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HARMONIC MAPPING

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Abstract. In this paper, we obtain some coefficient bounds of harmonic, orientation-preserving, univalent mappings defined on $\Delta = \{z : |z| > 1\}$.

1. Introduction

A continuous function $f = u + iv$ defined in a domain D $\subseteqq \mathbb{C}$ is harmonic in D if u and v are real harmonic in D. Let Σ be the set of all complex-valued, harmonic, orientation-preserving, univalent mappings

$$f(z) = h(z) + \overline{g(z)} + Alog|z| \tag{1.1}$$

of $\Delta = \{z : |z| > 1\}$, where

$$h(z) = z + \sum_{k=1}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in Δ and $A \in \mathbb{C}$. Hengartner and Schober[3] used the representation (1.1) to obtain some coefficient estimates and distortion theorems. Some coefficient bounds for $f \in \Sigma$ are also obtained by Jun[4] when $f(\Delta) = \Delta$.

In this article, we obtain some coefficient bounds of $f \in \Sigma$ by using properties of the analytic function $h - g$.

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2. Some coefficient bounds

Let S be the class of functions $H(z) = z + \sum_{n=0}^{\infty} c_n z^{-n}$ that are analytic and univalent for $|z| > 1$. The coefficient problems for this class appear to be difficult. One reason is that there can be no single extremal function for all coefficients. Sharp bounds for $|c_n|$ are known only for $1 \leq n \leq 3$:

$$|c_1| \leq 1 \ \ [2], \ \ |c_2| \leq \frac{2}{3} \ \ [5], \ \ |c_3| \leq \frac{1}{2} + e^{-6} \ \ [1]. \tag{2.1}$$

Although information beyond the third coefficient is incomplete, we have some useful results proved by Schober and Williams[6].

Theorem 2.1[6]. *Let $H \in S$.*

(1) *If c_1 is real, then $Re\{c_4 + 4c_1\} \leq 4$.*
(2) *If c_1 and c_2 are real, and $c_1 \geq \frac{1}{3}$, then $Re\{c_5 + c_3 + c_1\} \leq 1$.*
(3) *If c_1 and c_2 are real, then $Re\{c_5 + c_3 - 2c_1\} \leq 2$.*
(4) *If c_1 is real, and c_2 is imaginary, then $Re\{5c_1 - c_5\} \leq 5$.*

Corollary 2.2. *Let $H \in S$. Then we have*

(1) *$Re\{c_4\} \leq 8$ if c_1 is real,*
(2) $$Re\{c_5\} = \begin{cases} \leq \frac{7}{6} + e^{-6} & \text{if } c_1 \text{ and } c_2 \text{ are real, and } c_1 \geq \frac{1}{3} \\ \leq \frac{9}{2} + e^{-6} & \text{if } c_1 \text{ and } c_2 \text{ are real} \\ \geq -10 & \text{if } c_1 \text{ is real, and } c_2 \text{ is imaginary.} \end{cases}$$

Proof. By simple calculation, one can easily show this from (2.1) and Theorem 2.1. □

Theorem 2.3. *If $f \in \Sigma$ and $h - g$ is univalent, then*

$$|a_1 - b_1| \leq 1, \ \ |a_2 - b_2| \leq \frac{2}{3}, \ \ |a_3 - b_3| \leq \frac{1}{2} + e^{-6}.$$

Proof. Let $H(z) = h(z) - g(z) = z + \sum_{n=1}^{\infty} c_n z^{-n}$, where $c_n = a_n - b_n$. Then $H(z)$ is univalent and analytic. Thus $H(z) \in S$. By using the sharp bounds for $|c_n|$ in (2.1), we obtain our sharp estimates. □

Theorem 2.4. *For each $f \in \Sigma$ with univalent $h - g$, we have*

(1) $Re\{a_4 - b_4\} \leq 8$ *if* $a_1 - b_1$ *is real,*

$$(2)\ Re\{a_5 - b_5\} = \begin{cases} \leq \frac{7}{6} + e^{-6} \text{ if } a_1 - b_1 \text{ and } a_2 - b_2 \text{ are real,} \\ \qquad \text{and } a_1 - b_1 \geq \frac{1}{3} \\ \leq \frac{9}{2} + e^{-6} \text{ if } a_1 - b_1 \text{ and } a_2 - b_2 \text{ are real} \\ \geq -10 \quad \text{ if } a_1 - b_1 \text{ is real,} \\ \qquad \text{and } a_2 - b_2 \text{ is imaginary.} \end{cases}$$

Proof. $H(z) = h(z) - g(z) \in S$. Thus we have estimates by Corollary 2.2. □

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