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HAMILTONICITY OF QUASI-RANDOM GRAPHS

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Abstract. It is well known that a random graph $G_{1/2}(n)$ is Hamiltonian almost surely. In this paper, we show that every quasi-random graph $G(n)$ with minimum degree $(1 + o(1))n/2$ is also Hamiltonian.

1. Introduction

Let us consider the random graph model for graphs with n vertices and edge probability $p = 1/2$. Thus the probability space $\Omega(n)$ consists of all labeled graphs G of order n, and the probability $P(G)$ of G in $\Omega(n)$ is given by $P(G) = 2^{-\binom{n}{2}}$. For a graph property $\mathcal{P}$, it may happen that

$$P\{G \in \Omega(n) \mid G \text{ satisfies } \mathcal{P}\} \to 1 \text{ as } n \to \infty.$$

In this case, a typical graph in $\Omega(n)$, which we denote by $G_{1/2}(n)$, will have property $\mathcal{P}$ with overwhelming probability as n becomes large. We abbreviate this by saying that a random graph $G_{1/2}(n)$ has property $\mathcal{P}$ *almost surely*. For details of these concepts, see [1] or [8].

One would like to construct graphs that behave just like a random graph $G_{1/2}(n)$. Of course, it is logically impossible to construct a truly random graph. Thus Chung, Graham, and Wilson defined in [4] quasi-random graphs, which simulate $G_{1/2}(n)$ without much deviation. Among many equivalent quasi-random properties studied in [4] and [3], we list only three needed in this paper. Let $G(n)$ denote a graph on n vertices. A family $\{G(n)\}$ of graphs (or for brevity, a graph $G = G(n)$) is *quasi-random* if it satisfies any one of and hence all of the following.

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$P_1(s)$: For fixed s, each labeled graph $M(s)$ on s vertices occurs $(1+o(1))n^s/2^{\binom{s}{2}}$ times as an induced subgraph of G.

P_4: For each subset $S \subseteq V(G)$, the number $e(S)$ of edges in $G[S]$ is $e(S) = \frac{1}{4}|S|^2 + o(n^2)$. Here, $G[S]$ denotes the subgraph of G induced by S.

Q: For each subset $S \subseteq V(G)$, the number $e(S, \overline{S})$ of edges between S and $\overline{S}$ satisfies $e(S, \overline{S}) = \frac{1}{2}|S||\overline{S}| + o(n^2)$, where $\overline{S} = V(G) - S$.

Another property of $G(n)$, which is weaker than quasi-randomness, is the following.

P_0': All but $o(n)$ vertices have degree $(1+o(1))\frac{n}{2}$. In this case we say that $G(n)$ is *almost-regular*.

Note that the Paley graph Q_p on p vertices is quasi-random [4] and strongly regular with parameters $((p-1)/2, (p-5)/4, (p-1)/4)$ [1].

We showed in [7] how much quasi-random graphs deviate from random graphs $G_{1/2}(n)$ in connectedness. In this paper, we show the same in Hamiltonicity. All definitions and notation are the same as in [4] and [3].

2. The Main Result

In this section, we investigate the Hamiltonicity of quasi-random graphs. To do this we estimate the independence number $\beta(G(n))$ and the connectivity $\kappa(G(n))$ of a quasi-random graph $G(n)$. We know that $G_{1/2}(n)$ has independence number $r(n)-1$ or $r(n)$ almost surely for some integer $r(n)$ such that $r(n) \sim 2\log_2 n$ [1]. But quasi-random graphs satisfy the following.

THEOREM 1. *Let $G = G(n)$ be a quasi-random graph on n vertices. Then the independence number $\beta(G)$ of G satisfies $\beta(G) = o(n)$ and is bounded away from zero by any positive constant.*

Proof. Let S be any independent set of vertices of G. Then from property P_4, we have $e(S) = |S|^2/4 + o(n^2) = 0$ and so $|S| = o(n)$. Thus, $\beta(G) = o(n)$.

Let l be any fixed number. Then property $P_1(s)$ implies that for sufficiently large n, G contains a copy of $\overline{K_l}$, an empty graph of order l, as an induced subgraph. Therefore, $\beta(G) \geq (1+o(1))l$. $\square$

We know that $G_{1/2}$ has connectivity equal to the minimum degree almost surely [1]. For quasi-random graphs, we have the following.

Theorem 2. *Let $G = G(n)$ be a quasi-random graph on n vertices. If $\delta(G) = (1 + o(1))n/2$, then*

$$\kappa(G) = (1 + o(1))\frac{n}{2} = \delta(G).$$

Proof. Let $\delta(G(n)) = (1 + o(1))n/2$. Then we can see from Corollary 3 in [7] that G is connected. Let S be a subset of vertices of G such that the removal of all vertices in S results in a disconnected graph. Since $\kappa(G) \leq \delta(G)$, we may assume that $|S| \leq \delta(G)$. Therefore for a given $0 < \epsilon < 1$, there exists $n_0(\epsilon)$ such that

$$|S| \leq \delta(G) \leq (1 + \epsilon)\frac{n}{2}$$

for all $n \geq n_0$. Hence

$$|V - S| \geq n - (1 + \epsilon)\frac{n}{2} = (1 - \epsilon)\frac{n}{2}$$

for all $n \geq n_0$. Therefore, the induced subgraph $H = G[V - S]$ is quasi-random by Corollary 1 in [7] and is disconnected. Hence, by the Theorem in [7], a smallest component of H has order $o(n)$. But such a component together with S contains at least $\delta(G) + 1$ vertices, that is, $|S| + o(n) \geq \delta(G) + 1$ and so $|S| \geq \delta(G) + o(n)$. Hence, we have

$$\delta(G) + o(n) \leq \kappa(G) \leq \delta(G)$$

and so

$$\kappa(G) = \delta(G) + o(n) = (1 + o(1))\frac{n}{2}.$$

□

It is well known that every $G_{1/2}(n)$ is Hamiltonian almost surely. But as we have already seen in [7], there is a quasi-random graph $G(n)$ that is not even connected and hence not Hamiltonian. However, once again imposing appropriate degree restrictions on quasi-random graphs, this can be corrected. The Chvátal-Erdös theorem says that if a graph G has at least three vertices and $\beta(G) \leq \kappa(G)$, then G is Hamiltonian [5]. Hence the following is immediate from Theorems 1 and 2.

Corollary 3. *Let $G = G(n)$ be a quasi-random graph on n vertices. If $\delta(G) = (1 + o(1))n/2$, then G is Hamiltonian.* □

Even in case that a given quasi-random graph is not Hamiltonian, it contains a sufficiently large cycle.

COROLLARY 4. *Let $G = G(n)$ be a quasi-random graph on n vertices. Then G has a cycle of length $(1 + o(1))n$.*

Proof. Let $G = G(n) = (V, E)$ be a quasi-random graph and let $H = H(m) = (W, F)$ denote the subgraph of $G(n)$ induced by

$$S = \{v \in V |\, \deg_G(v) \geq (1 + o(1))\frac{n}{2}\}.$$

Then observe that

(1) $H(m)$ is a quasi-random graph in its own right by Corollary 1 in [7],

(2) $m = |W| = |S| = (1 + o(1))n$ and $\deg_H(v) \geq (1 + o(1))n/2 \geq (1 + o(1))m/2$ for all v in W, and

(3) $H(m)$ is connected for sufficiently large m by Corollary 3 in [7].

Therefore, $H(m)$ is a Hamiltonian subgraph with $m = (1 + o(1))n$ vertices. □

3. Examples

In this section, we find some examples of quasi-random graphs that are Hamiltonian.

EXAMPLE 5. Let p be a prime satisfying $p \equiv 1 \pmod 4$. Then the Paley graph Q_p on p vertices is quasi-random and $(p - 1)/2$-regular. Hence, by Corollary 3, both Q_p and the complement $\overline{Q_p}$ are Hamiltonian for sufficiently large p. Of course, it follows immediately from its definition that Q_p is Hamiltonian for all p.

EXAMPLE 6. Let F_n be a field with n elements (of course, n must be a positive power of a prime) and let $AP(F_n)$ be the affine plane of order n. Let S be a subset of "slopes" of the $n + 1$ parallel classes of lines such that $|S| \sim n/2$. We define a graph $G(n^2) = (V, E)$ as follows. Let V be the set of all points in $AP(F_n)$ and let $xy \in E$ if and only if the slope of the line in $AP(F_n)$ containing x and y belongs to S. Then $G(n^2)$ is a quasi-random graph of order n^2 [4], and every vertex of $G(n^2)$ has degree $(n - 1)|S| \sim n^2/2$. Hence, by Corollary 3, both $G(n^2)$ and the complement $\overline{G(n^2)}$ are Hamiltonian for sufficiently large n.

EXAMPLE 7. We define a graph $G_n = (V, E)$ as follows. Let V be the set of all n-subsets of a fixed $2n$-set and let $xy \in E$ iff $|x \cap y| \equiv 0$

(mod 2). Then G_n is a quasi-random graph of order $\binom{2n}{n}$ (see [4] or [2]). Every vertex v of G_n has degree

$$\deg(v) = \begin{cases} \frac{1}{2}\binom{2n}{n} & \text{if } n \text{ is odd} \\ \frac{1}{2}\binom{2n}{n} + \frac{(-1)^{n/2}}{2}\binom{n}{n/2} - 1 & \text{if } n \text{ is even} \end{cases}$$

and hence $\deg(v) \sim \frac{1}{2}\binom{2n}{n}$. Therefore both G_n and the complement $\overline{G_n}$ are Hamiltonian for sufficiently large n. Of course, it follows immediately from Dirac's theorem [6] that G_n is Hamiltonian when n is odd or when $n/2$ is even. However, it seems to be difficult to show without using quasi-randomness that G_n is Hamiltonian when $n/2$ is a sufficiently large odd integer.

Remark 8. Let $G = G(n)$ be a quasi-random graph. We showed in Theorem 1 that $\beta(G) = o(n)$ and $\beta(G)$ is bounded away from zero by any positive constant. Can we find better bounds for independence numbers of quasi-random graphs? Consider the following example. Let $G = G(n)$ be any quasi-random graph and let a_n be any sequence of positive numbers such that $a_n = o(n)$. Choose any $X \subseteq V(G)$ with $|X| = \lceil a_n \rceil$ and remove all edges in $G[X]$ from G. Then the resulting graph $H(n)$ is quasi-random and $\lceil a_n \rceil \leq \beta(H(n)) = o(n)$. This example shows that there is a quasi-random graph whose independence number is at least a_n for any $a_n = o(n)$.

Let $G = G(n)$ be any graph of order n and let $X \subseteq V(G)$. Then we know from [2] that

$$e(G) \geq \frac{n^2}{4}\Big(1 - (devG)^{\frac{1}{4}}\Big)$$

and that

$$devG[X] \leq \Big(\frac{n}{|X|}\Big)^4 devG.$$

Hence, we obtain

$$\begin{aligned} e(G(X)) &\geq \frac{|X|^2}{4}\Big(1 - (devG[X])^{\frac{1}{4}}\Big) \\ &\geq \frac{|X|^2}{4}\Big(1 - \frac{n}{|X|}(devG)^{\frac{1}{4}}\Big). \end{aligned}$$

Now, we assume that $X \subseteq V$ is an independent set of G. Then $e(G[X]) = 0$ and hence we have

$$\frac{|X|^2}{4}\Big(1 - \frac{n}{|X|}(devG)^{\frac{1}{4}}\Big) \leq 0.$$

Therefore, we have

$$|X| \leq n(devG)^{\frac{1}{4}}$$

and hence we have

$$\beta(G) \leq n(devG)^{\frac{1}{4}}$$

for any graph G of order n. We know from [2] that a quasi-random graph $G = G(n)$ has $devG = o(1)$. Hence the inequality above implies that $\beta(G) = o(n)$ if G is quasi-random and is thus stronger than the result in Theorem 1.

The problem of finding a better lower bound remains.

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