論文
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# Dynamics Oscillations in Suspension Bridges to Initial Conditions
# (현수교 다리에서의 초기치 문제에 대한 역학적 운동)

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## ABSTRACT

We model the torsional oscillation of a suspension bridge, which is the forced sine-Gordon equation on a bounded domain. We use finite difference method to solve nonlinear partial differential equation numerically. The partial differential equation has multiple periodic solutions. Whether the span oscillates with small or large amplitude depends only on its initial displacement and velocity. Moreover, we observe that the qualitative properties are consistent with the behavior observed at the Tacoma Narrows Bridge on the day of its collapse.

## 요 약

유계된 정의역에서 sine-Gordon 방정식인 현수교의 과격한 운동의 모델을 만든다. 유한 차분법을 이용하여 비선형 미분방정식을 수치 해석학적으로 푼다. 이 미분방정식은 다중 주기근을 가지고 있다. 다리가 큰 진폭이나 작은 진폭으로 진동하는 것은 초기의 변위와 속도에만 달려있다. 게다가, 많은 현상들이 Tacoma Narrows가 붕괴된 날에 관찰되는 것과 일치하고 있다.

## 1. Introduction

The Tacoma Narrows Bridge collapsed in 1940. There were the dramatic and destructive torsional oscillations of the Bridge at that time. Many scientists have tried to explain the cause of oscillations for over fifty years. We argue that nonlinear partial differential equations govern the motion of suspension bridges and that inherent nonlinearity gives rise to large amplitude oscillations. Theoretical and numerical evidence to support this claim for the vertical, torsional, and traveling wave motion of suspension bridges can be found in [2],[4],[5],[6], and [7].

We consider the entire length of the main span of the suspension bridge. More concretely, we will describe partial differential equation models for the torsional and coupled vertical-torsional motions of the center span. The equation which governs the torsional motion is the forced sine-Gordon equation on a bounded domain.

We use finite difference methods to compute periodic solutions to the torsional PDE. As in [5], we demonstrate that under small external forcing, the center span may oscillate periodically with small or large amplitude, depending only on its initial displacement and velocity. Moreover, we observe that the qualitative properties such as amplitude and

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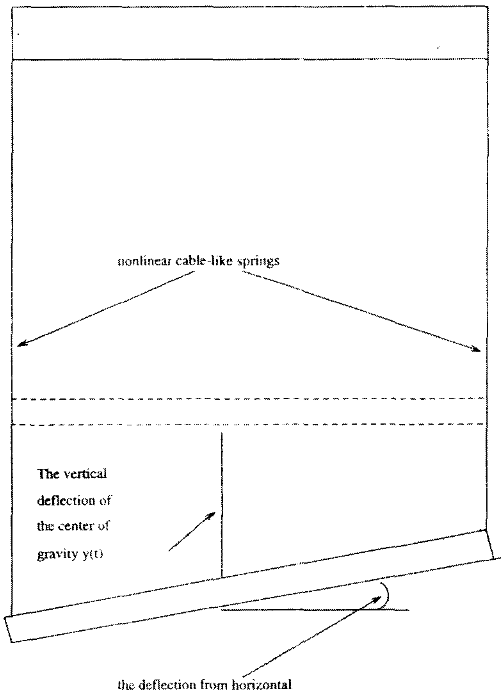
frequency of computed solutions are consistent with the behavior observed at the Tacoma Narrows Bridge on the day of its collapse.

## 2. Description of a Horizontal Cross Section of the Center Span

We will treat the center span of the bridge as a beam of length $L$ and width $2l$ suspended by cables. To model the motion of a horizontal cross section of the beam, we treat it as a rod of length $2l$ and mass $m$ suspended by cables. Let $y(t)$ denote the downward distance of the center of gravity of the rod from the unloaded state and let $\theta(t)$ denote the angle of the rod from horizontal at time $t$ [Fig. 1].




[Fig. 1] A Horizontal Cross Section of the Center Span

To model the torsional and vertical motion along the entire length of the center span, we consider the motion of the horizontal cross section located at position $x$ along the length of the span; thus $y$ and $\theta$ variables depend not only on time, but also on the position along the length of the span. Let $y(x,t)$ denote the downward distance from the unloaded state of the center of gravity of the cross section located at position $x$ at time $t$ and let $\theta(x,t)$ denote the angle of the cross section from horizontal.

The extension of the cable is $(y-l\sin\theta)$ in one side and $(y+l\sin\theta)$ in the other, and the force exerted by the cables is $-K(y-l\sin\theta)^+$ in one side and $-K(y+l\sin\theta)^+$ in the other. From [10] we have that the vertical and torsional motion satisfy the following equations,

$$\begin{aligned} \theta_{tt}-\varepsilon_1\theta_{xx} &= \frac{3K}{ml}\cos\theta[(y-l\sin\theta)^+-(y+l\sin\theta)^+] \\ &\quad -\delta_1\theta_t+f(x,t) \\ y_{tt}+\varepsilon_2 y_{xxxx} &= -\frac{K}{m}[(y-l\sin\theta)^++(y+l\sin\theta)^+] \\ &\quad -\delta_2 y_t+g \\ \theta(0,t) &= \theta(L,t)=y(0,t)=y(L,t)=y_{xx}(0,t) \\ &= y_{xx}(L,t)=0. \end{aligned} \tag{1}$$

Here, $\varepsilon_1, \varepsilon_2$ are physical constants related to the density and the shear modulus of elasticity of the beam, $\delta_1, \delta_2$ are damping constants, $f(x,t)$ is the external force, and $g$ is the force due to gravity. The boundary conditions reflect the fact that the ends of the span are hinged.

Assuming that the cables never lose tension, we have $y\pm l\sin\theta\geq 0$ and hence $(y\pm l\sin\theta)^+=y\pm l\sin\theta$. The equations (1) become uncoupled and the torsional and vertical motion satisfy the following equations (2) and (3), respectively.

$$\theta_{tt} - \varepsilon_1 \theta_{xx} = -\frac{6K}{m}\cos\theta\sin\theta - \delta_1\theta_t + f(x,t) \qquad (2)$$
$$\theta(0,t) = \theta(L,t) = 0$$

$$y_{tt} + \varepsilon_2 y_{xxxx} = -\frac{2K}{m}y - \delta_2 y_t + g \qquad (3)$$
$$y(0,t) = y(L,t) = y_{xx}(0,t) = y_{xx}(L,t) = 0$$

We observe that (2) is the forced, damped sine-Gordon equation, which arises in many applications. We approximate the periodic solutions using finite difference methods.

# 3. Numerical Results

Based on the previous experience in [1],[5], and [9], we expect to find interesting results. The interesting solutions are investigated when sufficient time has elapsed for the transient behaviors to have disappeared. Ahead of the experiments, the following section presents the physical constants and external forcing.

## 3.1 The Choice of Physical Constants and External Forcing

To determine the physical constants $K, m, l, L, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2$, and the external forcing terms $\overline{f}(t)$ and $f(x,t)$, we depend on [1],[5], and [9]. We choose $L = 1000, l = 6, m = 2500, \delta_1 = \delta_2 = 0.01$, $K = 1000, \varepsilon_1 = 0.01$, and $\varepsilon_2 = 0.0001$.

Then (2) becomes the following.

$$\theta_{tt} - \varepsilon\theta_{xx} + \delta\theta_t = -2.4\cos\theta\sin\theta + \lambda f(x,t)$$
$$\theta(0,t) = \theta(1,t) = 0 \qquad (4)$$
$$\theta(x,0) = \xi(x),\ \theta_t(x,0) = \eta(x)$$

Here, we choose $\varepsilon = \delta = 0.01$ and external forcing of the form

$$f(x,t) = \sin(\mu t)$$
$$f(x,t) = \sin(\mu t)\sin(2\pi x)$$

or

$$f(x,t) = \sin(\mu t)\sin(\pi x).$$

## 3.2 The Experiments

We apply the finite difference schemes to approximate solutions to the initial value problem

$$\theta_{tt} - \varepsilon\theta_{xx} + \delta\theta_t = -1.2\sin 2\theta + \lambda f(x,t)$$
$$\theta(0,t) = \theta(1,t) = 0 \qquad (5)$$
$$\theta(x,0) = \xi(x),\ \theta(x,\Delta t) = \psi(x)$$

In each experiment, we use specify the initial conditions $\overrightarrow{\theta_i^0} = \xi(x_i),\ \overrightarrow{\theta_i^1} = \psi(x_i),\ i = 1,\cdots,N-1$. We run the program for 400 periods of the forcing term $\lambda f(x,t) = \lambda\sin\mu t f(x)$, and examine the behavior of solution over the last 30 of the 400 periods.

In each of the experiments, we use 520 time steps per period of the forcing term, i.e., $\Delta t = \frac{2\pi}{520\mu}$ and we take $\Delta x = 0.025$.

For the discussion which follows, we define

$a$ = amplitude of the initial displacement $\overrightarrow{\theta^0}$

$a_p$ = amplitude of the resulting periodic solution.

### 3.2.1 Forcing which depends on time only

We begin by examining the response of the main span to small, time dependent forcing which is constant along the length of the span, specifically

$$\lambda f(x,t) = \lambda\sin\mu t$$

and initial conditions of the form

$$\theta(x,0)=\theta(x,\Delta t)=a\sin 2\pi x.$$

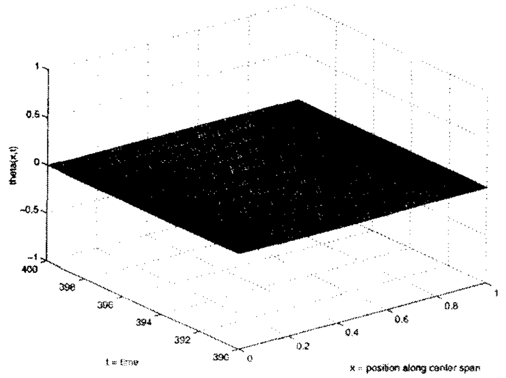
**Experiment** 1. In this experiment, we use $\lambda=0.04$ and $\mu=1.4$.

1a. We investigate the solution when $\theta(x,0)=\theta(x,\Delta t)=0.5\sin 2\pi x$.

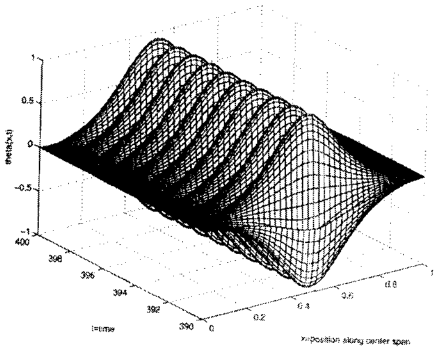
Despite the large initial displacement, we see in [Fig. 2] that by periods 390 through 400 of the forcing term, the span has settled down to no-noded, periodic oscillation of small amplitude (approximately 0.086 radians).

1b. We investigate the solution when $\theta(x,0)=\theta(x,\Delta t)=0.6\sin 2\pi x$.

We have increased the amplitude $a$ of the initial displacement only slightly different from 1a, but we see in [Fig. 3] that this small change has a dramatic impact on the motion of the span. As in 1a, by periods 390 through 400 of the forcing term, the span has settled down to periodic oscillation. But instead of settling to near equilibrium behavior, as in 1a, the amplitude of the oscillation is approximately 0.969 radians. This is close to the amplitude observed at the Tacoma Narrows Bridge on the day of the collapse, [1].




[Fig. 2] Experiment 1a: A small amplitude solution at $\lambda=0.04$, $\mu=1.4$. In all figures from [Fig. 2] to [Fig. 5], the $x$, $y$, and $z$ axis represent t=time, x=position along center span, and $\theta$(x,t), respectively.



[Fig. 3] Experiment 1b: A large amplitude periodic solution at $\lambda=0.04$, $\mu=1.4$.

### 3.2.2 One-noded forcing and initial conditions

The most prevalent motion observed at Tacoma Narrows was one-noded (no displacement at the center of the span) [1], so let us consider external forcing of the form

$$\lambda f(x,t)=\lambda\sin\mu t\sin 2\pi x$$

and initial conditions of the form

$$\theta(x,0)=\theta(x,\Delta t)=a\sin 2\pi x.$$

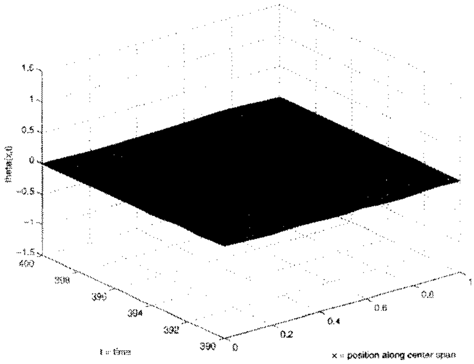
**Experiment** 2. In this experiment, we use $\lambda=0.06$ and $\mu=1.4$.

2a. We investigate the solution when $\theta(x,0)=\theta(x,\Delta t)=0.9\sin 2\pi x$.

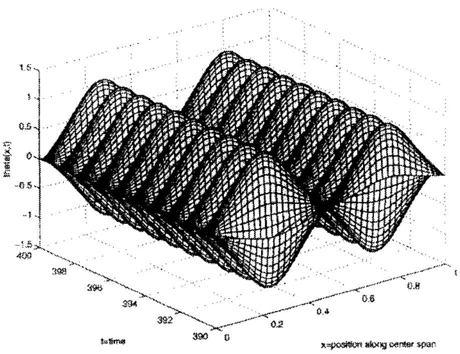
Despite the large initial displacement, we see in [Fig. 4] that by periods 390 through 400 of the forcing term, the span has settled down to one-noded, periodic oscillation of small amplitude (approximately 0.072 radians).

2b. We investigate the solution when $\theta(x,0)=\theta(x,\Delta t)=1.0\sin 2\pi x$.

We have increased the amplitude $a$ of the initial displacement only slightly different from 1a, but we see in [Fig. 5] that this change has a dramatic impact on the motion of the span. As in 1a, by periods 390 through 400 of the forcing term, the span has settled down to periodic oscillation. But instead of settling to near equilibrium behavior, as in 1a, the amplitude of the oscillation is approximately 1.117 radians. Again, we note that this is close to the amplitude observed at Tacoma Narrows on the day of the collapse, [5].



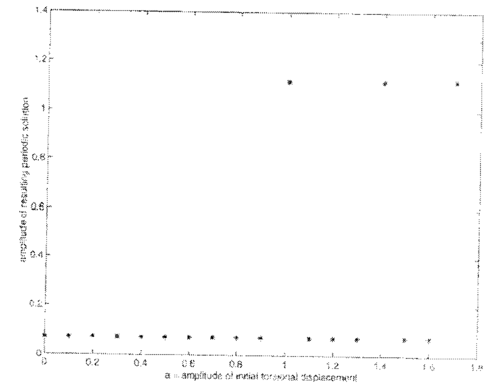

[Fig. 4] Experiment 2a: A small amplitude solution at $\lambda = 0.06$, $\mu = 1.4$.




[Fig. 5] Experiment 2b: A large amplitude periodic solution at $\lambda = 0.06$, $\mu = 1.4$.

2c. Based on our results in experiments 1 and 2, it is tempting to conjecture that the amplitude $a_p$ of the periodic solution increases with the amplitude $a$ of the initial displacement, but this is not the case. [Fig. 6] shows the amplitude $a_p$ of the periodic solution versus the amplitude $a$ of the initial displacement of the span for $a \in [0, 1.7]$.

We see in experiments 1 and 2 that a large initial displacement may result in large or small amplitude periodic oscillations.




[Fig. 6] Experiment 2c: Dependence on initial conditions. The $x$ and $y$ axis represent a=amplitude of initial torsional displacement and amplitude of resulting periodic solution, respectively.

## 4. Conclusion

All numerical results above were only approximately solved by finite difference method. From the various numerical experiment, there are rich phenomena associated with the oscillation of the Tacoma Narrows Bridge. In this section, we try to summarize the results of the numerical experiments which we have described in the previous section.

1. For $\mu \in [1.2, 1.4]$, under small external forcing $\lambda f(x,t) = \lambda \sin \mu t f(x)$, small or large amplitude periodic oscillation may result; the outcome depends on the amplitude $a$ of the initial displacement $\overrightarrow{\theta^0}$.

2. The amplitude $a_p$ of the periodic solution does not depend on $a$ in an intuitive way, for example, it does not increase with $a$.

3. The qualitative properties such as amplitude and frequency of large amplitude solutions are consistent with the behavior observed at the Tacoma Narrows Bridge on the day of its collapse.

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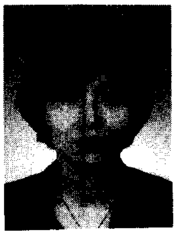

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