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비국소 경계 조건들을 가진 상미분 방정식들의 반무한 구간 상에서 근들의 존재성

Existence of Solutions on a Semi-Infinite Interval for Ordinary Differential Equation with Nonlocal Boundary Conditions

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<Abstract>

Motivated by the problem of steady-state heat conduction in a rod whose heat flux at one end is determined by observation of the temperature and heat flux at some point ξ in the interior of the rod, we consider the problem

$$y''(x) = a(x, y(x))\, y(x) \qquad (0 < x),$$
$$\lim_{x\to\infty} y(x) = 0, \quad y'(0) = g(y(\xi), y'(\xi))$$

for some fixed $\xi \in (0, \infty)$. We establish conditions guaranteeing existence and uniqueness for this problem on the semi-infinite interval $[0, \infty)$.

key words *: Existence, Semi-infinite Interval, Ordinary Differential Equations*

Introduction

In this paper we examine the question of existence of solutions for the problem

$$y''(x) = a(x, y(x))\, y(x) \quad (0 < x)$$
$$\lim_{x\to\infty} y(x) = 0, \quad y'(0) = g(y(\xi), y'(\xi)) \qquad (1)$$

for some fixed $\xi \in (0, \infty)$. Our approach to the problem on a semi-infinite domain is to consider the family of finite-interval problems

$$y''(x) = a(x, y(x))\, y(x) \qquad (0 < x < N)$$
$$y(N) = 0, \quad y'(0) = g(y(\xi), y'(\xi)) \qquad (2)$$

where ξ is fixed and $N > \xi$ is a parameter. Our goal is to show that a subsequence of the solutions of (2) converges uniformly on $[0, \infty)$ to a solution of (1) as $N \to \infty$; we use the Extended Ascoli-Arzela Lemma proved in [4]. The change of variable $x \to 1 - \frac{x}{N}$ reduces (2) to the problem studied in [5]:

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$$u_N''(x) = A(N, x, u_N(x))\, u_N(x) \quad (0<x<1)$$
$$u_N(0)=0, \quad u_N'(1) = G(u_N(\zeta), \; u_N'(\zeta)) \qquad (3)$$

Where $A(N,x,u_N)=N^2 a(N(1-x); u_N)$, $\zeta=1-\frac{\xi}{N}$ and $G(p,q)=-Ng(p,\frac{-q}{N})$.

Existence and Uniqueness on a Semi-infinite Interval

Theorem 1. *Let $\xi\in(0, \infty)$; let $a(x, y)$ be continuous and bounded below by some $\sigma^2>0$ on $[0,\infty)\times(-\infty,\infty)$; and let $g\in C(R\times R)$ satisfy one of*

i. g is nondecreasing in the first argument and nonincreasing in the second.

ii. g is nonincreasing in the first argument, nondecreasing in the second., and there exists γ, $0\leq\gamma<1$ such that for all z_i and x_i with $z_i\geq x_i$, $i=1, 2$, the inequality $0\leq g(x_1, -x_2)-g(z_1, -z_2)\leq\gamma(z_2-x_2)$ holds. Then (1) has a solution.

Proof. By Theorem 1 [5,I] in case hypothesis (i) holds and Theorem 1 [5,II] in the case of (ii), we see that (3) and hence (2) has solutions for each N> ξ. Extend any solution y_N of (2) to $[0,\infty)$ by setting $y_N(x)=0$ for x >N; then the extended solution is continuous on $[0,\infty)$. Let F be the set of these extended solutions y_N. To show that F is equicontinuous, it will suffice to show that $|y_N'(x)|$ is bounded uniformly on $[0, \infty)$. Suppose that $y_N(0)>0$ and y_N has a positive local maximum at $\hat{x}$. Then $y_N''(\hat{x})>0$, contradiction. Similarly, y_N has no negative local minimum either. Therefore, if $y_N(0)>0$, $y_N'\leq0$ and y_N' is nondecreasing. That is, $y_N'(0)\leq y_N'(x)\leq0$ and $y_N(0)\geq y_N(x)\geq0$ on [0, N]. Therefore, it is enough to bound $|y_N'(0)|$. First, let g satisfy (i). Then $y_N'(0)\geq g(0,0)$ independent of N. Hence, we obtain that for $x\in[0,\infty)$

$$0\geq y_N'(x)\geq y_N'(0)\geq g(0,0).$$

Now, let g satisfy (ii). Then $y_N'(0)\geq g(y_N(0), y_N'(0))$. Since g is nonincreasing in the first argument and $y_N(0)\leq -y_N'(0)N$, we have

$$-y_N'(0)+g(0,0)$$
$$\leq -g(y_N(0), y_N'(0))+g(0,0)$$
$$\leq -g(-y_N'(0)N, y_N'(0))+g(0,0)$$

or

$$-y_N'(0)\leq\frac{-g(0,0)}{1-\gamma}$$

For both conditions (i) and (ii), $y\equiv0$ for all x if $y(0)=0$ and a symmetric argument holds for the case $y_N(0)<0$. This proves that F is equicontinuous.

Continuing to consider in detail the case $y_N>0$, $y_N'<0$, we integrate $y_N' y_N''\leq\sigma^2 y_N y_N'$ over the interval [x, N] to conclude that

$$y_N'(N)^2 - y_N'(x)^2$$
$$\leq\sigma^2[y_N(N)^2 - y_N(x)^2]$$

so that $y_N'(x)^2\geq\sigma^2 y_N(x)^2$. From the preceding paragraph, it follows that there is a constant C_1 independent of N such that $|y_N'(x)|\leq C_1$; in fact, $C_1=|g(0,0)|$ if (i) holds and $C_1=|g(0,0)|/(1-\gamma)$ if (ii) holds. From $|y_N(x)|\leq|y_N'(x)|/\sigma\leq C_1/\sigma\equiv C_0$, we see that there is a constant C_0 independent of N such that $|y_N(x)|\leq C_0$ uniformly in N. By integrating $-y_N'(x)\geq\sigma\, y_N(x)$ over [0, x], we obtain

$$y_N(x)\leq y_N(0)\, e^{-\sigma x}\leq C_0\, e^{-\sigma x},$$

which implies that $y_N(x)$ is bounded by $\pm C_0 e^{-\sigma x}$. Since the functions that bound y_N approach 0 as $x\rightarrow\infty$, by the Extended Ascoli-Arzela Lemma (stated below), there exists a sequence, which we continue to denote by y_N, that converges uniformly on $[0,\infty)$ to a continuous function y satisfying $\lim_{x\to\infty} y(x)=0$. Without loss of generality $\{y_N'(0)\}$ also converges. Since

$$y_N'(x) = y_N'(0)+\int_0^x a(s, y_N(s))\, y_N(s)ds$$

it follows that $\{y_N'(x)\}$ also converges uniformly on each [0, X], say to $p(x)$.

Because differentiation is a closed operator, y is differentiable and $y'(x)=p(x)$. From the integral equation

$$y_N(x)= y_N(0)+x\, y_N'(0) + \int_0^x (x-s)a(s,\ y_N(s))\ y_N(s)ds,$$

we see on passing to the limit as $N\to\infty$ that y is a solution of the differential equation; passage to the limit in $y_N'(0)=g(\ y_N(\xi),\ y_N'(\xi))$ finishes the proof that (1) is satisfied.

Lemma. *(Extended Ascoli-Arzela Lemma) Let F be an infinite, equicontinuous set of functions defined on $[0,\ \infty)$, and suppose there exist continuous functions g and h such that $\lim_{x\to\infty} g(x)=\lim_{x\to\infty} h(x)=L$ and $g(x)\le f(x)\le h(x)$ on $[0,\infty)$ for all $f\in F$. Then F contains a sequence $\{f_n\}_{n=1}^{\infty}$ such that f_n converges uniformly on $[0,\infty)$ to some q satisfying $\lim_{x\to\infty} q(x)=L$.*

Finally, we examine uniqueness for the problem (1) on a semi-infinite interval.

Themrem 2. *Let $\xi\in(0,\ \infty)$; let a be continuous on $[0,\ \infty)\times(-\infty,\infty)$; let $a(\cdot,\ z)z$ be strictly increasing in z; let $g\in C(R\times R)$ satisfy either (i) or (ii) of Theorem 1. Then the solutions of (1) are unique.*

Proof. Suppose that u and v are two solutions of (1). Since any solution, y, of (1) satisfies $\lim_{x\to\infty} y(x)=0$, both u and v are solutions of the integro-differential equation

$$y'(x)=-\int_x^\infty a(s,y(s))y(s)ds.$$

We divide our argument into three cases.

Case 1: $v(x)>u(x)$ on $[0,\ \infty)$. Then we have

$$v'(x)=-\int_x^\infty a(s,v(s))v(s)ds < -\int_x^\infty a(s,u(s))u(s)ds=u'(x) \qquad (4)$$

First, let g satisfy the condition (i) of Theorem 1. Then

$$v'(0)=g(v(\xi),v'(\xi))>g(u(\xi),u'(\xi))=u'(0),$$

contradicting (4). Now, let g satisfy the condition (ii) of Theorem 1; then we have

$$\begin{aligned}0&\le u'(0)-v'(0)\\ &=g(u(\xi),u'(\xi))-g(v(\xi),v'(\xi))\\ &\le\gamma(u'(\xi)-v'(\xi)).\end{aligned}$$

Since

$$v'(\xi)-v'(0)=\int_0^\xi a(s,v(s))v(s)ds \ge \int_0^\xi a(s,u(s))u(s)ds=u'(\xi)-u'(0)$$

We have further that

$$\begin{aligned}u'(\xi)-v'(\xi)&\le u'(0)-v'(0)\\ &\le\gamma(u'(\xi)-v'(\xi)),\end{aligned}$$

which is impossible unless $u'(\xi)=v'(\xi)$. From this it follows from (4) that

$$\int_\xi^\infty [a(s,v(s))v(s)-a(s,u(s))u(s)]ds=0,$$

which implies that $v(x)\equiv u(x)$ on $[\xi,\infty)$, contradicting the hypothesis $v>u$ on $[0,\ \infty)$.

Case 2 : $v(x)>u(x)$ on $[0,\ \alpha)$ and $v(x)<u(x)$ on $(\alpha,\ \infty)$ for some α. Then

$$v(\alpha)=u(\alpha) \text{ and } v'(\alpha)\le u'(\alpha) \qquad (5)$$

Since u and v both solve the integral equation

$$y(x)=\int_x^\infty\int_t^\infty a(s,y(s))y(s)dsdt,$$

we have

$$v(\alpha)=\int_\alpha^\infty\int_t^\infty a(s,v(s))v(s)ds\,dt < \int_\alpha^\infty\int_t^\infty a(s,u(s))u(s)ds\,dt=u(\alpha).$$

This contradicts (5).

Case 3 : u and v intersect at least twice. Suppose that there is an interval $(\alpha_1,\ \alpha_2)\subset(0,\infty)$ such that $u(\alpha_i)=v(\alpha_i)$ and $u(x)<v(x)$ on $(\alpha_1,\ \alpha_2)$. Then

$$(v-u)'(\alpha_1)\ge0 \text{ and } (v-u)'(\alpha_2)\le0 \quad (6)$$

However,

$$(v-u)'(\alpha_1)-(v-u)'(\alpha_2)$$
$$=\int_{\alpha_1}^{\alpha_2}[a(s,u(s))u(s)-a(s,v(s))v(s)]ds<0$$

This final contradiction to (6) completes the proof of the theorem.

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