J. Korean Math. Soc. **39** (2002), No. 1, pp. 127–135

EQUIMULTIPLE GOOD IDEALS WITH HEIGHT 1

MEE-KYOUNG KIM

ABSTRACT. Let I be an ideal in a Gorenstein local ring A with the maximal ideal $\mathfrak{m}$. Then we say that I is an *equimultiple good ideal* in A, if I contains a reduction $Q = (a_1, a_2, \cdots, a_s)$ generated by s elements in A and $\mathrm{G}(I) = \oplus_{n \geq 0} I^n/I^{n+1}$ of I is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(I)) = 1 - s$, where $s = \mathrm{ht}_A I$ and $\mathrm{a}(\mathrm{G}(I))$ denotes the a-invariant of $\mathrm{G}(I)$. Let $\mathcal{X}_A^s$ denote the set of equimultiple good ideals I in A with $\mathrm{ht}_A I = s$, $\mathrm{R}(I) = A[It]$ be the Rees algebra of I, and $\mathrm{K}_{\mathrm{R}(I)}$ denote the canonical module of $\mathrm{R}(I)$. Let $a \in I$ such that $I^{n+1} = aI^n$ for some $n \geq 0$ and $\mu_A(I) \geq 2$, where $\mu_A(I)$ denotes the number of elements in a minimal system of generators of I. Assume that A/I is a Cohen-Macaulay ring. We show that the following conditions are equivalent.

(1) $\mathrm{K}_{\mathrm{R}(I)} \cong \mathrm{R}(I)_+$ as graded $\mathrm{R}(I)$-modules.
(2) $I^2 = aI$ and $aA : I \in \mathcal{X}_A^1$.

1. Introduction

Let A be a Gorenstein local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. Let t be an indeterminate over A. For an ideal I in A we put

$$\begin{aligned} \mathrm{R}(I) &= A[It] \subseteq A[t], \\ \mathrm{R}'(I) &= A[It, t^{-1}], \quad \text{and} \\ \mathrm{G}(I) &= \oplus_{n \geq 0} I^n/I^{n+1}, \end{aligned}$$

which we call the Rees algebra, the extended Rees algebra, and the associated graded ring of I, respectively. We denote by K_A, $\mathrm{K}_{\mathrm{R}(I)}$, and

Received October 1, 2000. Revised December 14, 2000.
2000 Mathematics Subject Classification: Primary 13A30; Secondary 13H10.
Key words and phrases: Rees algebra, associated graded ring, Cohen-Macaulay ring, Gorenstein ring, a-invariant.
The author was supported by Korea Research Foundation Grant (KRF-2000-015-DP0005).

$\mathrm{K}_{\mathrm{G}(I)}$ the canonical module of A, $\mathrm{R}(I)$, and $\mathrm{G}(I)$, respectively. Let $\mathrm{a}(\mathrm{G}(I))$ denote the a-invariant of $\mathrm{G}(I)$ ([5, (3.1.4)]).

Let I ($\neq A$) be an ideal in A with $\mathrm{ht}_A I = s$ and assume that I contains a reduction $Q = (a_1, a_2, \cdots, a_s)$ generated by s elements. Let $\mathrm{r}_Q(I) = \min\{n \geq 0 \mid I^{n+1} = QI^n\}$ be the reduction number of I with respect to Q. Suppose the ring $\mathrm{G}(I)$ is Cohen-Macaulay. Then the elements $a_1t, \cdots, a_st$ of $\mathrm{R}(I)$ form a regular sequence in $\mathrm{G}(I)$ and

$$\mathrm{G}(I)/(a_1t, \cdots, a_st)\mathrm{G}(I) \cong \mathrm{G}(I/Q)$$

as graded A-algebras ([9]). We have $\mathrm{a}(\mathrm{G}(I)) = \mathrm{r}_Q(I) - \mathrm{ht}_A I$ ([5, (3.1.6)]), since $\mathrm{G}(I/Q)$ is a Cohen-Macaulay ring with $\mathrm{a}(\mathrm{G}(I/Q)) = \mathrm{r}_Q(I)$. From this point of view S. Goto and M. Kim generalized the concept of good ideals in [4], which was first introduced by the first author, S. Iai, and K. Watanabe ([3]).

Let I ($\neq A$) be an ideal in A with $\mathrm{ht}_A I = s$. Then we say that I is an *equimultiple good* ideal in A if I contains a reduction $Q = (a_1, \cdots, a_s)$ generated by s elements in A and $\mathrm{G}(I)$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(I)) = 1 - s$. For each $0 \leq s \in \mathbb{Z}$ let $\mathcal{X}_A^s$ be the set of equimultiple good ideals I in A with $\mathrm{ht}_A I = s$. In [3] S. Goto, S. Iai, and K. Watanabe intensively studied $\mathfrak{m}$-primary good ideals I, that is, the case where $\mathrm{ht}_A I = d$, and gave many inspiring results. In [4] S. Goto and M. Kim successfully generalized some of results in [3] to those of equimultiple case with improvements. This research is in succession to [4]. The purpose of this paper is to prove the following.

THEOREM 1.1. *Let $a \in I$ such that $I^{n+1} = aI^n$ for some $n \geq 0$ and $\mu_A(I) \geq 2$. Suppose that A/I is a Cohen-Macaulay ring. Let $\mathrm{R} = \mathrm{R}(I)$. Then the following conditions are equivalent.*

(1) $\mathrm{K}_\mathrm{R} \cong \mathrm{R}_+$ *as graded* R*-modules.*
(2) $I^2 = aI$ *and* $aA : I \in \mathcal{X}_A^1$.

THEOREM 1.2. *Suppose that $I \in \mathcal{X}_A^1$ and $\mathrm{R} = \mathrm{R}(I)$. Then $\mathrm{K}_\mathrm{R} \cong \mathrm{R}_+$ as graded* R*-modules.*

Section 2 is devoted to some preliminaries. In Section 3, we will give the proofs of Theorem 1.1 and Theorem 1.2.

Before entering into details, let us fix again the standard notation in this paper. Throughout let $(A, \mathfrak{m})$ denote a Gorenstein local ring with $d = \dim A$. We denote by $\mu_A(*)$ the number of elements in a minimal system of generators and by $\ell_A(*)$ the length. Let $S = \oplus_{n\in\mathbb{Z}} S_n$ be a Noetherian graded ring and assume that S contains a unique graded maximal ideal $\mathfrak{M}$. We denote by $\mathrm{H}^i_{\mathfrak{M}}(*)$ $(i \in \mathbb{Z})$ the $i^{\underline{th}}$ local cohomology functor of S with respect to $\mathfrak{M}$. For each graded S-module E and $n \in \mathbb{Z}$, let $[\mathrm{H}^i_{\mathfrak{M}}(E)]_n$ denote the homogeneous component of the graded S-module $\mathrm{H}^i_{\mathfrak{M}}$ of degree n. If $S_n = (0)$ for all $n < 0$ and E is a finitely generated graded S-module, we have $[\mathrm{H}^i_{\mathfrak{M}}(E)]_n = (0)$ for all $n >> 0$ and $i \in \mathbb{Z}$. We put $\mathrm{a}(E) = \sup\{n \in \mathbb{Z} \mid [\mathrm{H}^s_{\mathfrak{M}}(E)]_n \neq (0)\}$ with $s = \dim_S E$ and call it the a-invariant of E ([5, (3.1.4)]). For each $n \in \mathbb{Z}$ let $E(n)$ stand for the graded S-module, whose underlying S-module coincide with that of E and whose graduation is given by $[E(n)]_i = E_{n+i}$ for all $i \in \mathbb{Z}$. We denote by K_S the graded canonical module of S, if it exists. We refer the reader to [1], [6], [7], or [8] for any unexplained notation or terminology.

2. Preliminaries

Let $s \in \mathbb{Z}$ and I $(\neq A)$ be an ideal in A with $\mathrm{ht}_A I = s$. The purpose of this section is to summarize some basic results on equimultiple good ideals.

Let us state the definition of equimultiple good ideals.

DEFINITION 2.1. We say that I is an *equimultiple good* ideal in A, if I contains a reduction $Q = (a_1, a_2, \cdots, a_s)$ generated by s elements a_i's in A and $\mathrm{G}(I)$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(I)) = 1 - s$.

Let $\mathcal{X}^s_A$ denote the set of equimultiple good ideals I in A with $\mathrm{ht}_A I = s$. In particular, if $s < 0$ or $s > d$, then $\mathcal{X}^s_A = \emptyset$. When I is an $\mathfrak{m}$-primary ideal, we denote $\mathcal{X}^d_A$ simply by $\mathcal{X}_A$ and we call it the set of good ideals in A ([3]).

Let us note here the next criterion, which gives us a basic characterization of equimultiple good ideals. For each Cohen-Macaulay local ring $(R, \mathfrak{n})$ with $\dim R = n$, let $\mathrm{r}(R) = \ell_R(\mathrm{Ext}^n_R(R/\mathfrak{n}, R))$ denote the Cohen-Macaulay type of R.

LEMMA 2.2. ([2, Proposition 2.4]) *Let I $(\neq A)$ be an ideal in A of height s and assume that I contains a reduction Q generated by s elements in A. Then the following conditions are equivalent.*

(1) $I \in \mathcal{X}^s_A$.
(2) $I^2 = QI$, $I = Q : I$, *and the ring A/I is Cohen-Macaulay.*

(3) *$I^2 = QI$, $IA_{\mathfrak{p}} \in \mathcal{X}^s_{A_{\mathfrak{p}}}$ for all $\mathfrak{p} \in \mathrm{Ass}_A A/I$, and the ring A/I is Cohen-Macaulay.*

(4) *The algebra $\mathrm{R}'(I)$ is a Gorenstein ring and $\mathrm{K}_{\mathrm{R}'(I)} \cong \mathrm{R}'(I)(2-s)$ as graded $\mathrm{R}'(I)$-modules.*

If $s > 0$, we may add the following.

(5) *$I^n = Q^n : I$ for all $n \in \mathbb{Z}$ and the ring A/I is Cohen-Macaulay.*

When this is the case, we have $\mathrm{K}_{A/I} \cong I/Q$, whence $\mathrm{r}(A/I) = \mu_A(I) - s \geq 1$.

PROPOSITION 2.3. *Let I ($\neq A$) be an ideal in A of $\mathrm{ht}_A I = s$ and assume that I contains a reduction Q generated by s elements in A. Assume that $d > s$ and $I = Q : I$. Then $\mathrm{depth} A/I > 0$.*

Proof. It is enough to show that $\mathfrak{m} \notin \mathrm{Ass}_A A/I$. Assume the contrary and choose an element x in A such that $\mathfrak{m} = I : x$. Then $\mathfrak{m}x \subseteq (Q : I)$ by our assumption. Since $\mathrm{depth} A/Q = d - s > 0$, $\mathfrak{m}/Q$ has a regular element on A/Q, so that $x \in (Q : I) = I$. This is impossible. $\square$

Let I be an $\mathfrak{m}$-primary ideal in A and assume that I contains a parameter ideal $Q = (a_1, a_2, \cdots, a_d)$ of A as a reduction. The ideal I is a good ideal in A if and only if $I^2 = QI$ and $I = Q : I$ ([3, Proposition 2.2]). The following Remark 2.4 is a consequence of Proposition 2.3, which shows that we do not need the condition of Cohen-Macaulayness of the ring A/I in the case where $\mathrm{ht}_A I = d - 1$.

REMARK 2.4. Let I ($\neq A$) be an ideal in A of $\mathrm{ht}_A I = s$ and assume that I contains a reduction Q generated by s elements in A. Let $d > 0$ and $s = d - 1$. Then

$$I \in \mathcal{X}^s_A \quad \text{if and only if} \quad I^2 = QI \text{ and } I = Q : I.$$

Proof. The ring A/I is a Cohen-Macaulay ring of $\dim A/I = 1$ by Proposition 2.3. Hence the assertion follows from Lemma 2.2. $\square$

We close this section with the following Theorem 2.5, which shows that the equimultiple good ideals behave well under flat ring extension.

THEOREM 2.5. *Let B be a Gorenstein local ring with a maximal ideal $\mathfrak{n}$ and let $\varphi : A \to B$ be a local homomorphism. Let I ($\neq A$) be an ideal in A of $\mathrm{ht}_A I = s$ and assume that I contains a reduction Q generated by s elements in A. Assume that B is flat over A. Then*

$$I \in \mathcal{X}_A^s \quad \text{if and only if} \quad IB \in \mathcal{X}_B^s.$$

Proof. The map $A/I \to B/IB$ is a local homomorphism and B/IB is flat over A/I, so that we have the equalities

$$\dim B = \dim A + \dim B/\mathfrak{m}B \quad \text{and} \quad \dim B/IB = \dim A/I + \dim B/\mathfrak{m}B$$

by [7, Theorem 15.1]. Hence we have $\mathrm{ht}_B IB = \mathrm{ht}_A I$. Since $B/\mathfrak{m}B$ is a Gorenstein local ring by [7, Theorem 23.4], we have that A/I is Cohen-Macaulay if and only if B/IB is Cohen-Macaulay by [7, Corollary to Theorem 23.3]. Therefore we conclude that

$$I \in \mathcal{X}_A^s \quad \text{if and only if} \quad IB \in \mathcal{X}_B^s$$

by Lemma 2.2 (2). □

3. Main theorems

Let I ($\neq A$) be an ideal in A with $\mathrm{ht}_A I = 1$ and assume that I contains an element a with $I^{n+1} = aI^n$ for some $n \geq 0$. Hence a is a nonzerodivisor of A. Let $K = Q(A)$ be the total quotient ring of A. We denote by $\mathcal{Y}_A$ the set of Gorenstein A-subalgebras C of K such that $C \supsetneq A$ but the A-module C is finitely generated. Let us note the next criterions, which play a key role in the present paper.

LEMMA 3.1. ([4, Lemma 2.1]) *Assume that $I^2 = aI$ but $I \neq aA$. Let $C = \{\frac{x}{a} \mid x \in I\}$ in K. Then*

(1) $A \subsetneq C = A[\frac{x}{a} \mid x \in I]$ and $I = aC$.
(2) $A :_K C = aA : I$ *and* $I :_K I = C$.
(3) $\mathrm{depth}_A C = d$ *if and only if the ring A/I is Cohen-Macaulay.*
(4) *C is a Gorenstein ring and $I = A :_K C$ if $I \in \mathcal{X}_A^1$.*

THEOREM 3.2. ([4, Theorem 2.2]) *The correspondence* $\mathcal{X}_A^1 \to \mathcal{Y}_A$, $I \mapsto I :_K I$ *is a bijection. It reverses the inclusion and takes back each* $C \in \mathcal{Y}_A$ *to* $A :_K C \in \mathcal{X}_A^1$. *We have* $\mathrm{K}_{A/A:_K C} \cong C/A$ *for all* $C \in \mathcal{Y}_A$.

As a consequence of Theorem 3.2 we get the following.

COROLLARY 3.3. *Assume that* $I^2 = aI$ *for some* $a \in A$. *Let* $\ell \geq 1$ *be an integer. Then* $\ell = 1$ *if* $I^\ell \in \mathcal{X}_A^1$.

Proof. Suppose $\ell \geq 2$. We have that $(I^\ell)^2 = a^\ell I^\ell$ and $A[\frac{I}{a}] = I^\ell/a^\ell$. We put $C = I^\ell/a^\ell$. Hence by Theorem 3.2 we see $C \in \mathcal{Y}_A$ so that $I^\ell = A :_K C$. However, $C = I/a$ because $I^\ell = a^{\ell-1}I$. Hence we have $I = aC$ and $aC \cdot C = aC = I \subseteq A$, whence $I \subseteq (A :_K C)$, which is impossible because $(A :_K C) = I^\ell$ and $\ell \geq 2$. Thus $\ell = 1$. □

We now prove the main Theorem, which gives a characterization of ideals I for which $aA : I \in \mathcal{X}_A^1$.

THEOREM 3.4. *Let* $a \in I$ *such that* $I^{n+1} = aI^n$ *for some* $n \geq 0$ *and* $\mu_A(I) \geq 2$. *Suppose that* A/I *is a Cohen-Macaulay ring. Let* $\mathrm{R} = \mathrm{R}(I)$. *Then the following conditions are equivalent.*

(1) $\mathrm{K_R} \cong \mathrm{R}_+$ *as graded* R*-modules.*
(2) $I^2 = aI$ *and* $aA : I \in \mathcal{X}_A^1$.

Proof. Let us apply the functor $\mathrm{H}^i_\mathfrak{m}(*)$ to the exact sequence $0 \to I \to A \to A/I \to 0$ of A-modules. Then we have $\mathrm{depth}_A I = d$, whence $\mathrm{depth}_A aI = d$ because $aI \cong I$ as A-modules. Applying the functor $\mathrm{H}^i_\mathfrak{m}(*)$ to exact sequence $0 \to aI \to A \to A/aI \to 0$ of A-modules, we get $\mathrm{depth}_A(A/aI) \geq d-1$, so that the ring A/aI is Cohen-Macaulay with $\dim(A/aI) = d-1$ because aI contains an A-regular element a^2. We need the following. □

CLAIM 1. $I^2 = aI$ *if* $\mathrm{K_R} \cong \mathrm{R}_+$.

Proof of Claim 1. Assume the contrary and choose $\mathfrak{p} \in \mathrm{Ass}_A(I^2/aI)$. Then $\mathfrak{p} \supseteq I$ and $\mathrm{ht}_A \mathfrak{p} = 1$, because $\mathfrak{p} \in \mathrm{Ass}_A(I^2/aI) \subseteq \mathrm{Ass}_A(A/aI) = \mathrm{Assh}_A(A/aI)$. Hence $I^2 A_\mathfrak{p} \supsetneq aIA_\mathfrak{p}$, so that we have $IA_\mathfrak{p} \supsetneq aA_\mathfrak{p}$. However we get that $\mathrm{K}_{\mathrm{R}_\mathfrak{p}} \cong (\mathrm{K_R})_\mathfrak{p} \cong (\mathrm{R}_+)_\mathfrak{p} \cong (\mathrm{R}_\mathfrak{p})_+$ by our assumption. By [3, Theorem 5.8] we have $I^2 A_\mathfrak{p} = aIA_\mathfrak{p}$, which is impossible. Thus $I^2 = aI$. This completes the proof of Claim 1. □

Thanks to Claim 1 we may assume that $I^2 = aI$. Let $C = I/a$ and we put $\mathcal{S} = \mathrm{R}_C(I) = \oplus_{n \geq o} I^n C$. We look at the exact sequence

$$(*) \qquad 0 \to \mathrm{R} \to \mathcal{S} \to \mathcal{S}/\mathrm{R} \to 0$$

of graded R-modules and note that $\mathcal{S}/\mathrm{R} = [\mathcal{S}/\mathrm{R}]_0 \cong C/A$ as A-modules.

CLAIM 2. *C/A is a Cohen-Macaulay A-module with* $\dim_A(C/A) \leq d-1$.

Proof of Claim 2. Since $I = aC \subseteq A$, we have $a \in (A : C)$, so that $\dim_A(C/A) \leq d-1$, because a is A-regular. Let us apply the functor $\mathrm{H}^i_{\mathfrak{m}}(*)$ to exact sequence

$$0 \to A \to C \to C/A \to 0$$

of A-modules. Then we get $\mathrm{depth}_A(C/A) \geq d-1$ by Lemma 3.1 (3), which implies the assertion of Claim 2.

Let $\mathcal{P} = \mathrm{R}(aA) = A[at]$. Let $\mathfrak{N} = \mathfrak{m}\mathcal{P} + \mathcal{P}_+$ and apply the functor $\mathrm{H}^i_{\mathfrak{N}}(*)$ to the exact sequence (*). By the Local Duality theorem for graded modules, we get the isomorphisms

$$\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{S}/\mathrm{R}, \mathrm{K}_{\mathcal{P}}) \cong (\mathrm{H}^{(d+1)-i}_{\mathfrak{N}}(\mathcal{S}/\mathrm{R}))^\vee \cong (\mathrm{H}^{(d+1)-i}_{\mathfrak{m}}(C/A))^\vee,$$

where $[*]^\vee$ denotes the Matlis Duality. Hence $\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{S}/\mathrm{R}, \mathrm{K}_{\mathcal{P}}) = 0$ for $i \neq 2$, by Claim 2. Let us apply the functor $\mathrm{Ext}^i_{\mathcal{P}}(*, \mathrm{K}_{\mathcal{P}})$ to the exact sequence (*). Then we get the isomorphism

$$\mathrm{Hom}_{\mathcal{P}}(\mathrm{R}, \mathrm{K}_{\mathcal{P}}) \cong \mathrm{Hom}_{\mathcal{P}}(\mathcal{S}, \mathrm{K}_{\mathcal{P}}).$$

Since $\mathcal{P} = A[at]$ is the polynomial ring over A, we have isomorphism

$$(**) \qquad \mathrm{K}_{\mathrm{R}} \cong [\mathrm{Hom}_{\mathcal{P}}(\mathcal{S}, \mathcal{P})](-1).$$

(1) $\Rightarrow$ (2) Suppose $\mathrm{K}_{\mathrm{R}} \cong \mathrm{R}_+$. Then we have that $\mathrm{K}_{\mathrm{R}} \cong \mathrm{R}_+ = \mathcal{S}_+ \cong \mathcal{S}(-1)$. Hence by (**) $\mathrm{Hom}_{\mathcal{P}}(\mathcal{S}, \mathcal{P}) \cong \mathrm{K}_{\mathrm{R}}(1)$, so that $\mathrm{Hom}_{\mathcal{P}}(\mathcal{S}, \mathcal{P}) \cong \mathcal{S}$. Thus $\mathrm{Hom}_A(C, A) \cong C$ as C-modules, whence $C_M \cong (\mathrm{K}_C)_M \cong \mathrm{K}_{C_M}$ for all $M \in \mathrm{Max}(C)$. Hence the ring C is Gorenstein. Therefore by Theorem 3.2 and Lemma 3.1 (2) $aA : I \in \mathcal{X}^1_A$.

(2) $\Rightarrow$ (1) Suppose $aA : I \in \mathcal{X}^1_A$. Then the ring C is Gorenstein by Lemma 3.1 (4). Hence $C \cong \mathrm{Hom}_A(C, A)$ as C-modules, so that $\mathcal{S} \cong \mathrm{Hom}_{\mathcal{P}}(\mathcal{S}, \mathcal{P})$. Thus by (**) we have that $\mathrm{K}_{\mathrm{R}} \cong \mathcal{S}(-1) \cong \mathcal{S}_+ = \mathrm{R}_+$. Therefore we complete the proof of Theorem 3.4. ☐

COROLLARY 3.5. ([3, Theorem 5.8]) *Let I be an $\mathfrak{m}$-primary ideal in A with* $\dim A = 1$. *Let $a \in A$ and assume that $I^{n+1} = aI^n$ for some $n \geq 0$. Let* $\mathrm{R} = \mathrm{R}(I)$. *Then the following conditions are equivalent.*

(1) $\mathrm{K_R} \cong \mathrm{R}_+$ *as graded* R*-modules.*
(2) $I^2 = aI$ *and* $aA : I \in \mathcal{X}_A$.

The following is an immediate consequence of Theorem 3.4. However we give a direct proof of Theorem 3.6.

THEOREM 3.6. *Suppose that $I \in \mathcal{X}_A^1$ and let* $\mathrm{R} = \mathrm{R}(I)$. *Then* $\mathrm{K_R} \cong \mathrm{R}_+$ *as graded* R*-modules.*

Proof. Let $a \in A$ such that $I^2 = aI$. We put $\mathcal{P} = \mathrm{R}(aA) = A[at]$. Then at is transcendental over A, because a is A-regular. Hence $\mathcal{P}$ is the polynomial ring over A, so that the ring $\mathcal{P}$ is Gorenstein with $\mathrm{a}(\mathcal{P}) = -1$. Thus $\mathrm{K}_\mathcal{P} \cong \mathcal{P}(-1)$, whence we have that

$$\begin{aligned} \mathrm{K_R} &\cong \mathrm{Hom}_\mathcal{P}(\mathrm{R}, \mathrm{K}_\mathcal{P}) \\ &\cong \mathrm{Hom}_\mathcal{P}(\mathrm{R}, \mathcal{P}(-1)) \\ &\cong [\mathrm{Hom}_\mathcal{P}(\mathrm{R}, \mathcal{P})](-1) \\ &\cong (\mathcal{P} : \mathrm{R})(-1), \end{aligned}$$

where $(\mathcal{P} : \mathrm{R})$ denotes the conductor. To get the conclusion, it suffices to claim the following. □

CLAIM. $(\mathcal{P} : \mathrm{R}) = I\mathrm{R} \cong \mathrm{R}_+(1)$.

If we could show this claim, we get that $\mathrm{K_R}(1) \cong (\mathcal{P} : \mathrm{R}) \cong \mathrm{R}_+(1)$, so that $\mathrm{K_R} \cong \mathrm{R}_+$.

Proof of Claim. We have $I\mathrm{R} = I\mathcal{P}$ as $I^{n+1} = a^n I$ for all $n \geq 0$, whence $I\mathrm{R} \subseteq (\mathcal{P} : \mathrm{R})$. Let $a^n t^n \in (\mathcal{P} : \mathrm{R})$ with $n \geq 0$. Then $a^n x = a^{n+1}$ for all $x \in I$. Hence by Proposition 2.2 (5) we have $a^n \in (a^{n+1} : I) = I^{n+1}$, so that $a^n t^n \in I\mathrm{R}$. Thus $(\mathcal{P} : \mathrm{R}) = I\mathrm{R} \cong \mathrm{R}_+(1)$. Therefore we complete the proof of Theorem 3.7. □

COROLLARY 3.7. ([3, Corollary 5.11]) *Let I be an $\mathfrak{m}$-primary ideal in A with* $\dim A = 1$. *Suppose that $I \in \mathcal{X}_A$ and let* $\mathrm{R} = \mathrm{R}(I)$. *Then* $\mathrm{K_R} \cong \mathrm{R}_+$ *as graded* R*-modules.*

ACKNOWLEDGMENT. The author would like to thank Professor S. Goto for his kind guidance.

References

[1] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge studies in advanced mathematics, vol. 39, Cambridge University, Cambridge·New York·Port Chester·Sydney, 1993.
[2] S. Goto, S. Iai, and M. Kim, *Good ideals in Gorenstein local rings obtained by idealization*, Proc. Amer. Math. Soc. (to appear)
[3] S. Goto, S. Iai, and K. Watanabe, *Good ideals in Gorenstein local rings*, Trans. Amer. Math. Soc. (to appear)
[4] S. Goto and M. Kim, *Equimultiple good ideals*, J. Math. Kyoto Univ. (to appear)
[5] S. Goto and K. Watanabe, *On graded rings, I*, J. Math. Soc. Japan **30** (1978), 179-213.
[6] J. Herzog and E. Kunz (eds.), *Der kanonische Modul eines Cohen-Macaulay-Rings*, Lecture Notes in Mathematics, vol. 238, Springer-Verlag, Berlin · Heidelberg · New York · Tokyo, 1971.
[7] H. Matsumura, *Commutative ring theory*, Cambridge University, Cambridge · London · Sydney, 1986.
[8] M. Nagata, *Local rings*, Interscience, 1962.
[9] P. Valabrega and G. Valla, *Form rings and regular sequences*, Nagoya Math. J. **72** (1978), 93–101.

Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Korea
E-mail: mkkim@math.skku.ac.kr