■ 論 文 ■

A new approach on Traffic Flow model using Random Trajectory Theory

확률경로 기반의 교통류 분석 방법론

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목 차

Key Words : Random Trajectories, Load Traffic Density, Instantaneous Traffic Flow, Velocity Field, Travel Time, Stochastic Differential Equation

요 약

교통량, 교통밀도, 교통류 속도 등, 교통류 변수에 대한 현재까지의 불확실한 정의와 연속적 파동방정식의 거시적 교통류 해석 상의 문제점을 지적하고 이를 개선하기 위해 교통류 변수들에 대한 새로운 확률적 정의를 제시하고 이들의 성격을 규명하였다. 이러한 새로운 교통류 변수들에 대한 새로운 정의를 바탕으로 미시적 운전자 행동을 세밀하게 수용할 수 있고 많은 교통환경에서 연속적 파동 방정식을 대체하여 교통류 변수들과 통행시간을 예측할 수 있는 미분방정식 체계를 확률 미분방적식을 이용하여 도출하였다. 도출된 미분 방정식을 단일 차량의 시공 괘적에 적용해 보았다.

This research has been partially performeded by the Advanced Highway Research Center funded by the Korea Ministry of Science and Technology, Korea Science and Engineering Foundation.

I. The Problem Statement

The Lighthill-Whitham(1955) and Richards(1956) proposed the kinematic wave equation of continuum. If we apply this to the traffic flow on a homogeneous road section, we have

$$\frac{\partial k(x,t)}{\partial t}+\frac{dq(k)}{dk}\frac{\partial k(x,t)}{\partial x}=0 \quad (1)$$

where $k(x, t)$, $\nu(x, t)$ and $q(x, t)$ are density, velocity, and flow(or volume) at location x and time t and $\frac{dq(k)}{dk}=\frac{dv(k(x,t))}{dk}k(x,t)+v(k(x,t))$. The equation(1) was constructed with the flow conservation rule,

$$\frac{\partial k(x,t)}{\partial t}+\frac{\partial q(x,t)}{\partial x}=0 \quad (2)$$

and the postulate that there exists some functional relation among flow, velocity, and density, the fundamental equation,

$$q(x,t)=k(x,t)\,v(x,t) \quad (3)$$

and the velocity is a function of density alone, given by,

$$v(x,t)=v(k(x,t)). \quad (4)$$

This partial differential equation(1) describes the evolution of density. With appropriate initial and/or boundary conditions, we may obtain the solution $k(x,t)$ of the equation(1) through the method of characteristics(Haberman, 1977). The modern macroscopic traffic flow theory adopts the wave equation of continuum as its theoretical foundation. However, since the flow of vehicles is not continuous and the motion of vehicles does not follow the same deterministic physical rule as the flow of continuum, it fails to provide the rigorous mathematical foundation for key features of traffic flows, such as local traffic density, instantaneous traffic flow, velocity field, and, in consequence, the functional relation among them, such as the equation(3), and the conservation rule, the equation(2).

In most of traditional approaches(Daganzo, 1997, May, 1990, Leutzbach, 1987, Haberman, 1977, and Newell, 1993), in defining macroscopic traffic flow variables, the location x and by a moment t, $N(x,t)$, is defined first, as a function of x and t. Then they define local traffic density $k(x,t)$ as a partial derivative of $N(x,t)$ with respect to x and instantaneous traffic flow $q(x,t)$ as a partial derivative of $N(x,t)$ with respect to t. However, since $N(x,t)$ is, in fact, a discontinuous surface, we cannot define $k(x,t)$ and $q(x,t)$ in a rigorous manner. Moreover, mainly due to discreteness of traffic flows, the traffic velocity field, $v(x,t)$, can not be properly defined in any manner. Therefore, the "smooth" conservation equation(2) and the equation(3) do not have its own mathematical foundation. In addition, the assumption (4) could be validated theoretically in traffic environment only through the finding(Gazis, 1959) of the liaison between the third GM car following microscopic model and Greenberg macroscopic model. The third GM car following microscopic model implies that, in a unidirectional single lane road, the velocity of a following vehicle is assumed to be determined deterministic and completely by the space headway to the lead vehicle. It leads to Greenberg model that is one particular form of (4). Hence, even though each individual driver's choice of velocity is quite random and, in reality, lane changing and overtaking occur occasionally, the current macroscopic traffic flow theory has no room for accommodating these various microscopic aspects of individual vehicle movement. And since the flow conservation equation(2) cannot be transformed into the wave equation(1) without the postulate (4), we may suspect the validity of application of wave equation to the traffic environments.

In this paper, we consider the probability space of a finite sequence of random trajectories, $\{\bar{a}(t); 0 \le t \le T\}$ whose realization is a finite sequence of non-decreasing continuous functions $a_i(t)$, $i=1,2,3,\cdots\cdots,N$ on $t \in [0, T]$, which may imply the trajectories of N vehicles on a homogeneous road section of $[0, E]$ during T units of time. With these random trajectories of vehicles, $\bar{a}(t)$, we define the expected cumulative number of vehicles′ having passed the location x by time t with initial condition $\bar{a}(s)=\bar{a}$, $u(x,t,s,\bar{a})$, which is smooth enough with respect to x and t. We redefine the local density $k(x,t,s,\bar{a})$ and instantaneous flow $q(x,t,s,\bar{a})$ as partial derivative of $u(x,t,s,\bar{a})$ with respect to x and t and investigate the nature of $k(x,t,s,\bar{a})$, $q(x,t,s,\bar{a})$, $v(x,t,s,\bar{a})$ and their relation carefully. In particular, we model microscopic car maneuvering behavior of drivers in a platoon into a finite dimensional system of stochastic differential equations, which generates a probability space of a sequence of random trajectories. The realization of each random trajectory is a continuous function on (x,t) domain. Using some results of stochastic calculus, we derive the partial differential equations for the conditional expected cumulative plot $u(x,t,s,\bar{a})$ which might replace the kinematic wave equation effectively for the better prediction of local traffic density, $k(x,t,s,\bar{a})$, instantaneous traffic flow, $q(x,t,s,\bar{a})$. We also derive a partial differential equation with appropriate boundary condition that leads to the estimation of the expectation of travel time of each vehicle to a certain location.

II. Random Trajectories, Local Traffic Density, Instantaneous Traffic Flow and Velocity Field

Consider a probability space of a sequence of random trajectories on a finite (x,t) domain whose realization is a set of sequences of non-decreasing differentiable functions $a_i(t)'s$, $i=1,2,3,\cdots\cdots,N$ on $t \in [0, T]$, which may imply the trajectories of vehicles on a road section of $[0, E]$ during T units of time. We define the cumulative plot which is the cumulative number of vehicles that are beyond a certain location x on the road section at time t, $A(x, \bar{a}(t))$:

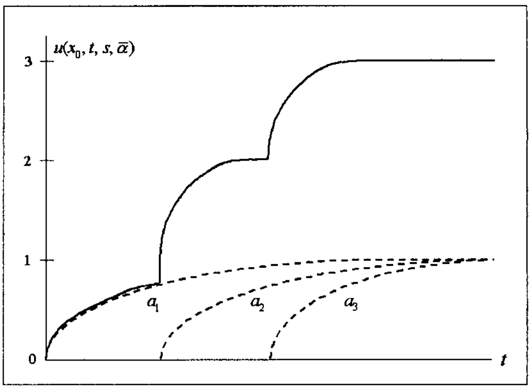
$$A(x,\bar{a}) = \sum_{i=1}^{N} \chi_{[x,\infty)}(a_i) \qquad (5)$$

where $\bar{a}$ a vector in R^N and $\chi_{[x,\infty)}(a)$ is a characteristic function, i.e. for real numbers a, $\chi_{[x,\infty)}(a)=1$ if $a \in [x,\infty)$, or $=0$ otherwise. We can define the expected cumulative plot, $u(x,t,s,\bar{a})$, for all x and t as follows :

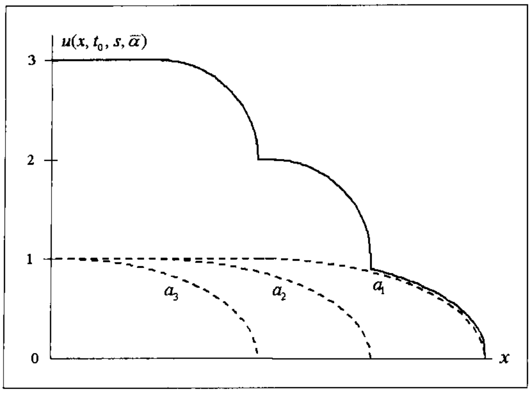
$$\begin{aligned} u,(x,t,s,\bar{a}) &= E[A(x,\bar{a}(t)) \mid \bar{a}(s)=\bar{a}] \\ &= \int A(x,\bar{a}) \Pr[\bar{a}(t) \in d\,\bar{a} \mid \bar{a}(s)=\bar{a}] \end{aligned} \qquad (6)$$

Then $u(x,t,s,\bar{a})$ can be interpreted as the summation of the conditional probability of each vehicle′s being beyond the location x at time t, given that $\bar{a}(s)=\bar{a}$:

$$\begin{aligned} u(x,t,s,\bar{a}) &= E[A(x,\bar{a}(t)) \mid \bar{a}(s)=\bar{a}] \\ &= E\Big[\sum_{i=1}^{N} \chi_{[x,\infty)}(a_i(t)) \mid \bar{a}(s)=\bar{a}\Big] \\ &= \sum_{i=1}^{N} \Pr[a_i(t) > x \mid \bar{a}(s)=\bar{a}] \end{aligned} \qquad (7)$$



〈Figure 1〉 The Graph of $u(x_0, t, s, \bar{a})$ and $\Pr[\bar{a}_i(t) > x_0 \mid \bar{a}(s)=\bar{a}]$, $i=1,2,3$



〈Figure 2〉 The Graph of $u(x, t_0, s, \bar{a})$ and $\Pr[a_i(t_0) > x \mid \bar{a}(s)=\bar{a}]$, $i=1,2,3$

$u(x, t, s, \bar{a})$ and its evolution with respect to time and location are represented in 〈Figure 1〉 and 〈Figure 2〉, respectively. $u(x, t, s, \bar{a})$ may be possibly assumed to be sufficiently smooth with respect to x, t and a_i, $i=1,2,3,\cdots\cdots,N$.

1. Novel definitions of instantaneous traffic flow, instantaneous traffic density and their nature

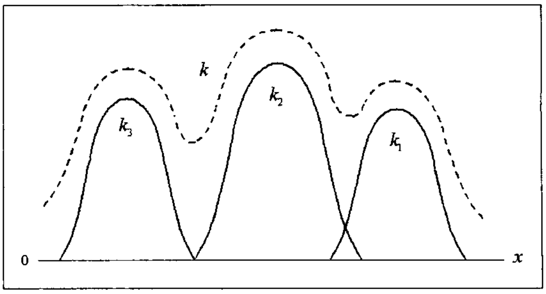
We may redefine instantaneous traffic flow, $q(x,t,s,\bar{a})$, and local traffic density, $k(x,t,s,\bar{a})$, at a specific location x and at a specific moment t as a partial derivative of $u(x,t,s,\bar{a})$ with respect to x and t :,

$$q(x,t,s,\bar{a}) = \frac{\partial u(x,t,s,\bar{a})}{\partial t} \quad (8)$$

$$k(x,t,s,\bar{a}) = \frac{\partial u(x,t,s,\bar{a})}{\partial x} \quad (9)$$

Moreover, the traffic density $k(x,t,s,\bar{a})$ has a quite natural interpretation as the summation of density functions of each vehicle's location at time t, which is a random variable:

$$k(x,t,s,\bar{a}) = -\frac{\partial u(x,t,s,\bar{a})}{\partial x}$$



〈Figure 3〉 The Graph of $k(x, t_0, s, \bar{a})$ and $k_i(x, t_0, s, \bar{a})$, $i=1,2,3$

$$\begin{aligned}
&= -\frac{\partial}{\partial x}\sum_{i=1}^{N}\Pr[a_i(t) > x \mid \bar{a}(s)=\bar{a}] \\
&= -\frac{\partial}{\partial x}\sum_{i}^{N}(1-\Pr[a_i(t) < x \mid \bar{a}(s)=\bar{a}]) \\
&= \frac{\partial}{\partial x}\sum_{i}^{N}F_{a_i(t)\mid\bar{a}(s)=\bar{a}}(x) \\
&= \sum_{i}^{N} f_{a_i(t)\mid\bar{a}(s)=\bar{a}}(x) \\
&\equiv \sum_{i=1}^{N} k_i(x,t,s,\bar{a}) \qquad (10)
\end{aligned}$$

where $F_{a_i(t)\mid\bar{a}(s)=\bar{a}}(x)$ and $f_{a_i(t)\mid\bar{a}(s)=\bar{a}}(x) \equiv k_i(x, t, s, \bar{a})$ are conditional cumulative distribution function and its density function of i vehicle's location at any given time t, $a_i(t)$. Note that $k_i(x,s,s,\bar{a})$ is a dirac delta function, centered at $x=a_i$. That is, at the beginning $t=s$, each $k_i(x,s,s,\bar{a})$ is a point mass at $x=a_i$ and, after (t_0-s) time units, each $a_i(t_0)$ is distributed with its density function, $k_i(x, t_0, s, \bar{a})$. The graphical representation of $k(x,t_0,s,\bar{a})$, $k_i(x,t_0,s,\bar{a})$, $i=1,2,3$, is in 〈Figure 3〉.

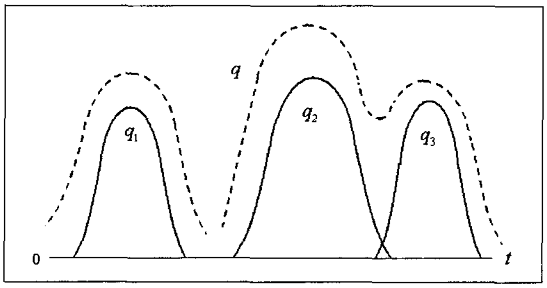
The instantaneous traffic flow $q(x,t,s,\bar{a})$ can be interpreted as traffic flow as the summation of conditional density functions of vehicle i's first passage time to a location x, given that $\bar{a}(s)=\bar{a}$, $\tau_i(x) = \inf\{t \geq s \mid a_i(t)=x, \bar{a}(s)=\bar{a}\}$:

$$\begin{aligned}
q(x,t,s,\bar{a}) &= \frac{\partial}{\partial t}\, u(x,t,s,\bar{a}) \\
&= \frac{\partial}{\partial t}\sum_{i=1}^{N}\Pr[a_i(t) > x \mid \bar{a}(s)=\bar{a}] \\
&= \frac{\partial}{\partial t}\sum_{i=1}^{N}\Pr[\tau_i(x) \leq t \mid \bar{a}(s)=\bar{a}]
\end{aligned}$$

since we assume that $a_i(t)$s' are non-decreasing functions. Then we have

$$= \frac{\partial}{\partial t}\sum_i^N F_{\tau_i(x)\mid \bar{a}(s)=\bar{a}}(t)$$
$$= \sum_i^N f_{\tau_i(x)\mid \bar{a}(s)=\bar{a}}(t)$$
$$\equiv \sum_{i=1}^N q_i(x,t,s,\bar{a}) \qquad (11)$$

where $F_{\tau_i(x)\mid \bar{a}(s)=\bar{a}}(t)$ and $f_{\tau_i(x)\mid \bar{a}(s)=\bar{a}}(t) \equiv q_i(x,t,s,\bar{a})$ are the conditional cumulative distribution function and its density function of vehicle i's first passage time to x, given that $\bar{a}(s)=\bar{a}$. Note that $q_i(a_i,t,s,\bar{a})$ is a dirac delta function, contered at $t=s$. That is, at the initial moment $t=s$, each $q_i(a_i,t,s,\bar{a})$ is a point mass at $t=s$ and, for some $x_0 > a_i$, the first passage time of vehicle i to x_0 is distributed with its density function, $q_i(x,t_0,s,\bar{a})$. The graphical representation of $q(x_0,t,s,\bar{a})$, $q_i(x_0,t,s,\bar{a})$, $i=1,2,3$ is in ⟨Figure 4⟩.



⟨Figure 4⟩ The Graph of $q(x_0,t,s,\bar{a})$ and $q_i(x_0,t,s,\bar{a})$, $i=1,2,3$

2. Compatibility to traditional definitions

The novel definitions of instantaneous traffic flow and local traffic density are compatible to the traditional ones in the sense that the integral of $k(x,t,s,\bar{a})$ over the space $[x_1,x_2]$ at a fixed t and the integral of $q(x,t,s,\bar{a})$ over the time interval $[t_1,t_2]$ at a fixed location x are the expected number of vehicles over the space $[x_1,x_2]$ at time t and the expected number of vehicles having passed the location x from t_1 to t_2, respectively.

$$\int_{x_1}^{x_2} k(x_1,t,s,\bar{a})dx$$
$$= u(x_1,t,s,\bar{a}) - u(x_2,t,s,\bar{a})$$
$$= E[A(x_1,\bar{a}(t)) \mid \bar{a}(s)=\bar{a}]$$
$$- E[A(x_2,\bar{a}(t)) \mid \bar{a}(s)=\bar{a}]$$
$$= E\Big[\sum_{i=1}^N \chi_{[x_1,x_2)}(a_i(t)) \mid \bar{a}(s)=\bar{a}\Big]$$

$$\int_{t_1}^{t_2} q(x,t,s,\bar{a})dt$$
$$= u(x,t_2,s,\bar{a}) - u(x,t_1,s,\bar{a})$$
$$= E[A(x,\bar{a}(t_2)) \mid \bar{a}(s)=\bar{a}]$$
$$- E[A(x,\bar{a}(t_1)) \mid \bar{a}(s)=\bar{a}]$$
$$= E[A(x,\bar{a}(t_2)) - A(x,\bar{a}(t_1)) \mid \bar{a}(s)=\bar{a}]$$

As long as the two cross partial derivatives of $u(x,t,s,\bar{a})$ are continuous, we can show $k(x,t,s,\bar{a})$ and $q(x,t,s,\bar{a})$ satisfy the flow conservation equation by Young's theorem(A similar argument can be found in p.283, Newell(1933)). That is, $\frac{\partial^2 u(x,t,s,\bar{a})}{\partial t \partial x} = \frac{\partial^2 u(x,t,s,\bar{a})}{\partial x \partial t}$ which leads to the flow conservation equation(2).

3. The stochastic nature of velocity field

With the above definitions of $u(x,t,s,\bar{a})$, $k(x,t,s,\bar{a})$ and $q(x,t,s,\bar{a})$, we may treat the traffic flow as the flow of continuum in a natural manner. Now, we have to answer to the questions "What is the velocity field in this novel stochastic sense?" and "What type of relation does exist between motion of each individual vehicle and the velocity field?".

It is quite natural to define the velocity field of the traffic flow as in the following manner:,

$$v(x,t,s,\bar{a}) \equiv \sum_{i=1}^n v_i(x,t,s,\bar{a})\frac{k_i(x,t,s,\bar{a})}{k(x,t,s,\bar{a})} \qquad (12)$$

where $v_i(x,t,s,\bar{a})$ is the conditional expectation of vehicle i's velocity, given that $a_i(t)=x$ and

$\bar{a}(s)=\bar{a}$, in the following sense :

$$v_i(x,t,s,\bar{a}) \equiv E\left[\frac{da_i(t)}{dt} \,\middle|\, a_i(t)=x,\ \bar{a}(s)=\bar{a}\right]$$

where $\frac{da_i(t)}{dt}$ stands for vehicle i's velocity at time t which is a random variable and whose possible values are the set of derivatives of all possible trajectories at time t. It implies that $v(x,t,s,\bar{a})$ is defined to be the weighted average of $v_i(x,t,s,\bar{a})$ and where the weight for each $v(x,t,s,\bar{a})$ is the ratio of $k_i(x,t,s,\bar{a})$ to $\sum_{i=}^{N} k_i(x,t,s,\bar{a})$. More specifically, the ratio may be considered as the likelihood ratio of vehicle i's being on x at time t to the summation of likelihood of the all other vehicle's being on x at time t. It could be shown that the fundamental equation(3) is still valid with the stochastic definition of $k(x,t,s,\bar{a})$, $q(x,t,s,\bar{a})$, and $v(x,t,s,\bar{a})$, *i. e.* $q(x,t,s,\bar{a})=k(x,t,s,\bar{a})\,v(x,t,s,\bar{a})$.

The proof is as follows(The main idea of this proof is similar to the one in p.74, Leutzbach(1987)) :

We could claim the following equality, ,

$$\begin{aligned}&\Pr[a_i(s')=x \text{ for } s'\in[t,t+dt) \\ &\quad \text{and } V_i(x,t,s,\bar{a})\in dv \mid \bar{a}(s)=\bar{a}] \\ &\Pr[a_i(t)\in[x,x+V_i(x,t,s,\bar{a})dt] \\ &\quad \text{and } V_i(x,t,s,\bar{a})\in dv \mid \bar{a}(s)=\bar{a}] \end{aligned} \quad (13)$$

where $V_i(x,t,s,\bar{a})$ is a random variable for each x and t, whose probability measure is defined to be.

$$dF_{x,t}(v)=\Pr\left[\frac{da_i(t)}{dt}\in dv \,\middle|\, a_i(t)=x,\ \bar{a}(s)=\bar{a}\right]$$

The first term of equality (13) is the probability that the vehicle i appears at a location x during the time interval $[t,t+dt)$, having speed $V_i(x,t,s,\bar{a})$, and the right-side term is the probability that the vehicle is located in the distance interval $[x,\ x+V_i(x,t,s,\bar{a})dt)$ at time t, having speed $V_i(x,t,s,\bar{a})$.

By definition of $q_i(x,t,s,\bar{a})$ and $k_i(x,t,s,\bar{a})$ in (8) and (9),

$$\begin{aligned}&\Pr[a_i(s')=x \text{ for some } s'\in[t,t+dt) \\ &\quad \text{and } V_i(x,t,s,\bar{a})\in dv \mid \bar{a}(s)=\bar{a}] \\ &\quad = q_i(x,t,s,\bar{a})dt\ dF_{x,t}(v) \text{ and ;} \\ &\Pr[a_i(t)\in[x,x+V_i(x,t,s,\bar{a})dt) \\ &\quad \text{and } V_i(x,t,s,\bar{a})\in dv \mid \bar{a}(s)=\bar{a}] \\ &\quad = k_i(x,t,s,\bar{a})v\,dt\ dF_{x,t}(v)\end{aligned}$$

Integrating both terms with respect to, we obtain the following equality :

$$\begin{aligned}&q_i(x,t,s,\bar{a}) \\ &= k_i(x,t,s,\bar{a})E[V_i(x,t,s,\bar{a})] \\ &= k_i(x,t,s,\bar{a}) \\ &\quad E\left[\frac{da_i(t)}{dt}\in dv \,\middle|\, a_i(t)=x,\ \bar{a}(s)=\bar{a}\right] \\ &= k_i(x,t,s,\bar{a})v_i(x,t,s,\bar{a})\end{aligned} \quad (14)$$

By summation (14) over each vehicle, the fundamental equation(3) can be obtained by the definition (12).

4. A simple model

Consider a probability space that is generated by two random trajectories, $a_i(t)=\alpha_i+\beta_i t, i=1,2$ $t\in[0,\infty)$ where α_i, β_i are independent uniform random variables on $[0,1]$. It leads to the independence of $a_i(t)$s. Suppose that $a_i(0)=\alpha_i^0$ is initially known for each i. Then we may derive the expected cumulative plot, $u(x,t,0,\bar{a}^0)$, $u(x,t,0,\bar{a}^0)$, as follows :

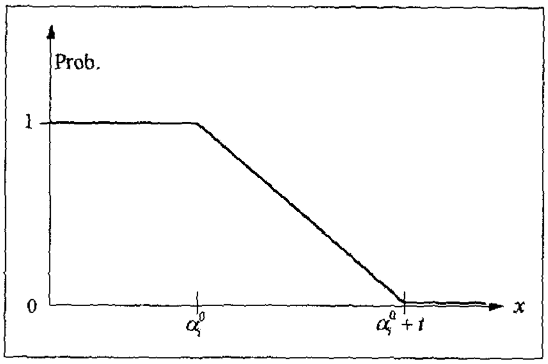
$$\begin{aligned}u(x,t,0,\bar{a}^0) &= \sum_{i=1}^{2}\Pr[a_i(t)>x \mid \bar{a}(0)=\bar{a}^0] \\ &= \sum_{i=1}^{2}\Pr[a_i(t)>x \mid \bar{a}_i(0)=\alpha_i^0]\end{aligned}$$

since $a_i(s)$s are independent. For each i,

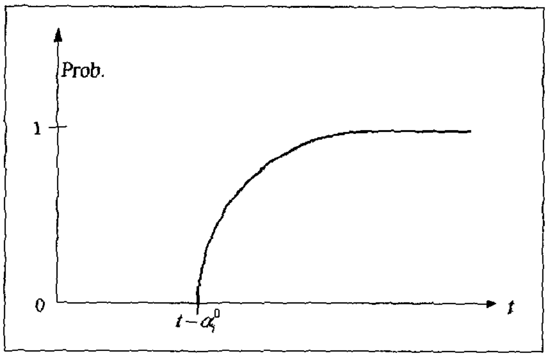
$$\Pr[a_i(t) > x \mid a_i(0) = \alpha_i^0] = \Pr[\alpha_i^0 + \beta_i t > x] = \Pr\left[\beta_i > \frac{x-\alpha_i^0}{t}\right] = \begin{cases} 0, & \alpha_i^0 + t \le x \\ 1 - \frac{x-\alpha_i^0}{t}, & \alpha_i^0 \le x \le \alpha_i^0 + t \\ 1, & x < \alpha_i^0 \end{cases} \quad (15)$$

For a fixed t, we may view it with respect to x as in ⟨Figure 5⟩. For a fixed x, (1) is represented with respect to t as in (16) and graphically in ⟨Figure 6⟩.

$$= \begin{cases} 1 - \frac{x-\alpha_i^0}{t}, & \text{if } x - \alpha_i^0 \le t \\ 0, & x - \alpha_i^0 > t \end{cases} \quad (16)$$

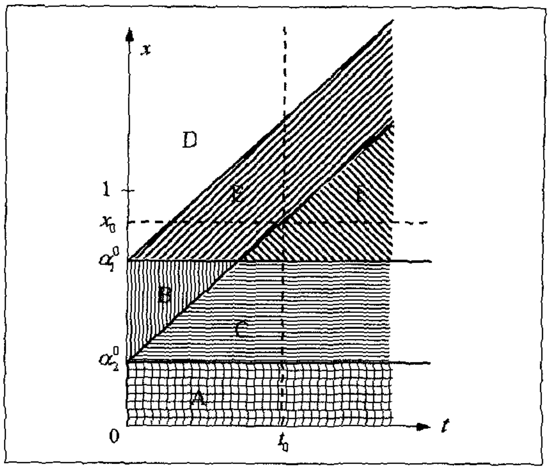


⟨Figure 5⟩ The Graph of $\Pr[a_i(t) > x \mid a_i(0) = \alpha_i^0]$ on x



⟨Figure 6⟩ The Graph of $\Pr[a_i(t) > x \mid a_i(0) = \alpha_i^0]$ on t

Without loss of generality, we assume that $\alpha_1^0 > \alpha_2^0$. Then we can calculate the expected cumulative plot $u(x, t, 0, \overline{\alpha}^0)$ on separated areas as in ⟨Figure 7⟩.

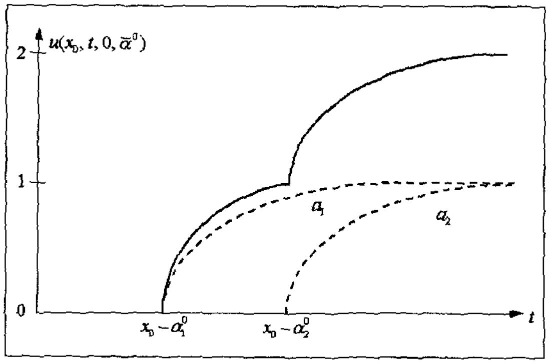


⟨Figure 7⟩ The Separated Areas

$$u(x, t, 0, \overline{\alpha}^0) = \begin{cases} 2 & on\ A \\ 1 & on\ B \\ 0 & on\ D \\ 2 - \frac{x-\alpha_2^0}{t} & on\ C \\ 1 - \frac{x-\alpha_1^0}{t} & on\ E \\ 2 - \frac{x-\alpha_2^0}{t} - \frac{x-\alpha_1^0}{t} & on\ F \end{cases} \quad (17)$$

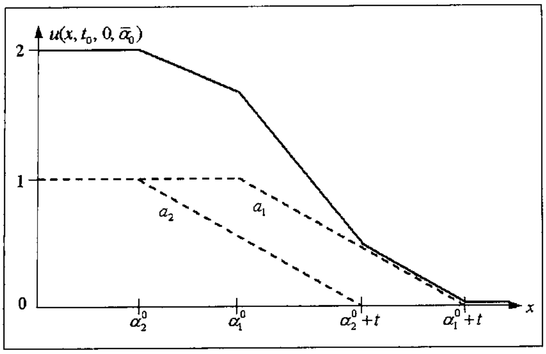
At $x = x_0$, $u(x_0, t, 0, \overline{\alpha}^0)$ can be graphically represented with respect to t by ⟨Figure 8⟩ and at $t = t_0$, $u(x, 0, \overline{\alpha}^0)$ can be graphically viewed as in ⟨Figure 9⟩.

By the definition (8) and (9), we may obtain $q(x, t, 0, \overline{\alpha}^0)$, $q(x, t, 0, \overline{\alpha}^0)$ and the ratio $\frac{q}{k}$ from (17) as in ⟨Table 1⟩. The $\frac{q}{k}$ ratio can be reproduced as v by definition (12) as in ⟨Table 2⟩. The ⟨Table 1⟩



⟨Figure 8⟩ The Graph of $u(x_0, t, 0, \overline{\alpha}^0)$ on t

and ⟨Table 2⟩ confirm the fundamental equation(3) in this simple model.



⟨Figure 9⟩ The Graph of $u(x,t,0,\bar{a}^0)$ on x

⟨Table 1⟩ The Values of $q(x,t,0,\bar{a}^0)$, $k(x,t,0,\bar{a}^0)$ and the Ratio $\frac{q}{k}$ in the simple model

Area	q	k	$\frac{q}{k}$
A, B and D	0	0	N/A
C	$\frac{x-a_2^0}{t^2}$	$\frac{1}{t}$	$\frac{x-a_2^0}{t}$
E	$\frac{x-a_1^0}{t^2}$	$\frac{1}{t}$	$\frac{x-a_1^0}{t}$
F	$\frac{2x-a_1^0-a_2^0}{t^2}$	$\frac{2}{t}$	$\frac{2x-a_1^0-a_2^0}{t}$

Ⅲ. System of Stochastic Differential Equations for Vehicle Maneuvering Behavior in a Platoon

In the previous sections, we have novel definitions of basic instantaneous traffic flow variables on a general probability space of a sequence of random trajectories. This part is to establish a model of traffic flow on this probability space which has a room for accommodating the randomness and microscopic maneuvering behavior of individual drivers.

We model microscopic car maneuvering behavior of drivers in a platoon into a finite dimensional system of stochastic differential equations in a general form. This model may generate a well-defined probability space of a sequence of random trajectories which implies trajectories of vehicles on a unidirectional finite road section for a certain finite period of time.

We import some stochastic calculus for the solution approaches in predicting the traffic flow variables of this model: by using some formulae, we derive some partial differential equations under appropriate initial and/or boundary condition for the expected cumulative plot and travel time.

1. The model

In this model, maneuvering behavior of a group of drivers, especially in their choice of velocity on a homogeneous unidirectional road section, is modeled into a system of stochastic differential equations as follows ;

$$d\,a_i(t) = g_i(\bar{a}(t))dt + \sigma_i(\bar{a}(t))d\,\varepsilon_i(t),\quad i=1,2,\cdots,N,$$
$$\bar{a}(t) = [a_1(t), a_2(t), \cdots\cdots a_N(t)],\quad 0 \le t \le T,\ 0 \le x \le E \qquad (18)$$

⟨Table 2⟩ The Values of k, k_i, $i=1,2$ and v, v_i, $i=1,2$

Area	k_1	k_2	k	v_1	v_2	v
A, B and D	0	0	0	N/A	N/A	N/A
C	0	$\frac{1}{t}$	$\frac{1}{t}$	N/A	$x-a_2^0$	$\frac{x-a_2^0}{t}$
E	$\frac{1}{t}$	0	$\frac{1}{t}$	$x-a_1^0$	N/A	$\frac{x-a_1^0}{t}$
F	$\frac{1}{t}$	$\frac{1}{t}$	$\frac{2}{t}$	$x-a_1^0$	$x-a_2^0$	$\frac{2x-a_1^0-a_2^0}{t}$

for a sufficiently large N, T, and E with a set of appropriate initial, boundary conditions, and/or a set of given deterministic trajectories $a_k(\cdot)$ for some k s among N vehicles. In the equation(18), each $\varepsilon_i(\cdot)$ is assumed to be a independent standard Brownian motion and $a_i(t)$ represents the position of vehicle i at time t. $g_i(a_1(t), a_2(t), \cdots\cdots, a_N(t))$ is the drift coefficient function which represents the average velocity of vehicle i at time t and $\sigma_i(a_1(t), a_2(t), \cdots\cdots, a_N(t))$ is the diffusion coefficient function which reflects the randomness of the actual velocity.

Let $g_i(a_1(t), a_2(t), \cdots\cdots, a_N(t)) = g(a_i(t), \overline{a(t)})$ and $\sigma_i(a_1(t), a_2(t), \cdots\cdots, a_N(t)) = \sigma(a_i(t), \overline{a}(t))$ where $g(l, a_1, a_2, \cdots\cdots, a_N)$ and $\sigma(l, a_1, a_2, \cdots\cdots, a_N)$ are decreasing functions as $\overline{a}$ gets denser around l in an appropriate sense. The system of the stochastic differential equations(18) implies that, in a homogeneous road section, as each driver in a group of vehicles feels denser traffic around his or her vehicle, he or she tends, in general, to decrease the speed of his or her vehicle and the degree of freedom in his or her choice of decrement(increment) in speed at time t is assumed to decrease(increase). But the degree of freedom in speed decrement (increment) at time t and location l is random and its distribution is determined by the term $\sigma(a_i(t), \overline{a}(t))d\varepsilon_i(t)$

For example, we consider the car - following situation along a unidirectional road single lane road in which $g(a_i(t), \overline{a}(t))$ is defined to be $g(a_i(t), \overline{a}(t)) = \alpha_0[\ln(a_{i-1}(t) - a_i(t))] + \alpha_0 \ln k_j$, where α_0 is the 'free-flow' speed and k_j is the jam density of the road. And we may possibly assume that $\sigma(a_i(t), \overline{a}(t))$ be a small fraction of $g(a_i(t), \overline{a}(t))$. For example, if a vehicle is supposed to run 60km per hour, then we may reasonably assume that the car would be between 750m and 1,250m after one minute with 95% probability. In the case, $\sigma(a_i(t), \overline{a}(t))$ is less than 2% of $g(a_i(t), \overline{a}(t))$ under some reasonable assumptions.

The probability space of random vehicle trajectories, $\overline{a}(\cdot)$, may be well-defined with the system of equations(18). That is, with appropriately defined $g_i(\cdot)$s and $\sigma_i(\cdot)$s (Malliars, 1984 and Schuss, 1980), we could construct the probability structure of the diffusion vector process, $\overline{a}(\cdot)$, with its transition probability function with given initial positions of vehicles, $\overline{a}$, at time s, $\Pr[\overline{a}(t) \in G \mid \overline{a}(s) = \overline{a}]$ for any Borel-measurable set $G \subset R^N$. To the contrary to the assumption of non-decreasing trajectories in II, it has been known that for a diffusion model like (18), the probability of non-decreasing trajectories is zero(See p.119. Schuss(1980)). However, in all road traffic situations, $\sigma(a_i(t), \overline{a}(t))$ is a small fraction of $g(a_i(t), \overline{a}(t))$. And, moreover, as time passes, the drift terms are increasing in the order of t and the diffusion terms are increasing in the order of $\sqrt{t}$. Hence, as long as we are interested in the traffic flow after at least couple of minutes from the initial moment, we don't need to concern about the probability of backward movement of trajectories. For example, if the average speed of the vehicle i is 60km per hour, we may possibly assume that after 1 minute its location is between 0.72km and 1.28km with probability 0.95. The corresponding diffusion coefficient is about 1.1. With this diffusion coefficient, it is not necessary to take the fact about "non-decreasing" into account. Therefore, we possibly assume that $\Pr$[vehicle positions at time $t \in G$ | initial vehicle positions $= \overline{a}$] can be approximated by $\Pr[\overline{a}(t) \in G \mid \overline{a}(s) = \overline{a}]$ for any Borel-measurable set $G \subset R^N$ in the rest of this paper.

2. A solution approach for expected cumulative plot, local traffic density, and instantaneous traffic flow

Let $f(\overline{x})$ denote a continuous real-valued bounded function and for $s < t$, t fixed, $\overline{x} \in R^N$. Let

$$\begin{aligned} u_f(t, s, \overline{a}) &= E[f(\overline{a}(t) \mid \overline{a}(s) = \overline{a}] \\ &= \int f(\overline{a}) \Pr[\overline{a}(t) \in d\overline{a} \mid \overline{a}(s) = \overline{a}] \end{aligned} \quad (19)$$

and if $u_f(t,s,\overline{a})$ has continuous bounded partials, $\frac{\partial u_f}{\partial x_i};\ \frac{\partial^2 u_f}{\partial x_i \partial x_j}$, $\forall i,\ j \in N$ then $u_f(t,s,\overline{a})$ is differentiable with respect to s and satisfies the following partial differential equation :

$$\frac{\partial u_f}{\partial s}(t,s,\overline{a}) + Mu_f(t,s,\overline{a}) = 0 \qquad (20)$$

where

$$Mu_f(t,s,\overline{a}) \equiv \sum_{i=1}^{N} g_i(\overline{a}) \frac{\partial u_f}{\partial a_i}(t,s,\overline{a}) + \frac{1}{2}\sum_{i=1}^{N} \sigma_i^2(\overline{a}) \frac{\partial^2 u_f}{\partial a_i^2}(t,s,\overline{a})$$

with boundary condition $u_f(t,s,\overline{a}) \to f(\overline{a})$ as $s \to t$. They call it "Kolmogorov's backward equation". (For the proof of this theorem see Gihman and Skorohod(1969)) However, since we have discontinuous $a(x,\overline{a}) \equiv \sum_{i=1}^{N} \chi_{[x,\infty 0)}(a_i)$ for $f(\overline{a})$ in (19), we can not apply equation(20) to find $u(x,t,s,\overline{a})$ directly. Instead, we propose a continuous approximation $A_\gamma(x,\overline{a})$ for $A(x,\overline{a})$ such that $E_{\overline{a}}[A_\gamma(x,\overline{a}) - A(x,\overline{a})] \to 0$ as $\gamma \to \infty$. Any function in the form of cumulative probability distribution function, $F_{x,\gamma}(a)$, symmetrically centered at x with sufficiently small variance, parameterized by a large γ, might be a good candidate for the replacement of the step function, $\chi_{[x,\infty]}(a)$. Let $A_\gamma(x,\overline{a}) \equiv \sum_{i=1}^{N} F_{x,\gamma}(a_i)$ and $u_\gamma(x,t,s,\overline{a}) = E[A_\gamma(x,\overline{a}(t)) | \overline{a}(s) = \overline{a}]$. Then, by (20), we may have the following partial differential equation for $u_\gamma(x,t,s,\overline{a})$ for each x and t :

$$\frac{\partial u_\gamma}{\partial t}(x,t,s,\overline{a}) + \sum_{i=1}^{N} g_i(\overline{a}) \frac{\partial}{\partial a_i} u_\gamma(x,t,s,\overline{a}) + \frac{1}{2}\sum_{i=1}^{N} \sigma_i^2(\overline{a}) \frac{\partial^2}{\partial a_i^2} u_\gamma(x,t,s,\overline{a}) = 0$$

with $u_\gamma(x,t,s,\overline{a}) \to A_\gamma(x,\overline{a})$ as $t \to s$. For a sufficiently large γ, we may obtain a good approximation $u_\gamma(x,t,s,\overline{a})$ for $u(x,t,s,\overline{a})$. With appropriately designed $g_i(\cdot)$ and $\sigma_i(\cdot)$ that describe each driver's driving behavior, we may find the solution $u_\gamma(x,t,s,\overline{a})$ for each x and t by numerical methods and, in turn, $k_\gamma(x,t,s,\overline{a})$ and $q_\gamma(x,t,s,\overline{a})$. Note that

$$u_\gamma(x,t,s,\overline{a}) = u_\gamma(y, t-s, 0, \overline{a}-x+y) \qquad (21)$$

for any y, since $g_i(\cdot)$ and $\sigma_i(\cdot)$ are function of the initial position, $\overline{a}$, only.

3. A solution approach for prediction of travel time

Let $\overline{a}(t)$ be the solution of (18) in an fixed open domain $\Omega \subset R^n$ and let $\tau_{\overline{a}} = \inf\{s \geq 0 | \overline{a}(s) \in \partial\Omega, \overline{a}(0) = \overline{a} \in \Omega]$ be the travel time of $\overline{a}(t)$ to the boundary of Ω, $\partial\Omega$, given that each vehicle's initial position is known, $\overline{a}(0) = \overline{a} \in \Omega$. We can find the expected travel time, $E[\tau_{\overline{a}}]$ as follows(cf. p.118, Schuss(10)). Let $l(\overline{x})$ be the solution of the following boundary value problem

$$Ml(\overline{x}) = -1, \text{ if } \overline{x} \in \Omega, \quad l(\overline{x}) = 0, \text{ if } \overline{x} \in \partial\Omega \qquad (22)$$

Then the expected travel time $E[\tau_{\overline{a}}]$ of $\overline{a}(t)$ to $\partial\Omega$ from the initial position $\overline{a}(0) = \overline{a}$ is equal to $l(\overline{a})$.

Consider the travel time of vehicle i to a certain location x_0, given that each vehicle's initial position is known, $\overline{a}(0) = \overline{a}$, $\tau^i_{\overline{a}} = \inf\{s \geq 0 | a_i(s) = x_0, \overline{a}(0) = \overline{a}\}$. Then, by letting $\Omega = \{\overline{x} \in R^n : x_i < x_0\}$, we may obtain the following ordinary differential equation for $l(\overline{x})$

$$\sum_{j=1}^{N} g_j(\overline{x}) \frac{\partial}{\partial x_j} l(\overline{x}) + \frac{1}{2}\sum_{j=1}^{N} \sigma_j^2(\overline{x}) \frac{\partial^2}{\partial x_j^2} l(\overline{x}) = -1, \text{ if } x_i < x_0, \ l(\overline{x}) = 0, \text{ if } x_i = x_0 \text{ and } x_i \to -\infty \qquad (23)$$

Then the expected travel time $E[\tau^i_{\overline{a}}]$ of $a_i(t)$ to x_0 from the initial position $\overline{a}(0) = \overline{a}$ is equal to $l(\overline{a})$.

4. An application to a single vehicle trajectory model

Consider a trajectory of a single vehicle that is represented by the following stochastic differential equation.

$$da(t) = \mu\, dt + \sigma d\varepsilon(t)$$
$$\text{and } a(s)=0,\ 0<s<t \qquad (24)$$

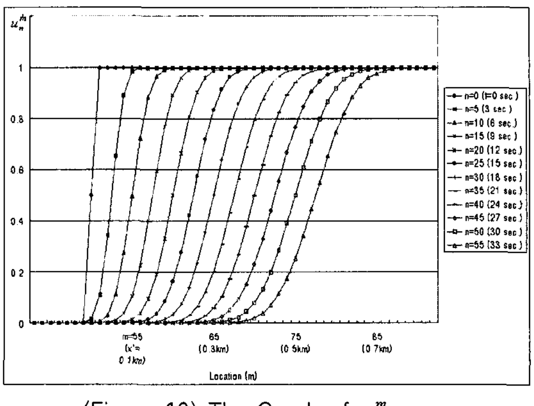
where μ and σ are positive constants and $\varepsilon(t)$ is a standard Brownian motion. It implies that the average velocity of this vehicle is μ km per hour and the expected value and variance of the vehicle location after t hours are μt and $\sigma^2 t$. In consequences, we are assuming that probability of the vehicle being between $\mu t - 2\sigma\sqrt{t}$ and $\mu t + 2\sigma\sqrt{t}$ is about 95%. Now, we want to figure out $u(x,t,s,a) = E[\chi_{x,\infty)}(a(t)) \mid a(s)=a]$ that is equal to $\Pr[a(t) > x \mid a(s)=a]$. Let $F_{x,\gamma}(a) = \dfrac{1}{1+\exp[-\gamma(a-x)]}$ that can be a continuous approximation of the characteristic function $\chi_{[x,\infty)}(a)$ for a sufficiently large γ. Let $\mu_\gamma(x,t,s,a) = E[F_{x,\gamma}(a(t)) \mid a(s)=a]$. Then, for a fixed x and a fixed t, Equation(19) holds, i.e.

$$\frac{\partial u_\gamma}{\partial t}(x,t,s,a) + \mu \frac{\partial}{\partial a} u_\gamma(x,t,s,a) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial a^2} u_\gamma(x,t,s,a) = 0$$
$$\text{with } u_\gamma(x,t,s,a) \rightarrow F_{x,\gamma}(a) \text{ as } s \rightarrow t. \qquad (25)$$

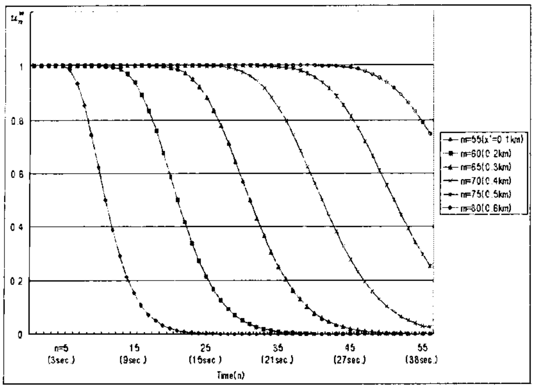
To obtain a numerical solution of Equation(25) with respect to s and a, we apply implicit finite difference method, using the backward-difference approximation for the $\frac{\partial u_\gamma}{\partial t}(x,t,s,a)$ term and the symmetric central-difference approximation for the $\frac{\partial u_\gamma}{\partial a}(x,t,s,a)$ term and the $\frac{\partial^2}{\partial a^2} u_\gamma(x,t,s,a)$ term. This leads to the following finite difference equation.

$$\frac{u_n^m - u_{n-1}^m}{\delta s} = -(1/2)\,\sigma^2 \frac{u_n^{m+1} - 2u_n^m + u_n^{m-1}}{(\sigma a)^2} - \mu \frac{u_n^{m+1} - u_n^{m-1}}{2(\delta a)} \qquad (26)$$

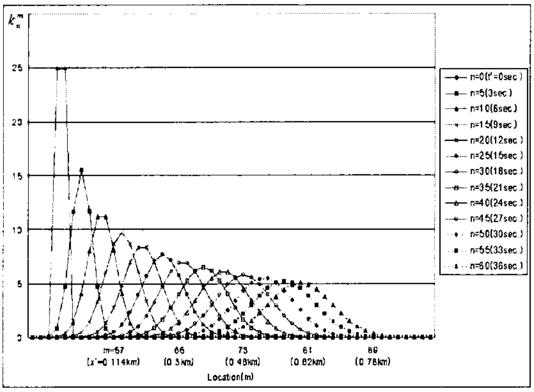
where $u_n^m = u_\gamma(x,t,s_n,a_m)$, $s_n = n\,\delta s$, $\delta s = \frac{t}{N}$, $n = 0,1,2,\cdots,N$, and $a_m = m\delta a$, $\delta a = \frac{x-a_o}{M}$ $m = 0,1,2,\cdots,M$. Equation(26) can be solved by backward substitution with the terminal condition $u_N^m = f_{x,\gamma}(a_m) = \dfrac{1}{1+\exp[-\gamma(a_m - x)]}$, $m = 0,1,2,\cdots,3,M$. By Equality (21), we know that $u_n^m = u_\gamma(x,t,s_n,a_m)$ is equal to $u_\gamma(x-a_m, t-s_n, 0,0) \equiv u_\gamma(x_m', t_n', 0,0)$. It implies that u_n^m is the conditional probability of the vehicle beyond the location, x_m', by time t_n', given that the initial position was 0 at time 0. The numerical results of u_n^m are graphically presented with respect to m for several fixed n in ⟨Figure 10⟩ with respect to n for several fixed m in ⟨Figure 11⟩ for specific values of parameters, $\mu = 60$km per hour, $\sigma = 1.1$, $t = \frac{1}{60}$ hour, $x = 1$km, $N = 100$, $M = 50$, and $\gamma = 300$. With these parameters, we assume that the average speed of the vehicle is 60km per hour and after 1 minute its location is between 0.72km and 1.28km with probability 0.95. From the result above, we obtain the numerical values of instantaneous traffic flow $q_n^m \equiv q_\gamma(x_m', t_n', 0,0)$ and the local traffic density $k_n^m \equiv k_\gamma(x_m', t_n', 0,0)$ and present them in ⟨Figure 12⟩ and ⟨Figure 13⟩, respectively.



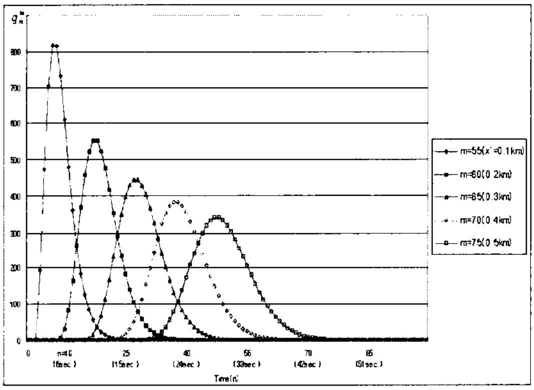
⟨Figure 10⟩ The Graph of u_n^m on m



⟨Figure 11⟩ The Graph of u_n^m on n



⟨Figure 12⟩ The Graph of $k_n^m = k(x_n^{'}, t_n^{'}, 0, 0)$ on m



⟨Figure 13⟩ The Graph of $q_n^m = q(x_n^{'}, t_n^{'}, 0, 0)$ on n

Consider the travel time of the vehicle to a certain location x_0, given that the initial position is known, $a(0) = \alpha$, $\tau_\alpha = \inf\{s \geq 0 | a(s) = x_0, a(0) = \alpha\}$. Then the expected travel time $E[\tau_\alpha]$ of $a(t)$ to x_0 from the initial position $a(0) = \alpha \leq x_0$ is equal to $l(\alpha)$.

We may obtain the following ordinary differential equation for $l(x)$ by applying (21) for this simple single trajectory model.

$$\mu \frac{\partial l(x)}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 l(x)}{\partial x^2} = -1, \text{ if } x < x_0,$$
$$l(x) = 0, \text{ if } x = x_0 \text{ and } x \to -\infty \quad (27)$$

The solution of the equation(27) is $l(x) = \frac{1}{\mu}(x_0 - x)$, $x \leq x_0$. Hence, for $\alpha \leq x_0$, the expected travel time, $E[\tau_\alpha]$, from α to x_0 is $\frac{1}{\mu}(x_0 - \alpha)$, which is the mean travel time of the vehicle from α to x_0.

We may easily extend the results of the single trajectory model to a general traffic flow in a light traffic situation on a homogeneous road section in which each driver choose her or his velocity, not interfered by other drivers.

Ⅳ. Concluding Remarks

This paper is to develop a novel concept of traffic flow variables such as local traffic density, instantaneous traffic flow, velocity field, and to investigate their nature on a general probability space of a sequence of random trajectories. We view observed trajectories of a vehicle platoon as one realization of a finite sequence of random trajectories and model the sequence with a system of stochastic differential equations. Each equation of the system may represent microscopic random maneuvering behavior of each vehicle with properly designed drift coefficient functions and diffusion coefficient functions. Currently extensive researches are being done for designing drift coefficient functions and diffusion coefficient functions that are appropriate in describing the microscopic behavior of individual driver. In parallel with studying these coefficient functions, numerical experiments for the suggested solution approaches proceed.

Even though we propose the system of stochastic differential equation as a general framework for analyzing motion of a platoon of vehicles, we are not arguing that this is "the approach" to the analysis of traffic flow in the stochastic framework of random trajectories. Instead, we present it as just one particular candidate. There can be many other alternative models that can be constructed in this novel stochastic framework, proposed in this paper.

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♧ 주 작 성 자 : 박영욱
♧ 논문투고일 : 2002. 1. 10
논문심사일 : 2002. 4. 10 (1차)
2002. 6. 7 (2차)
2002. 9. 5 (3차)
2002. 9. 11 (4차)
심사판정일 : 2002. 9. 11
♧ 반론접수기간 : 2003. 2. 28