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Characterization of Tightness for Fuzzy Random Variables

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Abstract

In this paper, we establish some characterizations of tightness for a sequence of random elements taking values in the space of upper-semicontinuous fuzzy sets with compact support in R.

Keywords : fuzzy random variables, tightness.

1. Introduction

The notion of tightness plays an important role both in the theory of weak convergence and in its applications. The relationships between weak convergence and tightness can be also found in Billingsley[1].

The concept of a fuzzy random variable was introduced by Puri and Ralescu[11] as a natural generalization of a random set in order to represent relationships between the outcomes of random experiment and inexact data due to the subjectivity. Joo and Kim [6] introduced a new metric d_s on the space $F(R)$ of fuzzy numbers in R so that $F(R)$ is separable and topologically complete, and Ghil et.al.[3] characterized compact subsets of $F(R)$. Also, Kim [9] proved that a fuzzy mapping is measurable if and only if it is measurable when considered as a function into the metric space $F(R)$ endowed the metric d_s.

In this paper, motivated by the works of Joo and Kim [6] and Kim [9], we establish the theory of tightness for fuzzy random variables. Section 2 is devoted to describe some preliminary results, and the main results are given in section 3.

2. Preliminary Results

In this section, we describe some basic concepts of fuzzy numbers. Let R denote the real

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line. A fuzzy number is a fuzzy set $\tilde{u} : R \to [0, 1]$ with the following properties;

(1) $\tilde{u}$ is normal, i.e. there exists $x \in R$ such that $\tilde{u}(x) = 1$,

(2) $\tilde{u}$ is upper semicontinuous,

(3) supp $\tilde{u} = cl\{ x \in R \, ; \, \tilde{u}(x) > 0\}$ is compact,

(4) $\tilde{u}$ is a convex fuzzy set, i.e., $\tilde{u} \, (\lambda x + (1 - \lambda)y) \geq \min \, (\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0,1]$.

where $cl\,(A)$ denote the closure of A.

We denote the family of all fuzzy numbers by $F(R)$. For a fuzzy set $\tilde{u}$, the α-level set of $\tilde{u}$ is defined by

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\} & \text{if } \; 0 < \alpha \leq 1, \\ \text{supp } \tilde{u} & \text{if } \; \alpha = 0. \end{cases}$$

Then it follows that $\tilde{u}$ is a fuzzy number if and only if $L_1 \tilde{u} \neq \phi$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \in [0,1]$. From this characterization of fuzzy numbers, a fuzzy number $\tilde{u}$ is completely determined by the end points of the intervals $L_\alpha \tilde{u} = [u_\alpha^1, u_\alpha^2]$.

Furthermore, by Theorem 1.1 of Goetschel and Voxman [4], we can identify a fuzzy number $\tilde{u}$ with the parametrized representation $\{(u_\alpha^1, u_\alpha^2) \,|\, 0 \leq \alpha \leq 1\}$ where u_α^1 and u_α^2 are considered as functions of $\alpha \in [0,1]$.

Now, we define the metric d_∞ on $F(R)$ by

$$d_\infty(\tilde{u}, \tilde{v}) = \sup_{0 \leq \alpha \leq 1} h(L_\alpha \tilde{u}, L_\alpha \tilde{v}) \tag{2.1}$$

where h is the Hausdorff metric defined as

$$h(L_\alpha \tilde{u}, L_\alpha \tilde{v}) = \max \, (|u_\alpha^1 - v_\alpha^1|, |u_\alpha^2 - v_\alpha^2|)$$

Also, the norm $\| \tilde{u} \|$ of fuzzy number $\tilde{u}$ will be defined as

$$\| \tilde{u} \| = d(\tilde{u}, \tilde{0}) = \max \, (|u_0^1|, |u_0^2|)$$

Then it is well-known that $F(R)$ is complete but nonseparable with respect to the metric d_∞. Joo and Kim [6] introduced a metric d_s on $F(R)$ which makes it a separable metric space as follows;

Definition 2.1. Let T be the class of strictly increasing, continuous mapping of $[0,1]$ onto itself. For $\tilde{u}, \tilde{v} \in F(R)$, we define

$$d_s(\tilde{u}, \tilde{v}) = \inf\{\varepsilon > 0 \, ; \text{ there exists a } t \in T \text{ such that } \sup_{0 \leq \alpha \leq 1} |t(\alpha) - \alpha| \leq \varepsilon \text{ and } d_\infty(\tilde{u}, t \circ \tilde{v}) \leq \varepsilon\},$$

where $t \circ \tilde{v}$ denotes the composition of $\tilde{v}$ and t.

Then it follows immediately that d_s is a metric on $F(R)$ and $d_s(\tilde{u},\tilde{v}) \le d(\tilde{u},\tilde{v})$. The metric d_s will be called the Skorokhod metric. Note that a sequence $\{\tilde{u}_n\}$ in $F(R)$ converges to a limit $\tilde{u}$ in the metric d_s if and only if there exists a sequence of functions $\{t_n\}$ in T such that

$$\lim_{n\to\infty} t_n(\alpha) = \alpha \text{ uniformly in } \alpha, \text{ and } \lim_{n\to\infty} d_\infty(t_n(\tilde{u}_n), \tilde{u}) = 0.$$

If $d_\infty(\tilde{u}_n, \tilde{u}) \to 0$, then $d_s(\tilde{u}_n, \tilde{u}) \to 0$. But the converse is not true.

Now we define, for $\tilde{u} \in F(R)$ and $0 < \delta < 1$, $0 \le \alpha < \beta \le 1$,

$$\begin{aligned} w_{\tilde{u}}(\alpha,\beta) &= h(L_{\alpha^+}\tilde{u}, L_\beta \tilde{u}) \\ &= \max(u^1_\beta - u^1_{\alpha^+}, u^2_{\alpha^+} - u^2_\beta), \end{aligned} \tag{2.2}$$

where $L_{\alpha^+}\tilde{u}$ denotes the closed interval $[u^1_{\alpha^+}, u^2_{\alpha^+}]$ with convention $u^i_{\alpha^+}$ the right-limit of u^i at α. If we define

$$w'(\tilde{u},\delta) = \inf \max_{1\le i\le r} w_{\tilde{u}}(\alpha_{i-1}, \alpha_i), \tag{2.3}$$

where the infimum is taken over all partitions $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ of $[0,1]$ satisfying $\alpha_i - \alpha_{i-1} > \delta$ for all i, then Lemma 3.2 of Joo and Kim [6] implies that

$$\lim_{\delta\to 0} w'(\tilde{u},\delta) = 0 \text{ for each } \tilde{u} \in F(R). \tag{2.4}$$

3. Main Results

Throughout this section, we assume that $F(R)$ is considered as the metric spaces endowed with the the Skorokhod metric d_s. Let (Ω, Σ, P) be a probability space. A function $\tilde{X} : \Omega \to F(R)$ is called a fuzzy random variable if it is measurable, i.e.

$$\tilde{X}^{-1}(B) = \{\omega \ in\ \Omega : \tilde{X}(\omega) \in B\} \in \Sigma \text{ for every } B \in B_s,$$

where B_s denotes the Borel σ-field of $F(R)$ generated by the metric d_s.

If we denote $\tilde{X}(\omega) = \{(X^1_\alpha(\omega), X^2_\alpha(\omega)) | 0 \le \alpha \le 1\}$, then it is well-known that $\tilde{X}$ is a fuzzy random variable if and only if for each $\alpha \in [0,1]$, X^1_α and X^2_α are random variables in the usual sense(See Kim [9]).

Since $F(R)$ is separable and topologically complete, we can apply the notion of tightness for random elements in a complete separable metric to the case of fuzzy random variables.

Definition 3.1. (1) Let $\tilde{X}_n$ be a sequence of fuzzy random variables. then $\{\tilde{X}_n\}$ is said to be tight if for every $\varepsilon > 0$, there exists a compact subset K of $F(R)$ such that

$$P(\tilde{X}_n \notin K) < \varepsilon \text{ for all } n.$$

We wish to characterize the tightness of fuzzy random variables. To this end, we need the characterization of compact subsets of $F(R)$.

Theorem 3.1. Let K be a subset of $F(R)$. Then K is relatively compact in the d_s-topology if and only if

$$\sup\{\|\tilde{u}\| : \tilde{u} \in K\} < \infty$$

and

$$\lim_{\delta \to 0} \sup\{ w'_{\tilde{u}}(\delta) : \tilde{u} \in K\} = 0.$$

Proof : See Ghil et. al. [4].

The main result is as follows;

Theorem 3.2 $\{\tilde{X}_n\}$ is tight if and only if

(1) For each $\eta > 0$, there exists a $\lambda > 0$ such that for all n,

$$P(\{\omega : \|\tilde{X}_n(\omega)\| > \lambda\}) \leq \eta. \tag{3.1}$$

(2) For each $\varepsilon > 0$ and $\eta > 0$, there exists a $\delta \in (0,1)$ such that for all n,

$$P(\{\omega : w'(\tilde{X}_n(\omega), \delta) \geq \varepsilon\}) \leq \eta. \tag{3.2}$$

Proof. (Necessity) Suppose that $\tilde{X}_n$ is tight. For given $\varepsilon > 0$ and $\eta > 0$, there exists a compact subset K of $F(R)$ such that

$$P(\tilde{X}_n \notin K) < \eta \text{ for all } n.$$

By Theorem 3.1, we have that

$$K \subset \{\tilde{u} : \|\tilde{u}\| \leq \lambda\} \text{ for large enough } \lambda,$$

and

$$K \subset \{\tilde{u} : w'(\tilde{u}, \delta) < \varepsilon\} \text{ for small enough } \delta.$$

Therefore, (1) and (2) follows.

(Sufficiency). Suppose that (1) and (2) hold. For given $\eta > 0$, we choose $\lambda > 0$ so that

$$P(\{\omega : \|\tilde{X}_n(\omega)\| > \lambda\}) \leq \frac{\eta}{2} \text{ for all } n.$$

Then for each k, we choose δ_k so that for all n,

$$P\left\{\omega : w'(\tilde{X}_n(\omega), \delta_k) \geq \frac{1}{k}\right\} \leq \frac{\eta}{2^{k+1}} \text{ for all } n.$$

Let $A=\{\tilde{u}: \|\tilde{u}\| \leq \lambda\}$ and $A_k=\left\{\tilde{u}: w'(\tilde{u}, \delta_k) < \frac{1}{k}\right\}$. If K is the closure of $A \cap (\bigcap_{k=1}^{\infty} A_k)$, then K is compact by Theorem 3.1. Since

$$P(\tilde{X}_n \notin K) \leq P(\tilde{X}_n \notin A) + \sum_{k=1}^{\infty} P(\tilde{X}_n \notin A_k) < \eta$$

for all n, we conclude that $\tilde{X}_n$ is tight.

Since $F(R)$ is separable and topologically complete, a single fuzzy random variable $\tilde{X}$ is tight. By the above theorem, for given $\varepsilon > 0$ and $\eta > 0$, there exist a $\lambda \in (0,1)$ and $\delta \in (0,1)$ such that

$$P(\{\omega: \|\tilde{X}(\omega)\| > \lambda\}) \leq \eta,$$

and

$$P(\{\omega: w'(\tilde{X}(\omega), \delta) \geq \varepsilon\}) \leq \eta.$$

Thus, if (3.1) and (3.2) are satisfied except for infinitely many n, we may ensure that (3.1) and (3.2) hold for all n by increasing λ and decreasing δ if necessary.
Therefore, we have the modified form of Theorem 3.2.

Corollary 3.1. $\{\tilde{X}_n\}$ is tight if and only if

(1) $\lim_{\lambda \to \infty} \limsup_{n \to \infty} \mathrm{P}(\{\omega: \|\tilde{X}_n(\omega)\| > \lambda\}) = 0.$

(2} For each $\varepsilon > 0$,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathrm{P}(\{\omega: w'(\tilde{X}_n(\omega), \delta) \geq \varepsilon\}) = 0.$$

Now let $j(\tilde{u}) = \sup_\alpha j_{\tilde{u}}(\alpha)$. Then by Lemma 3.2 of Joo and Kim [6], $j(\tilde{u})$ is well-defined. Since $\|\tilde{u}\|$ is controlled by $\|\max(|u_1^1|, |u_1^2|)\|$ and $j(\tilde{u})$, we have alternative form of corollary 3.1.

Corollary 3.2. The following two conditions can be substituted for (1) in corollary 3.1:

(1′) $\lim_{\lambda \to \infty} \limsup_{n \to \infty} \mathrm{P}(\{\omega: \max(|X_{n1}^1(\omega)|, |X_{n1}^2(\omega)|) > \lambda\}) = 0.$

(1″) $\lim_{\lambda \to \infty} \limsup_{n \to \infty} \mathrm{P}(\{\omega: j(\tilde{X}_{n(\omega)}) > \lambda\}) = 0.$

Proof. Since $\|L_1\tilde{X}_n(\omega)\| \leq \|\tilde{X}_n(\omega)\|$ and $j(\tilde{X}_n(\omega)) \leq \|\tilde{X}_n(\omega)\|$, it follows that (1) implies (1′) and (1″).

Assume that $(1'), (1'')$ and (2) of Corollary 3.1 hold. Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ be a partition of [0,1] satisfying $\min_i(\alpha_i - \alpha_{i-1}) > \delta$ such that

$$w_{\tilde{u}}(\alpha_{i-1}, \alpha_i] < w'(\tilde{u}, \delta) + 1 \quad \text{for all } i = 1, \cdots, r.$$

Then $h(L_{\alpha_i}\tilde{u}, L_{\alpha_{i-1}}\tilde{u}) < w'_{\tilde{u}}(\delta) + 1 + j(\tilde{u})$, and since $\delta r \leq 1$, we have

$$\begin{aligned} \|\tilde{u}\| = \max(|u_0^1|, |u_0^2|) &\leq \|\max(|u_1^1|, |u_1^2|)\| + r(w'(\tilde{u}, \delta) + 1 + j(\tilde{u})) \\ &\leq \|\max(|u_1^1|, |u_1^2|)\| + \frac{1}{\delta}(w'(\tilde{u}, \delta) + 1 + j(\tilde{u})). \end{aligned}$$

Therefore, (1) of Corollary 3.1 follows from $(1'), (1''), (2)$ and the inequality

$$\begin{aligned} &P(\{\omega: \|\tilde{X}_n(\omega)\| > 2\lambda\}) \\ &\leq P(\{\omega: \max(|X^1_{n1}(\omega)|, |X^2_{n1}(\omega)|) > \lambda\}) + P(\{\omega: w'(\tilde{X}_n(\omega), \delta) + 1 + j(\tilde{X}_n(\omega)) > \lambda\delta\}) \\ &\leq P(\{\omega: \max(|X^1_{n1}(\omega)|, |X^2_{n1}(\omega)|) > \lambda\}) + P(\{\omega: w'(\tilde{X}_n(\omega), \delta) \geq 1\}) \\ &\qquad + P(\{\omega: j(\tilde{X}_n(\omega)) > \lambda\delta - 2\}). \end{aligned}$$

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