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Product of Irreducible Characters

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Abstract

In this paper we prove that the property of product irreducible characters that is ,if $\phi \in Irr(H)$ and $\theta \in Irr(K)$ are faithful, then $\phi \times \theta$ is faithful if and only if $|Z(H)|$ and $|Z(K)|$ are relative primes where $G = H \times K$.

0. Introductions

Character theory provide a powerful tool for proving theorems about finite groups. Complex representations and their characters were first studied nearly one hundred years ago by Frobenius and his theorem on transitive permutation groups was the first major achievement of the theory; it remains to this day, along with Burnsides $p^a q^b$ theorem, one of the highlights of any first course on character theory.

Burnside theory is very pretty and very useful. This theorem was proved in early years of 20 century as an application of the character theory of finite groups. The original proof of Burnside is very short and clear.

In the first edition of his book "Theory of groups of finite order" (1897), Burnside presented group theoretic argument which proved the theorem for many special choice of the integers a, b but it was only after studying Frobenius' new theory of group representations that he was able to prove the theorem in general. Indeed, many later attempts to find a proof which does not use character theory were unsuccessful, until such a character-free proof was finally obtained by the combined works of Thompson, Goldschmidt, Bender, and Matsuyama. Frobenius theorem remains untouched by noncharacter-theoretic methods.

Both of these results may be regarded as non-simplicity criteria. The study and application of character theory, since Brauer proposed a systematic program to classify the finite simple groups at the Amsterdam International Congress in 1954, can not be divorced from the classification itself. Although purely group theoretic methods have dominated the major part of that work which took place between 1970 and 1980, the classification could never have been carried out without the early progress using character theory. The reason for this is quite simple.

Character theory provides a means of applying ring theoretic techniques to the study of finite groups. Although much of the theory can be developed in other ways, it seams more natural to approach character via ring or more accurately, algebra.

Let G be a finite group and let $\mathbb{C}$ be the set of complex numbers. A $\mathbb{C}$-representation of G is a homomorphism $T: G \to GL(n, \mathbb{C})$ for some integer n where $GL(n, \mathbb{C})$ is the general linear group of nonsingular $n \times n$ matrices over $\mathbb{C}$. The integer n is the degree of T. Two representations T, S of degree n are similar if there exists a nonsingular $n \times n$ matrix P, such that $T(g) = P^{-1}S(g)P$ for all $g \in G$. A representation $T(g)$ is said to be reducible if $T(g)$ is similar to the form $\begin{pmatrix} T_1(g) & T_2(g) \\ 0 & T_3(g) \end{pmatrix}$, $(g \in G)$ where $T_i(g)$ is representation of G. Otherwise $T(g)$ is called irreducible. Let T be an $\mathbb{C}$-representation of G. Then the $\mathbb{C}$-character χ of G afforded by T is the function given by $\chi(g) = trT(g)$ (trace of $T(g)$). Note that $\chi(1) = \deg T$ and characters of degree 1 are called linear characters. Note that $g \in \ker T$ iff $\chi(g) = \chi(1)$ [7]. Thus $\ker \chi = \{g \in G \mid \chi(g) = \chi(1)\}$ is defined. A character χ of G is said to be faithful if $\ker \chi = 1$. Throughout this paper, a group G is finite and the characters of G are complex characters. Let $Irr(G)$ be the set of all irreducible complex characters. In this paper, we prove that the property of product of two irreducible faithful characters(Theorem 7).

2. Preliminaries and main result

Lemma 1. Let G be a finite group and let χ be the $\mathbb{C}$-representation T of G afforded by $\mathbb{C}$-representation T of G. If for $g \in G$, $n = o(g)$, the order of g, then

(a) $T(g)$ is similar to diagonal matrix $diag(\varepsilon_1, \cdots, \varepsilon_f)$

(b) $\varepsilon_i^n = 1$

(c) $\chi(g)=\sum\varepsilon_i$ and $|\chi(g)|\le\chi(1)$

(d) $\chi(g^{-1})=\overline{\chi(g)}$

Proof. The restriction of T to the cyclic group $\langle g\rangle$ is a representation of $\langle g\rangle$ and hence it is no loss to assume $G=\langle g\rangle$. By Maschkes Theorem, it follow that T is similar to a representation in block diagonal form, with irreducible representations of G appearing on the diagonal blocks. Since $G=\langle g\rangle$ is abelian this irreducible representations have degree 1, and thus T is similar to a diagonal represenation. Now (a) follows, and we may assume that T is diagonal.

We define $I=T(g^n)=T(g)^n=diag(\varepsilon_1^n,\cdots,\varepsilon_f^n)$. Therefore (b) is proved. It follows that $|\varepsilon_i|=1$ and $|\sum\varepsilon_i|\le\sum|\varepsilon_i|=f=\chi(1)$. It is clear that $\chi(g)=\sum\varepsilon_i$ so that (c) follows. Now $T(g^{-1})=T(g)^{-1}=diag(\varepsilon_1^{-1},\cdots,\varepsilon_f^{-1})$ so that $\chi(g^{-1})=\sum\varepsilon_i^n$. Since $|\varepsilon_i|=1$, we have $\varepsilon_i^{-1}=\overline{\varepsilon_i}$ and $\chi(g^{-1})=\overline{\chi(g)}$. The proof is complete.

Definition 2. Let χ be a character of G. Then $Z(\chi)=\{g\in G\mid|\chi(g)|=\chi(1)\}$ is called the center of χ.

Lemma 3. Let χ be a character afforded by a C-representation T of G. Then

(a) $Z(\chi)=\{g\in G\mid T(g)=\varepsilon I \text{ for some } \varepsilon\in \mathrm{C}\}$

(b) $Z(\chi)$ is a subgroup of G.

(c) $\chi_{Z(\chi)}=\chi(1)\lambda$ for some linear character λ of $Z(\chi)$

(d) $Z(\chi)/\ker\chi$ is cyclic

(e) $Z(\chi)/\ker\chi\subseteq Z(G/\ker\chi)$: central

Furthermore, if $\chi\in Irr(G)$, then

(f) $Z(\chi)/\ker\chi=Z(G/\ker\chi)$

Proof. By Lemma1, $T(g)$ is similar to $diag(\varepsilon_1,\cdots,\varepsilon_f)$ with $|\varepsilon_i|=1, 1\le i\le f$. Since $\chi(g)=\sum\varepsilon_i$, it follows that $|\chi(g)|=f$ iff all ε_i are equal. Since the onlyy matrix similar to εI is εI itself, conclusion (a) follows. Define the function $\lambda:Z(\chi)\to\mathrm{C}$ by $T(z)=\lambda(z)I$ for $z\in Z(\chi)$. It follows for $z,w\in Z(\chi)$ that $T(zw)=\lambda(z)\lambda(w)I$ and hence $Z(\chi)$ is a subgroup and λ is a homomorphism(linear character) of $Z(\chi)$. We have that

$\chi(z)=\chi(1)\lambda(z)$ of $g\in Z(\chi)$ and (b) and (c) have been proved. Clearly $\ker\chi=\ker\lambda$ and thus $Z(\chi)/\ker\chi$ is isomorphic to the image of λ, a finite multiplicative subgroup of the field $\mathbb{C}$. This subgroup is necessarily cyclic and (d) follows. Also, $\ker\chi=\ker T$ and $T(Z(\chi))\subseteq Z(T(G))$ and (e) is an immediate consequence. Finally, if $g(\ker\chi)\in Z(G/\ker\chi)$, then $T(g)\in Z(T(G))$. If $\chi\in Irr(G)$, then we conclude that $T(G)=\varepsilon I$ for some $\varepsilon\in\mathbb{C}$. Now (f) follows from (a) and the proof is complete.

Lemma 4. Let $\chi\in Irr(G)$ and $z\in Z(\chi)$. Then

(a) $\chi(gz)=\varepsilon\chi(g),\ \varepsilon=\dfrac{\chi(z)}{\chi(1)}$ for $g\in G$

(b) $\chi(z^n)=\varepsilon^n\chi(1)$ for integer n.

Proof. (a) By Lemma3, we have $\chi(z)=trT(z)=tr\varepsilon I=n\varepsilon=\chi(1)\varepsilon$ and

$$\chi(gz)=trT(gz)=trT(g)\circ T(z)=trT(g)\circ\varepsilon I=tr\varepsilon T(g)=\varepsilon\chi(g).$$

Hence $\chi(gz)=\varepsilon\chi(g)$ and $\varepsilon=\dfrac{\chi(z)}{\chi(1)}$.

(b) We prove induction on n. Assume it holds for $n-1$, that is, $\chi(z^{-1})=\varepsilon^{n-1}\chi(1)$. Then by (a) we have $\chi(z^n)=\chi(z^{n-1}z)=\varepsilon\chi(z^{n-1})=\varepsilon^n\chi(1)$. The proof is complete.

Definition 5. Let $G=H\times K$ and let ϕ and θ be the characters of H and K, respectively. Define $\chi=\phi\times\theta$ by $\chi(hk)=\phi(h)\theta(k)$ for $h\in H$ and $k\in K$.

Under the isomorphism $H\cong G/K$, there is a corresponding character $\widehat{\phi}$ of G with $K\subseteq\ker\widehat{\phi}$ and $\widehat{\phi}(hk)=\phi(h)$. Similarly, there is a corresponding character $\widehat{\theta}$ of G with $\widehat{\theta}(hk)=\theta(k)$. It follows that $\phi\times\theta=\widehat{\phi}\,\widehat{\theta}$ is a character of G.

Proposition 6. Let $G=H\times K$. Then character $\phi\times\theta$ for $\phi\in Irr(H)$ and $\theta\in Irr(K)$ are exactly the irreducible characters of G.

Proof. Let $\phi,\phi_1\in Irr(H)$ and $\theta,\theta_1\in Irr(K)$. Let $\chi=\phi\times\theta$ and $\chi_1=\phi_1\times\theta_1$, then

$$[\chi,\chi_1]=\frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\chi_1(g)}$$

$$=\frac{1}{|H||K|}\sum_{h\in H,k\in K}\phi(h)\theta(k)\overline{\phi_1(h)\,\theta(k)}$$

$$=\left(\frac{1}{|H|}\sum_{h\in H}\phi(h)\overline{\phi_1(h)}\right)\left(\frac{1}{|K|}\sum_{k\in K}\theta(k)\overline{\theta_1(k)}\right)$$
$$=[\phi,\phi_1][\theta,\theta_1].$$

It follows that the $\phi\times\theta$ are all distinct and irreducible. Now

$$\sum_{\substack{\phi\in Irr(H)\\ \theta\in Irr(K)}}\{(\phi\times\theta)(1)\}^2=\sum_{\substack{\phi\in Irr(H)\\ \theta\in Irr(K)}}\phi(1)^2\theta(1)^2=\Big(\sum_{\phi\in Irr(H)}\phi(1)^2\Big)\Big(\sum_{\theta\in Irr(K)}\theta(1)^2\Big)$$
$$=|H||K|=|G|$$

and thus the $\phi\times\theta$ are all of $Irr(G)$.

Theoerm 7. Let $G=H\times K$. Let $\phi\in Irr(H)$ and $\theta\in Irr(K)$ be faithful. Then $\phi\times\theta$ is faithful if and only if $|Z(H)|$ and $|Z(K)|$ are relatively primes.

Proof. Since $\phi\in Irr(H)$, $\theta\in Irr(K)$ and ϕ,θ are faithful, by Lemma2, we have $Z(\phi)=Z(H)$ and $Z(\theta)=Z(K)$ and by proposition 6, $\phi\times\theta\in Irr(G)$. Suppose that $|Z(H)|$ and $|Z(K)|$ are relatively prime. Then we want $\ker(\phi\times\theta)=\{(1,1)\}$. Let $(h,k)\in\ker(\phi\times\theta)$. So $\phi(h)\theta(k)=\phi(1)\theta(1)$. Taking modulus we get $|\phi(h)||\theta(k)|=\phi(1)\theta(1)$. Now by Lemma 1, $\phi(h)\le\phi(1)$ and $\theta(k)\le\theta(1)$. Multiplying these relations, then $\phi(h)\theta(k)\le\phi(1)\theta(1)$. The equality holds wherever $|\phi(h)|=\phi(1)$ and $|\theta(k)|=\theta(1)$. Thus $h\in Z(\phi)$ and $k\in Z(\theta)$ and so $o(h)$ and $o(k)$ are relatively prime. On the other hand, $h\in Z(\phi)$ implies $\phi(h)=\varepsilon_1\phi(1)$, where ε_1 is a $o(h)$th-root of unity. Similarly $\theta(k)=\varepsilon_2\theta(1)$, where ε_2 is a $o(k)$th-root of unity. It follows that $\phi(h)\theta(k)=\varepsilon_1\varepsilon_2\phi(1)\theta(1)$. So $\varepsilon_1\varepsilon_2=1$. It follows that $\varepsilon_1=\varepsilon_2^{-1}$ is both $o(h)$th and $o(k)$th-root of unity. Thus since $o(h)$ and $o(k)$ are relatively prime, we have $\varepsilon_1=\varepsilon_2=1$. Hence $\phi(h)=\phi(1)$ and $\theta(k)=\theta(1)$ that is $h\in\ker\phi$ and $k\in\ker\theta$. But ϕ and θ are faithful, so $h=1$ and $k=1$.

Conversely, assume that $\phi\times\theta$ is faithful and we want that $Z(\phi)$ and $Z(\theta)$ are relatively prime. If not, then there is a prime p with $p/|Z(\phi)|$ and $p/|Z(\theta)|$. Thus there are $h\in Z(\phi),k\in Z(\theta)$ both of order p. As before $\phi(h)=\varepsilon\phi(1)$ and $\theta(k)=\delta\theta(1)$ for some ε,δ roots of unity of order p. Note that $o(h)=o(k)=p$ implies $a\neq1,b\neq1$ and by faithfulness $\varepsilon\neq1$ and $\delta\neq1$. Since ε and δ are roots of unity, there is n such that $\varepsilon^n=\omega^{-1}$ and note that $p\nmid n$.

By Lemma 4, it follows that $\phi(h^n)=\varepsilon^n\phi(1)=\delta^{-1}\phi(1)$ with $h^n\neq1$ since $p\nmid n$. So we have $(\phi\times\theta)(h^n, k)=\phi(h^n)\theta(k)=\delta^{-1}\phi(1)\delta\theta(1)=\phi(1)\theta(1)=(\phi\times\theta)(1,1)$.

Thus $(h^n, k)\in\ker(\phi\times\theta)$ and $(h^n, k)\neq(1,1)$ which contradicts the faithfulness of $\phi\times\theta$. Therefore $|Z(\phi)|$ and $|Z(\theta)|$ are relatively prime. The proof is complete.

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