Stability of a Generalized Quadratic Type Functional Equation

I. Introduction

Functional equations are a useful tool for narrowing the possible models for a phenomenon in that at least one more not very restrictive equations can formulate a model and when paired with an empirical or logical constraint of a general character those equations lead to precise quantitative relationships. In this paper we will deal one of the functional equations problem.

In 1940, S. M. Ulam [9] gave a wide ranging talk before a mathematical. Colloquium at the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms:

Let $G_1$ be a group and let $G_2$ be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h: G_1 \rightarrow G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$ then there is a homomorphism $H: G_1 \rightarrow G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

for all $x \in G_1$?

For the case where the answer is affirmative, the functional equation for homomorphisms will be called stable. The first result concerning the stability of functional equations was presented by D. H. Hyers [1]. He has excellently answered the question of Ulam for the case where $G_1$ and $G_2$ are Banach spaces. In 1978, a generalized version of the theorem of Hyers for
approximately linear mappings was given by Th. M. Rassias. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors.

The quadratic function \( f(x) = x^2 \) is a solution of the functional equation

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y).
\]

So, every solution of the functional equation

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]

is said to be a quadratic function. S. H. Lee\([3]\) proved the stability of the equation

\[
a^2 f\left( \frac{x + y}{a} \right) + a^2 f\left( \frac{x - y}{a} \right) = 2f(x) + 2f(y).
\]

In this paper we deal with a generalized quadratic functional equation

\[
a^2 f\left( \frac{x + y}{a} \right) + b^2 f\left( \frac{x - y}{b} \right) = 2f(x) + 2f(y).
\]

where \( a \) and \( b \) are nonzero real constants.

In Section 2 we solve a generalized quadratic functional equation. In Section 3 we prove the stability of a generalized quadratic functional equation. Throughout this paper \( a \) and \( b \) are nonzero real constants.

II. A solution of a generalized quadratic functional equation

Throughout this section \( X \) and \( Y \) will be real linear spaces. Given a function \( f : X \to Y \), consider the following equation

\[
a^2 f\left( \frac{x + y}{a} \right) + b^2 f\left( \frac{x - y}{b} \right) = 2f(x) + 2f(y)
\]

for all \( x, y \in X \).

Theorem 2.1. If a function \( f : X \to Y \) satisfies (2.1) for all \( x, y \in X \), then \( f(x) - f(0) \) is quadratic.

proof. We consider first the case \( a^2 + b^2 - 4 \neq 0 \). Putting \( x = y = 0 \) in (2.1) we have

\[
a^2 f(0) + b^2 f(0) = 4 f(0).
\]

Hence \( f(0) = 0 \) since \( a^2 + b^2 - 4 \neq 0 \). Putting \( y = x \) in (2.1) we have

\[
a^2 f\left( \frac{2x}{a} \right) = 4 f(x)
\]

for all \( x \in X \). Putting \( y = 0 \) in (2.1) we have

\[
a^2 f\left( \frac{x}{a} \right) + b^2 f\left( \frac{x}{b} \right) = 2 f(x)
\]

for all \( x \in X \). Putting \( x = 0 \) and \( y = x \) in (2.1) we have

\[
a^2 f\left( \frac{x}{a} \right) + b^2 f\left( \frac{x}{b} \right) = 2 f(x)
\]

for all \( x \in X \). Hence we obtain \( f(x) = f(-x) \) for all \( x \in X \).

Putting \( y = -x \) in (2.1) we have

\[
b^2 f\left( \frac{2x}{b} \right) = 4 f(x)
\]

for all \( x \in X \). Putting \( x = 0 \) and \( y = 2x \) in (2.1) we have

\[
a^2 f\left( \frac{2x}{a} \right) + b^2 f\left( \frac{-2x}{b} \right) = 2 f(2x)
\]

for all \( x \in X \). Using (2.2), (2.5), (2.6) and evenness of \( f \) we have

\[
f(2x) = 4 f(x)
\]

for all \( x \in X \). By (2.2), (2.5), (2.7), we have

\[
a^2 f\left( \frac{x}{a} \right) = f(x) = b^2 f\left( \frac{x}{b} \right)
\]

for all \( x \in X \). From (2.1) and (2.8) we have

\[
f(x + y) + f(x - y) = 2 f(x) + 2 f(y)
\]

for all \( x, y \in X \).

Now, we prove the case \( a^2 + b^2 - 4 = 0 \). Let \( Q(x) = f(x) - f(0) \) for all \( x \in X \). Then \( Q(0) = 0 \) and \( Q \) satisfies (2.1).

As a similar way to the case \( a^2 + b^2 - 4 \neq 0 \), we have \( Q \) is quadratic. \( \Box \)
III. Stability of a generalized quadratic functional equation

Let \( R^+ \) denote the set of nonnegative real numbers. Recall that a function \( H: R^+ \times R^+ \to R^+ \) is homogeneous of degree \( p > 0 \) if it satisfies \( H(tu, tv) = t^p H(u, v) \) for all nonnegative real numbers \( t, u \) and \( v \). Throughout this section, \( X \) and \( Y \) will be a real normed space and a real Banach space, respectively. We may assume that \( H \) is homogeneous of degree \( p \). Given a function \( f: X \to Y \), we set
\[
Df(x, y) = \frac{x+y}{a} f(x+y) + \frac{x-y}{b} f(x-y) - 2 f(x) - 2 f(y).
\]

Theorem 3.1. Assume that \( \delta \geq 0 \), \( p \in (0, \infty) \setminus \{2\} \) and \( \delta = 0 \) when \( p > 2 \). Let an even function \( f: X \to Y \) satisfy
\[
\Vert D f(x, y) \Vert \leq \delta + H(\|x\|, \|y\|)
\]
for all \( x, y \in X \) and \( f(0) = 0 \). Then there exists a unique quadratic function \( Q: X \to Y \) such that
\[
\Vert f(x) - Q(x) \Vert \leq \frac{1}{2} \delta + \frac{1}{4 \|x\|} \delta + \frac{1}{2 \|x\|^2} \h(x)
\]
for all \( x \in X \), where \( \h(x) = 2H(\|x\|, \|x\|) + H(\|2x\|, 0) \).

Proof. Putting \( y = x \) in (3.1) we have
\[
\Vert a^2 f\left(\frac{2x}{a}\right) - 4 f(x) \Vert \leq \delta + H(\|x\|, \|x\|)
\]
for all \( x \in X \). Putting \( x = 2x \) and \( y = 0 \) in (3.1) we have
\[
\Vert a^2 f\left(\frac{2x}{a}\right) + b^2 f\left(\frac{2x}{b}\right) - 2 f(2x) \Vert \leq \delta + H(\|2x\|, 0)
\]
for all \( x \in X \). Putting \( y = -x \) in (3.1) we have
\[
\Vert b^2 f\left(\frac{2x}{b}\right) - 4 f(x) \Vert \leq \delta + H(\|x\|, \|x\|)
\]
for all \( x \in X \) since \( f \) is even. From (3.3), (3.4) and (3.5) we have
\[
\Vert 8 f(x) - 2 f(2x) \Vert \leq 3 \delta + \h(x)
\]
for all \( x \in X \), where
\[
\h(x) = 2H(\|x\|, \|x\|) + H(\|2x\|, 0).
\]
Hence, we have
\[
\Vert f(x) - \frac{1}{4} f(2x) \Vert \leq \frac{3}{8} \delta + \frac{1}{8} \h(x)
\]
for all \( x \in X \). We divide the remaining proof by two cases.

1. The case \( 0 < p < 2 \). Using (3.6) we have
\[
\left| \frac{Q(2^n x)}{4^n} - \frac{Q(2^{n+1} x)}{4^{n+1}} \right| 
\]
\[
= \frac{1}{4^n} \left| f(2^n x) - \frac{1}{4} f(2 \cdot 2^n x) \right|
\]
\[
\leq \frac{1}{4^n} \frac{3}{8} \delta + \frac{1}{8} \frac{1}{2^{p-2}} \h(x)
\]
for all \( x \in X \) and all positive integers \( n \). From (3.6) and (3.7) we have
\[
\left| \frac{Q(2^n x)}{4^n} - \frac{Q(2^n x)}{4^n} \right| 
\]
\[
\leq \sum_{k=n}^{\infty} \frac{1}{4^k} \frac{3}{8} \delta + \sum_{k=n}^{\infty} \frac{1}{8} \frac{1}{2^{p-2}} \h(x)
\]
for all \( x \in X \) and all nonnegative integers \( m \) and \( n \) with \( m < n \).

This show that \( \left\{ \frac{Q(2^n x)}{4^n} \right\} \) is a Cauchy sequence for all \( x \in X \). Consequently, we can define a function
\[
Q: X \to Y \text{ by } Q(x) = \lim_{n \to \infty} \frac{Q(2^n x)}{4^n}
\]
for all \( x \in X \). We have \( Q(0) = 0 \) and
\[
\| DQ(x, y) \| = \lim_{n \to \infty} 4^{-n} \| DQ(2^n x, 2^n y) \|
\]
\[
\leq \lim_{n \to \infty} (4^{-n} \delta + \frac{1}{2^{n+p-2}}) H(\|x\|, \|y\|) = 0
\]
for all \( x, y \in X \). By Theorem 2.1, it follows that \( Q \) is quadratic. Putting \( m = 0 \) in (3.8) and letting \( n \to \infty \) we have (3.2). Now, let \( Q': X \to Y \) be another quadratic function satisfying (3.2). Then we have
\[ \|Q(x) - Q'(x)\| = \|Q(2^n x) - Q'(2^n x)\| \leq 4^{-n} \left( \|Q(2^n x) - Q(2^n x)\| + \|Q'(2^n x) - Q'(2^n x)\| \right) \leq 4^{-n} \delta + \frac{2^{n(\rho - 2)}}{4 - 2^n} h(x) \]

for all \( x \in X \) and all positive integers \( n \).

Since \( \lim_{n \to \infty} \left( 4^{-n} \delta + \frac{2^{n(\rho - 2)}}{4 - 2^n} h(x) \right) = 0 \), we can conclude the \( Q(x) = Q'(x) \) for all \( x \in X \). This proves the uniqueness of \( Q \).

(2) The case \( \rho > 2 \). Replacing \( x \) by \( \frac{x}{2} \) in (3.6) and multiplying both sides of (3.6) by 4 we have
\[ \|f(x) - 4 f(\frac{x}{2})\| \leq 2^{-\rho - 1} h(x) \]
for all \( x \in X \). Using (3.9) we have
\[ \|4^s f(2^{-n} x) - 4^{s+1} f(2^{-(s+1)} x)\| \leq 2^{-s-1} \frac{2^{s-\rho}}{2^{(s-\rho)n}} h(x) \]
for all \( x \in X \) and all positive integers \( n \). From (3.9) and (3.10) we have
\[ \|4^s f(2^{-n} x) - f(x)\| \leq \sum_{s=0}^{\infty} 2^{-s-1} 2^{s-\rho} h(x) \]
for all \( x \in X \) and all positive integers \( n \). The rest of the proof is similar to the corresponding part of the case \( 0 < \rho < 2 \).

Theorem 3.2. Assume that \( \delta \geq 0 \). Let an odd function \( f: X \to Y \) satisfy
\[ \|Df(x, y)\| \leq \delta + H(||x||, ||y||) \]
for all \( x, y \in X \). Then
\[ \|f(x)\| \leq \frac{1}{b^2} \delta + \frac{1}{b^2} h(x) \]
for all \( x \in X \), where
\[ h(x) = H\left( \left\| \frac{b}{2} x \right\|, \left\| \frac{b}{2} x \right\| \right) \]

Proof. Putting \( y = -x \) in (3.11) we have
\[ \left\| f\left( \frac{2x}{b} \right) \right\| \leq \delta + H(||x||, ||x||) \]
for all \( x \in X \). Replacing \( x \) by \( \frac{b}{2} x \) in (3.13) and then dividing both sides of its result by \( b^2 \) yields (3.12).

\[ \Box \]

Theorem 3.3. Let \( \delta \geq 0 \) and \( \rho \in (0, \infty) \setminus \{2\} \). Assume that \( \delta = 0 \) if \( \rho > 2 \) and
\[ ||(a^2 + b^2 - 4) f(0)|| = 0 \] if \( \rho > 2 \).

If a function \( f: X \to Y \) satisfies (3.1) for all \( x, y \in X \) then there exists a unique quadratic function \( Q: X \to Y \) such that
\[ ||f(x) - f(0) - Q(x)|| \leq \left( \frac{1}{2} + \frac{1}{b^2} \right) \delta + \frac{1}{2} (||a^2 + b^2 - 4|| f(0)||) \]

and
\[ ||f(x) - f(0) - Q(x)|| \leq \left( \frac{1}{2} + \frac{1}{b^2} \right) \delta + \frac{1}{2} (||a^2 + b^2 - 4|| f(0)||) \]

and
\[ ||f(x) - f(0) - Q(x)|| \leq \left( \frac{1}{2} + \frac{1}{b^2} \right) \delta + \frac{1}{2} (||a^2 + b^2 - 4|| f(0)||) \]

Proof. Let \( q_1(x) = Q_1(x) + f(-x) \) for all \( x \in X \).

Then \( q_1(0) = f(0) \), \( q_1(-x) = q_1(x) \) and
\[ ||Dq_1(x, y)|| \leq \delta + H(||x||, ||y||) \] for all \( x, y \in X \).

Let \( q_x = q_1(x) - q_1(0) \) for all \( x \in X \).

Then \( q(0) = 0 \), \( q(-x) = q(x) \) and
\[ \| Dq(x, y) \| = \| Dq(x, y) - (a^2 + b^2 - 4) f(0) \| \]
\[ \leq \| Dq(x, y) \| + \| (a^2 + b^2 - 4) f(0) \| \]
\[ \leq \delta + \| (a^2 + b^2 - 4) f(0) \| + H(||x||, ||y||) \]
for all \( x, y \in X \).

By Theorem 3.1, there exists a unique quadratic function \( Q: X \to Y \) satisfying (3.15).

Let \( g(x) = \frac{f(x) - f(-x)}{2} \) for all \( x \in X \).

Then \( g(-x) = -g(x) \) and
\[ \| Dg(x, y) \| \leq \delta + H(||x||, ||y||) \]
for all \( x, y \in X \). By Theorem 3.2, we have (3.16).

Clearly, we have (3.14) for all \( x \in X \).

Define a mapping \( H: R^+ \times R^+ \to R^+ \) by
\[ H(a, b) = \delta (a^2 + b^2) \]
Then \( H \) is homogeneous of the degree \( p \). Thus we have the following corollaries.

**Corollary 3.4.** Assume that \( \delta \geq 0 \), \( p \in (0, \infty) \setminus \{2\} \)
and \( \delta = 0 \) when \( p > 2 \). Let an even function \( f: X \to Y \)
satisfy
\[ \| Df(x, y) \| \leq \delta + \theta ||x||^p + ||y||^p \]
for all \( x, y \in X \) and \( f(0) = 0 \). Then there is a unique quadratic function \( Q: X \to Y \) such that
\[ \| f(x) - Q(x) \| \leq \frac{1}{2} \delta + \frac{4 + 2^p}{2(4 - 2^p)} \theta ||x||^p \]
for all \( x \in X \).

**Corollary 3.5.** Assume that \( \delta \geq 0 \). Let an odd function \( f: X \to Y \)
satisfy
\[ \| Df(x, y) \| \leq \delta + \theta ||x||^p + ||y||^p \]
for all \( x, y \in X \). Then
\[ \| f(x) \| \leq \frac{1}{b^2} \delta + \frac{b^{p-2}}{2^{p-1}} \theta ||x||^p \]
for all \( x \in X \).

**Corollary 3.6.** Let \( \delta \geq 0 \) and \( p \in (0, \infty) \setminus \{2\} \).

Assume that \( \delta = 0 \) if \( p > 2 \) and
\[ \| (a^2 + b^2 - 4) f(0) \| = 0 \] if \( p > 2 \).

If a function \( f: X \to Y \) satisfies
\[ \| Df(x, y) \| \leq \delta + \theta \left( ||x||^p + ||y||^p \right) \]
for all \( x, y \in X \), then there exists a unique quadratic function \( Q: X \to Y \) such that
\[ \| f(x) - Q(0) - Q(x) \| \leq \frac{1}{2} \delta + \frac{4 + 2^p}{2(4 - 2^p)} \theta ||x||^p \]
and for all \( x \in X \):
\[ \| f(x) - Q(-x) \| \leq \frac{1}{2} \delta + \frac{b^{p-2}}{2^{p-1}} \theta ||x||^p \]

**IV. Conclusion**

So far we investigated a generalized quadratic functional equation
\[ a^2 f \left( \frac{x + y}{a} \right) + b^2 f \left( \frac{x - y}{b} \right) = 2f(x) + 2f(y) \]
where \( a \) and \( b \) are nonzero real constants.

We also solve a generalized quadratic functional equation and then prove the stability of a generalized quadratic functional equation.

This functional equation can be used for a variety of applications in some areas of the behavioral sciences such as sensory psychology (psychophysics), utility theory under uncertainty, and aggregation of inputs where proves the stability of this equation and outputs in an economic or social context.
It is expected that more functional equations in the various forms such as exponential, logarithmic and multiplicative functional equations would be applicable soon.

REFERENCES