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On Gödel's Program from Incompleteness to Speed-up

연세대학교 **현우식**

Abstract

Gödel's metamathematical program from Incompleteness to Speed-up theorems shows the necessity of ever higher systems beyond the fixed formal system and devises the relative consistency.

0. Formal Mathematical Systems

According to Gödel[8], a formal mathematical system is a system of symbols with rules for using them. This system basically consists of two parts: *Language part* and *Transformation rule part.*

Fomal Language: The individual symbols are called *undefined terms*. *Formulas* are finite sequences of the undefined terms.

Transformation Rule: There is a class of formulas called *meaningful formulas*, and a class of meaningful formulas called *axioms*. Rule of inference is a specified list of rules.

If such a rule be called R, it defines the relation of *immediate consequence* by R between a set of meaningful formulas $M_1, \cdots, M_k$ called the *premises* and a meaningful formula N, called the conclusion.

For each rule of inference there is a finite procedure for determining whether a given formula B is an immediate consequence by that rule of given formulas $A_1, \cdots, A_n$ and there is a finite procedure for determining whether a given formula A is a meaningful formula or an axiom. Therefore, rule of inference must be effective.

A formula N is called an immediate consequence of $M_1, \cdots, M_n$ if N is an immediate consequence of $M_1, \cdots, M_n$ by any one of the rules of inference. A finite sequence of formulas is a *proof*, i.e., the last formula of the sequence, if each formula of the sequence is either an axiom or an immediate consequence of one or more of the preceding formulas. A formula is *provable* if a proof of it exists.

Let the symbol $\neg A$ be one of the undefined terms and express negation. Then the formal system is said to be *complete* if for every meaningful formula A either A or $\neg A$ is provable. A formal system is said to be *consistent* if and only if there is no formula A such that both A and $\neg A$ are provable. A formal system is said to be *decidable* if and only if there exists a mechanical procedure for determining whether any given formula is provable.

1. Gödel's First Incompleteness Theorem

E. Post[13] viewed the Gödel's incompleteness theorem as a fundamental discovery in the limitations of the mathematicizing powers of Homo Sapiens. According to Post, Gödel's theorem concerning the incompleteness of first order logic can be transformed into conclusions concerning all formal logics and all methods of solvability.

In 1931, Gödel proved that a system, in which all propositions of arithmetic can be expressed as meaningful formulas, is not complete. This means that he established the incompleteness of a sufficiently strong axiomatic formal system. By introducing Gödel numbering of formal expressions, he arithmetized the syntax of primitive recursive extension of the system P from *Principia Mathematica* by B. Russell and A. N. Whitehead. Gödel then employed a diagonal argument and constructed a Gödel sentence G that is not provable in P.

We say that a formal system S is *sound* for a formula Π^0_1 in S whenever Π^0_1 is true in the system of natural numbers if S proves Π^0_1. If follows that a formal system S is consistent if and only if S is sound for Π^0_1. Here, we may consider Π^0_1 formulas as $\forall y P(x, y)$ with one free variable x and a decidable predicate P.[1] Let $Pf_s(x, y)$ be a decidable binary predicate expressing that y is a proof of the formula x in S.[2] Then

1. The class of Π^0_1 formulas are one of the form $\forall P(x)$ representing for all variables x the predicate P holds, where the predicate P has a decidable property of natural numbers.

$\forall y \neg Pf_s(x, y)$ is in the class of Π_1^0 formulas. Gödel constructed a Π_1^0 sentence such as

$$G(S) \equiv \forall y \neg Pf(\gamma, y)$$

where γ is the Gödel number of $G(S)$.

Gödel's First Incompleteness Theorem(1931) says that if a formal system S is consistent then neither $G(S)$ nor $\neg G(S)$ is provable in S. Here, let M be a mathematical system that makes only true statements. Suppose that we obtain a Gödel sentence such that

$M \Leftrightarrow$ 「M never proves G is true.」

(1) If G is provable in M, then 「M never proves G is true」 is true. Hence, M never proves G. (2) If G is not provable in M, then 「M never proves G is true」 is false. Hence, M proves G. Both contradicts the consistency of M.[3]

2. Gödel's Second Incompleteness Theorem

By formalizing the proof of the first incompleteness theorem, Gödel stated the second incompleteness theorem in his same work[6]. Gödel's Second Incompleteness Theorem says that if a formal system S is consistent then its own consistency, denoted by $Con(S)$, is not provable in S. The second incompleteness theorem may be restated from various perspectives. Versions of other scholars are as follows:

1. S. Kleene [12] *If the number-theoretic formal system is consistent, then not prove Con; i.e., if the system is consistent, then there is no consistency proof for it by methods formalizable in the system.*
2. P. Cohen [2] *$Con(Z_1)$ cannot be proved in Z_1 (where Z_1 is the formal system for*

2. $Pf_s(x, y)$ is a decidable relation between two natural numbers x and y, that is, an algorithm exists to decide, for each choice of value x and y, whether or not $Pf_s(x, y)$ holds.
3. Gödel's first incompleteness theorem differs from Liar's Paradox in that the theorem employs metamathematical method. Let α be the class of formulas that are not expressed in a system S, and β be the class of formulas that are expressed in S. Suppose that $\beta \subseteq \alpha$, where β represents the class of formulas provable in S. Then Gödel proved that $\beta \subset \alpha$. By the soundness theorem, there exists a true sentence but not provable. Hence, Gödel's incompleteness theorem is not a paradox. For more on this issue, see [8].

elementary arithmetic).

3. T. Jech [11] *It is unprovable in set theory, (unless it is inconsistent) that there exists a model of set theory.*

4. H. DeLong [3] *No consistent human computer can prove his own consistenty.*

It is said that Gödel's incompleteness theorem was a fatal blow to the Hilbert's formalist program for the foundations of mathematics. However, this is not an trivially supported statement. For Gödel himself did not deny the possibility of a finitist proof of the consistency of mathematics. As Gödel concluded,

> The entire proof of Theorem XI (i.e., the Second Incompleteness Theorem) carries over word for word to the axiom system of set theory, M, and to that of classical mathematics, A, and here, too, it yields the result: There is no consistency proof for M, or A, respectively, provided M, or A is consistent. **I wish to note expressly that Theorem XI** (and the corresponding results for M and A) **do not contradict Hilbert's formalist viewpoint**. For this viewpoint presupposes only the existence of a consistency proof in which nothing but finitary means of proof is used, and it is conceivable that there exist finitary proofs that *cannot* be expressed in the formalism of P(i.e., the formal system of Principia Mathematica) (or of M or A). (Italics in original, Gödel[6], p. 195, Bold in mine.)

According to Gödel, the second incompleteness theorem does not contradict Hilbert's program, and there may be finitary proofs of $Con(P)$ that cannot be representable in P. Surpassing the formal systems in representational power could obtain finitary consistency proofs.

3. Gödel's Speed-up Theorems

Gödel[9] maintained that higher types allow one to prove undecidable Gödel sentences in systems of lower levels.[4] Let S_i be the system of logic of the ith order, the natural

4. The term *speed-up* is not due to Gödel himself, but was introduced by M. Blum(A machine-independent theory of the complexity of recursive functions, *Journal of the Association for Computing Machinery* *14*(1967), 322-336.) in terms of complexity theory in computer science. This phenomenon became a major topic in theoretical computer science and, indeed, the

numbers being taken as individuals. Then S_i contains the appropriate logical axioms, variables and quantifiers for natural numbers, for classes of natural numbers, for classes of classes of natural numbers, and so on, up to classes of the ith type, but no variables of a higher type. Then, there are properties of S_i that are provable in S_{i+1} but not in S_i. On the other hand, if we consider those formulas A that are provable in S_i as well as S_{i+1}, then the following holds: For each function Φ that is computable in S_i there exist infinitely many formulas A such that if k is the length of a shortest proof of A in S_i and l is the length of a shortest proof of A in S_{i+1}, then $k > \Phi(l)$. He concluded that

> *Thus, passing to the logic of the next higher order has the effect, not only of making provable certain propositions that were not provable before, but also of making it possible to shorten, by an extraordinary amount, infinitely many of the proofs already available.* (Italics in original, Gödel[9], p. 397)

The main result is that there are formulas that can be proved both in S_n and S_{n+1} but whose shortest proof in S_{n+1} is much shorter than that in S_n. Thus, by successively adding a higher type one arrives at a sequence of ever stronger systems. In fact, Gödel already addressed the speed-up issue in his previous papers[6,7]. However, there is a critical difference between "On Formally Undecidable Propositions of Principia Mathematica and Related Systems I(1931)"[6] and "On the Length of Proofs(1936)"[9]. In his 1936 paper, Gödel allowed *infinitely* many proofs with a given number of symbols in his 1931 work.

According to M. Arbib[1], Gödel's incompleteness theorem can be removed incrementally in a mechanical way in what is referred to as a speed-up, which is analogous to adding of an undecidable sentence, such as a Gödel sentence, to an incomplete logic system. Arbib introduced the proof-measure system to prove that if a system L_1 can do some things arbitrarily quicker than L then L_1 can do some things that L cannot do at all. Gödel's speed-up theorem is relevant for increasing the range of the computing machine by adding new instructions. Arbib remarks:

> If we add an undecidable axiom to an incomplete logic, not only are there truths that become theorems for the first time, but also theorems that were already provable in the old system may have shorter proofs in the new

celebrated $P=NP$? problem can itself be thought of as a speed-up problem.

system[1, p. 187]

Arbib's contribution here is the combination of adding new axiom to a logic system with adding new instructions to a computing machine. It seems to me a proper application of Gödel's second incompleteness theorem to computer science. Nevertheless, the finite description issue and the consistency issue in effect remains unresolved.

4. Gödel's Program

The Speed-up theorem implies that a certain function has no best argorithms. S. Feferman[5] articulates the speed-up aspects of Gödel's incompleteness theorems from Gödel's own footnotes 48a in [6], in which Gödel clarified:

> As will be shown in Part II of this paper, the true reason for the incompleteness inherent in all formal systems of mathematics is that the formation of ever higher types can be continued into the transfinite (see Hilbert 1926 "Über das Unendliche," *Mathematische Annalen 95*, 161-190), while in any formal system at most denumerably many of them are available. For it can shown that the undecidable propositions constructed here become decidable whenever appropriate higher types are added (for example, the type ω to the system P). An analogous situation prevails for the axiom of set theory.

Feferman then calls this *Gödel's doctrine*: wherein the unlimited transfinite iteration of the power-set operation is necessary to account for finitary mathematics. He asserts that the true reason for the incompleteness phenomena is that the formation of ever higher types can be continued into the transfinite, both in systems explicitly using types and in systems of set theory, such as Zermelo-Fraenkel set theory, for which the (cumulative) type structure is implicit in the axioms[5, p. 229]. Thus one can obtain a new Π^0_1 sentence by adding a higher type to the fixed system. Along this line of thought, Feferman in 1962 studied the transfinite recursive and concluded that even a higher type formation cannot be established by the incompleteness property alone[4]. Following Gödel[10, p. 150],

> … there exist certain negative results, such as incompleteness of every

> formalism or the paradox of Richard. But closer examination shows that *these results do not make a definition of the absolute notions concerned impossible under all circumstances*, but only exclude certain ways of defining them, or at least, that certain very closely related concepts may be definable in an absolute sense. (Italics in mine).

Thus, Gödel's incompleteness theorem does not absolutely close the window to prove consistency if Hilbert's restrictions with respect to finite procedures are abandoned. Rather, Gödel's theorems open the possibility of further attempts to prove the consistency of formal systems by adding new axioms and theorems. From this perspective, consistency proof is not a matter of ultimate mathematical truth, but of relative truth. In his *The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis*(1940), therefore, Gödel proved that if the set theory - whose axioms are those of the von Neumann-Bernays system except the axiom of choice - is consistent, then the theory obtained by adding a strong from of the axiom of choice and the generalized continuum hypothesis to these axioms is also consistent.

In his "On Completeness and Consistency"[7], Gödel generalized his incompleteness theorem and its mathematical meaning as follows:

> To be sure, all the propositions thus constructed are expressible in Z (hence are number-theoretic propositions); they are, however, not decidable in Z, but only in higher systems, for example, in that of analysis. In case we adopt a type-free construction of mathematics, as is done in the axiom system of set theory, axioms of cardinality (that is, axioms postulating the existence of sets of ever higher cardinality) take the place of the type extensions, and it follows that certain arithmetic proposition that are undecidable in Z become decidable by axioms of cardinality, for example, by the axiom that there exist sets whose cardinality is greater than every α_n, where $\alpha_0 = \aleph_0$, $\alpha_{n+1} = 2^{\alpha_n}$. (Gödel[7], p. 237)

In essence, considerations of much higher systems with respect to the relative consistency may help us to understand how we can actually approach Gödel about his metamathematical program from incompleteness to speed-up theorems.

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