}{\partial \xi \partial \lambda}} \\ &= (1-e^{-\lambda\tau})^{-1/2} n^{-1/2}\sum_{i=1}^{n} e^{\lambda\tau/2}\{\lambda(t_i-\tau)-1\} I(t_i-\tau) \end{aligned} \tag{9}$$

where $I(x)=1,\ x\ge 0$ and $0,\ x<0$.

Similarly, consider the change-point model with increasing hazard :

$$h(t)=\begin{cases}\lambda = \lambda_0, & 0\le t\le \tau \\ (1+\xi)\lambda=\lambda_1, & t>\tau.\end{cases}$$

Then we have

$$Z_n(\tau) = (1-e^{-\lambda\tau})^{-1/2} Z_n(\tau,\lambda) .$$

Also $\widehat{Y}_n(\tau)=(1-e^{-\lambda\tau})^{-1/2}Z_n(\tau,\widehat{\lambda}_n)$, where the mle $\widehat{\lambda}_n=n/\sum_{i=1}^{n} t_i$

$$\hat{\tau}_S=\inf\{t\in[\tau_l,\tau_u]: \sup\ |\widehat{Y}_n(\tau)|\ \text{ occurs}\}. \tag{10}$$

Finally we consider the partial score-statistic process with $\widehat{\lambda}_0$ in (3) and use the following process

$$\widehat{Z}_n(\tau)=(1-e^{-\lambda\tau})^{-1/2}Z_n(\tau,\widehat{\lambda}_0)$$

where $\widehat{\lambda}_0$ is the maximum likelihood estimator of λ_0 in the model (1).

For our change-point hazard model (1), we suggest the change-point estimator as

$$\hat{\tau}_{PS}=\inf\{t\in[\tau_l,\tau_u]: \sup\ |\widehat{Z}_n(\tau)|\ \text{ occurs}\}. \tag{11}$$

The following theorem shows that the limiting distribution of the suggested partial score-statistic process is the absolute Brownian bridge.

Theorem 1. With $\widehat{\lambda}_0=r/\{\sum_{i=1}^{r} t_i+(n-r)\tau\}$ in (3), $\widehat{Z}_n(t)$ converges to

$$|Z(t)| = \frac{|W(t) - tW(1)|}{\{t(1-t)\}^{1/2}}, \quad 0 \le t \le 1 \tag{12}$$

where W is standard Brownian motion.

proof. Consider the partial score-statistic process

$$\begin{aligned} Z_n(\tau) &= (1-e^{-\lambda\tau})^{-1/2} n^{-1/2} \sum_{i=1}^{n} e^{\lambda\tau/2}\{\lambda(t_i - \tau) - 1\} I(t_i - \tau) \\ &\approx (1-e^{-\lambda\tau})^{-1/2} (ne^{\lambda\tau})^{1/2} \int_\tau^\infty \{(x-\tau)\lambda - 1\} dF_n(x) \\ &= (1-e^{-\lambda\tau})^{-1/2} Z_n(\tau, \lambda) \end{aligned}$$

where $Z_n(\tau, \lambda) = (ne^{\lambda\tau})^{1/2} \int_\tau^\infty \{(x-\tau)\lambda - 1\} dF_n(x)$.

Using $\int_o^\infty U_n^F(x)\, dx = n^{1/2}\left[\frac{1}{\lambda} - \frac{1}{\hat{\lambda}_0}\right]$, $\int_\tau^\infty (x-\tau) n^{-1/2} dU_n^F(s) = -\int_\tau^\infty n^{-1/2} U_n^F(x)\, dx$,

$\int_\tau^\infty (x-\tau)\, dF(x) = \frac{1}{\lambda}[1 - F(\tau)]$ and $F(\tau) = 1 - e^{-\lambda\tau}$, we have

$$\begin{aligned} &\int_\tau^\infty \{(x-\tau)\hat{\lambda}_0 - 1\} dF_n(x) \\ &= \int_\tau^\infty \{(x-\tau)\lambda - 1\} dF_n(x) + (\hat{\lambda}_0 - \lambda) \int_\tau^\infty (x-\tau)\, dF_n(x) \\ &= (ne^{\lambda\tau})^{-1/2} Z_n(\tau, \lambda) + (\hat{\lambda}_0 - \lambda) \int_\tau^\infty n^{-1/2}(x-\tau) n^{1/2} d(F_n(x) - F(x)) \\ &\qquad + (\hat{\lambda}_0 - \lambda) \int_\tau^\infty (x-\tau)\, dF(x) \\ &= (ne^{\lambda\tau})^{-1/2} Z_n(\tau, \lambda) - (\hat{\lambda}_0 - \lambda) \int_\tau^\infty n^{-1/2} U_n^F(x)\, dx + \hat{\lambda}_0(1 - F(\tau)) \int_0^\infty n^{-1/2} U_n^F(x)\, dx \end{aligned}$$

where $U_n^F(x) = n^{1/2}\{F_n(x) - F(x)\}$ denotes the empirical process based on n independent random variables from F.

$$\begin{aligned} \hat{Z}_n(\tau) &= (1-e^{-\lambda\tau})^{-1/2} Z_n(\tau, \lambda)|_{\lambda = \hat{\lambda}_0} = (1-e^{-\lambda\tau})^{-1/2} (ne^{\lambda\tau})^{1/2} \int_\tau^\infty \{(x-\tau)\lambda - 1\} dF_n(x) \\ &= (1-e^{-\lambda\tau})^{-\frac{1}{2}} e^{\frac{1}{2}\tau(\hat{\lambda}_0 - \lambda)} \cdot \\ &\quad \left[Z_n(\tau, \lambda) + \hat{\lambda}_0 e^{-\frac{1}{2}\lambda\tau} \int_0^\infty U_n^F(x)\, dx + (\hat{\lambda}_0 - \lambda)\lambda^{-1} n^{-\frac{1}{2}} \{-Z_n(\tau, \lambda) + e^{\frac{\lambda\tau}{2}} U_n^F(\tau)\}\right] \\ &\to (1-e^{-\lambda\tau})^{-\frac{1}{2}} \left[Z_n(\tau, \lambda) + \lambda e^{-\frac{\lambda\tau}{2}} \int_0^\infty U_n^F(x)\, dx\right] \end{aligned}$$

as far as $\hat{\lambda}_0 \xrightarrow{p} \lambda$ as $n \to \infty$.

Let $U_n(u) = n^{1/2}\{F_n(u) - u\}$, $0 \le u \le 1$ be the uniform empirical process and $U \equiv \{U(u) : 0 \le u \le 1\}$ be the Brownian bridge. It is known that $U_n^F = U \circ (1-F) = U \circ F$ and $U_n \xrightarrow{d} U$.

After applying integration by parts we get

$$Z_n(\tau,\lambda) = -e^{\lambda\tau/2}\left\{\lambda\int_\tau^\infty U_n^F(x)\,dx - U_n^F(\tau)\right\}$$

and $\lambda\int_\tau^\infty U_n^F(x)\,dx = \int_0^t u^{-1}U_n(u)\,du.$

By the changes of variable $1-F(x)=u$ and $1-F(\tau)=t$, we obtain

$$Z^*(t) = Z_n\left(-\frac{1}{\lambda}\ln t\right) = t^{-1/2}\left\{U_n(t) - \int_0^t u^{-1}U_n(u)\,du\right\}$$

which converges to $t^{-1/2}W(t)$ where $W(t)=U(t)-\int_0^t u^{-1}U(u)du$ is a standard Brownian motion. Then

$$\begin{aligned}
&(1-e^{-\lambda\tau})^{-1/2}\left[Z_n(\tau,\lambda)+\lambda e^{-\frac{\lambda\tau}{2}}\int_0^\infty U_n^F(x)\,dx\right] \\
&\to (1-t)^{-1/2}\left[Z^*(t)+\sqrt{t}\int_0^1 u^{-1}U(u)du\right] \\
&\to (1-t)^{-1/2}t^{-1/2}[W(t)-tW(1)]
\end{aligned}$$

Therefore $\widehat{Z_n}(\tau)$ converges to $Z(t)=\dfrac{W(t)-tW(1)}{\{t(1-t)\}^{1/2}}$. And our process $|\widehat{Z_n}(\tau)|$ converges to $|Z(t)|=\dfrac{|W(t)-tW(1)|}{\{t(1-t)\}^{1/2}}$.

Theorem 2. $\widehat{\tau_{PS}}$ converges to

$$\inf\{t\in[\tau_l,\tau_u] : \sup|Z(t)| \text{ occurs}\}, \quad [\tau_l,\ \tau_u]\in[0,1]$$

where $|Z(t)|$ is in Theorem 1.

proof. It is obvious from Theorem 1.

4. Simulation

A random sample $T_1,\ldots,T_n$ were generated from the density function

$$f(t)=\begin{cases}\lambda_0\exp(-\lambda_0 t), & t\le\tau\\ \lambda_1\exp(-\lambda_0 t-\lambda_1(t-\tau)), & t>\tau\end{cases}$$

for various $(\lambda_0=1,\lambda_1,\tau)$ values. The sample size 100 were used in 1,000 repetitions.

To illustrate and compare the behavior of the change-point estimators, the mean, the mean squared error(MSE) and two hazard rate estimates were calculated.

Considered are the maximum likelihood estimator $\widehat{\tau_L}$, Henderson's modified mle's $\widehat{\tau_{HU}}$, $\widehat{\tau_H}$ and Nguyen, Rogers and Walker estimator $\widehat{\tau_N}$, Akman and Raftery estimator $\widehat{\tau_A}$

, $\widehat{\tau_S}$ with the partial score and $\widehat{\lambda}=n/\sum_{i=1}^{n}t_i$ and the proposed estimators $\widehat{\tau_{PS}}$ based on the mle of the first hazard rate, $\widehat{\lambda_0}=r/\{\sum_{i=1}^{r}t_i+(n-r)\tau\}$.

The statistics were computed for the fixed bounds considered as the plausible range of the change-point. For the true change-point $\tau=0.25,\ 0.5$, the range $\tau_l=0.1$ and $\tau_u=1.0$ were considered and for $\tau=1.0$, the range $\tau_l=0.5,\ \tau_u=1.5$ were used in the simulation. Akman and Raftery (1986) process is also restricted to $[\tau_l,\ \tau_u]$. For the decreasing hazard rates in Table 1, the proposed estimator has smaller mse's than others when the change-point occurs former part of ordered data. And for the increasing hazard rates in Table 2, the proposed estimator has smaller mse's than others and even far better than $\widehat{\tau_S}$. When the hazard rates increase after the change-point, no other considered estimators beat the proposed change-point estimator.

4. Concluding Remarks

We considered estimation of the change-point when there is change in the hazard. Comparison was done with the previously suggested method with the likelihood considered. The proposed change-point estimator with the partial score-process and it has a good performance especially for increasing hazard rates. Likelihood can be modified to add the information about the situation and the other adjusted techniques can be made.

Chang, Chen and Hsiung (1994) proposed the nonparametric change-point estimation with the Nelson-Aalen estimator for censored data. We expect further study for change-point estimation for the censored data and developing the nonparametric techniques.

References

[1] Jung G. M. and Han, M. H. (1998). 위험률의 변화점모형에 대한 추론, 「한국통계학회논문집」, 제5권 2호, 477-489.

[2] Akman, V. E. and Raftery, A. E. (1986). Asymptotic Inference for a Change-point Poisson Process, Annals of Statistics, 1583-1590.

[3] Chang, I-S., Chen, C-H. and Hsiung, C. A. (1994). Estimation in Change-point Hazard Rate Models with Random Censorship, IMS Lecture Notes Vol 23 - Change-point Problems, 78-92.

[4] Henderson, R. (1990). A Problem with the Likelihood Ratio Test for a Change-point Hazard Rate Model. *Biometrika* 77, 4, 835-843.

[5] Liang, K-Y., Self, S. G. and Liu, X. (1990). The Cox Proportional Hazards Model with Change-point: An Epidemiologic Application, Biometrics 46, 783-793.

[6] Matthews, D. E. and Farewell, Vernon, T. (1982). On Testing for Constant Hazard against a Change-point Alternative. *Biometrics* 38, 463-468.

[7] Matthews. D. E., Farewell, V. T. and Pyke, R. (1985). Asymptotic Score-statistic Processes and Tests for Constant Hazard Against a Change-point Alternative. *Annals of Statistics* 13, 2, 583-591.

[8] Nguyen, H. T., Rogers, G. S. and Walker, E. A. (1984). Estimation in Change-point Hazard Rate Models. *Biometrika* 71, 299-304.

[9] Yao, Y. C. (1986). Maximum Likelihood Estimation in Hazard Rate Models with a Change-point, *Communications in Statistics - Theo. and Meth.*. 15(8), 2455-2466.

[2001년 9월 접수, 2002년 2월 채택]

Table1. Comparison of Change-point Estimator for τ in the Decreasing Hazard Rate with the sample size 100 in 1000 Repetitions

		$\lambda_0=1.5$		$\lambda_1=1$		$\lambda_0=2$		$\lambda_1=1$	
τ		mean	mse	$\widehat{\lambda_0}$	$\widehat{\lambda_1}$	mean	mse	$\widehat{\lambda_0}$	$\widehat{\lambda_1}$
0.25	$\widehat{\tau_L}$	0.289	0.0283	1.584	0.901	0.260	0.0120	2.111	1.179
	$\widehat{\tau_H}$	0.291	0.0284	1.576	0.900	0.267	0.0134	2.103	1.177
	$\widehat{\tau_{HU}}$	0.318	0.0342	1.565	0.897	0.266	0.0150	2.111	1.179
	$\widehat{\tau_N}$	0.183	0.0246	1.545	0.958	0.216	0.0211	2.030	1.254
	$\widehat{\tau_{NA}}$	0.442	0.0984	1.437	0.932	0.322	0.0481	2.048	1.233
	$\widehat{\tau_A}$	0.403	0.4907	1.490	0.877	0.118	0.0913	2.091	1.180
	$\widehat{\tau_S}$	0.179	0.0162	1.706	0.932	0.174	0.0113	2.237	1.228
	$\widehat{\tau_{PS}}$	0.255	0.0223	1.639	0.903	0.238	0.0106	2.153	1.875
0.50	$\widehat{\tau_L}$	0.458	0.0525	1.777	1.154	0.483	0.0274	2.083	0.824
	$\widehat{\tau_H}$	0.465	0.0534	1.763	1.153	0.493	0.0276	2.074	0.819
	$\widehat{\tau_{HU}}$	0.479	0.0482	1.769	1.151	0.511	0.0316	2.059	0.811
	$\widehat{\tau_N}$	0.348	0.0811	1.709	1.234	0.409	0.0536	2.046	0.926
	$\widehat{\tau_{NA}}$	0.669	0.1371	1.672	1.236	0.566	0.0815	2.033	0.835
	$\widehat{\tau_A}$	0.495	0.0254	1.785	1.138	0.419	0.0178	2.137	0.862
	$\widehat{\tau_S}$	0.293	0.0853	1.890	1.213	0.327	0.0690	2.242	0.942
	$\widehat{\tau_{PS}}$	0.443	0.0569	1.828	1.146	0.481	0.0361	2.097	0.827
1.00	$\widehat{\tau_L}$	0.950	0.0859	1.645	0.864	0.932	0.0622	2.134	1.203
	$\widehat{\tau_H}$	0.980	0.0854	1.642	0.849	0.965	0.0644	2.127	1.187
	$\widehat{\tau_{HU}}$	0.928	0.0937	1.639	0.883	0.980	0.0740	2.116	1.208
	$\widehat{\tau_N}$	0.759	0.1391	1.622	1.000	0.831	0.0939	2.114	1.355
	$\widehat{\tau_{NA}}$	1.227	0.1845	1.590	0.860	1.131	0.1594	2.086	1.407
	$\widehat{\tau_A}$	0.706	0.1129	1.684	0.975	0.598	0.1765	2.176	1.475
	$\widehat{\tau_S}$	0.905	0.0989	1.663	0.859	0.982	0.1014	2.122	1.208
	$\widehat{\tau_{PS}}$	0.975	0.0865	1.653	0.831	1.053	0.0974	2.107	1.196

Table 2. Comparison of Change-point Estimator for τ in the Increasing Hazard Rate with the sample size 100 in 1000 Repetitions

		$\lambda_0=1$	$\lambda_1=1.5$			$\lambda_0=1$	$\lambda_1=2$		
τ		mean	mse	$\widehat{\lambda_0}$	$\widehat{\lambda_1}$	mean	mse	$\widehat{\lambda_0}$	$\widehat{\lambda_1}$
0.25	$\widehat{\tau_L}$	0.333	0.0363	0.947	1.413	0.284	0.0129	0.983	2.463
	$\widehat{\tau_H}$	0.317	0.0328	0.931	1.410	0.276	0.0117	0.974	2.446
	$\widehat{\tau_{HU}}$	0.336	0.0374	0.925	1.407	0.262	0.0116	0.948	2.406
	$\widehat{\tau_N}$	0.235	0.0274	1.055	1.339	0.258	0.0172	1.097	2.309
	$\widehat{\tau_{NA}}$	0.516	0.1080	1.054	1.419	0.507	0.1058	1.261	2.839
	$\widehat{\tau_A}$	0.616	0.1432	1.256	1.261	0.618	0.1416	1.579	2.517
	$\widehat{\tau_S}$	0.225	0.0174	0.856	1.382	0.192	0.0085	0.889	2.230
	$\widehat{\tau_{PS}}$	0.367	0.0399	0.954	1.436	0.330	0.0206	1.064	2.564
0.50	$\widehat{\tau_L}$	0.514	0.0522	0.948	1.464	0.521	0.0184	1.015	2.962
	$\widehat{\tau_H}$	0.494	0.0534	0.935	1.458	0.510	0.0197	1.006	2.936
	$\widehat{\tau_{HU}}$	0.498	0.0505	0.924	1.457	0.488	0.0252	0.980	2.845
	$\widehat{\tau_N}$	0.417	0.0730	1.049	1.360	0.497	0.0270	1.093	2.710
	$\widehat{\tau_{NA}}$	0.802	0.1464	1.046	1.517	0.776	0.1247	1.214	3.872
	$\widehat{\tau_A}$	0.856	0.1485	1.176	1.311	0.876	0.0156	1.442	3.214
	$\widehat{\tau_S}$	0.318	0.0742	0.847	1.378	0.315	0.0631	0.867	2.374
	$\widehat{\tau_{PS}}$	0.562	0.0445	0.958	1.486	0.542	0.0146	1.039	3.008
1.00	$\widehat{\tau_L}$	1.014	0.0814	1.024	2.125	1.024	0.0340	1.084	2.488
	$\widehat{\tau_H}$	1.004	0.0818	1.021	2.109	1.019	0.0341	1.083	2.183
	$\widehat{\tau_{HU}}$	0.978	0.0794	1.020	2.092	1.028	0.0386	1.086	2.487
	$\widehat{\tau_N}$	0.899	0.1161	1.070	1.825	0.985	0.0494	1.129	2.226
	$\widehat{\tau_{NA}}$	1.304	0.1576	1.096	2.225	1.289	0.1392	1.182	2.515
	$\widehat{\tau_A}$	1.068	0.1178	1.193	1.584	1.158	0.1292	1.292	1.773
	$\widehat{\tau_S}$	0.893	0.0925	0.987	2.017	0.869	0.0619	1.033	2.313
	$\widehat{\tau_{PS}}$	1.012	0.0700	1.015	2.135	1.000	0.0262	1.077	2.466