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DOMAIN OF EXISTENCE OF A PERTURBED CAUCHY PROBLEM OF ODE

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Abstract. In this paper we consider the Cauchy problem

$$dx(t)/dt = x(t)^n,\ x(0) = x_0.$$

The domain of existence is lower semicontinuous for perturbations of the data. We present a simple formula for the time of existence which is exact when there exists no perturbations.

1. Introduction

The simple example of the Cauchy problem

$$\frac{dx(t)}{dt} = x(t)^n, x(t) = x_0 > 0, n \geq 2 \tag{1.1}$$

shows that one cannot expect in general to have global solutions of a nonlinear differential equations. In fact, the solution of (1.1) is

$$x(t) = \frac{x_0}{\sqrt[n-1]{1-(n-1)x_0^{n-1}t}}, \tag{1.2}$$

so a solution exists in the interval where $(n-1)x_0^{n-1}t < 1$ but it becomes unbounded as $(n-1)x_0^{n-1}t \nearrow 1$.

2. Perturbation of the Cauchy problem

Lemma 2.1[2]. *Let $f(t,x)$ and $\frac{\partial f}{\partial x}(t,x)$ be continuous in an open set $\Omega \subset \mathbb{R}^{n+1}$. Assume that the Cauchy problem*

$$\begin{cases} \frac{dx(t)}{dt} &= f(t,x(t)), \\ x(t_0) &= x_0 \end{cases} \tag{2.1}$$

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has a solution with graph in Ω *for* $t_1 \le t \le t_2$ *where* $t_1 < t_0 < t_2$. *Then there is a neighborhood* U *of* x_0 *in* $\mathbb{R}^n$ *such that for every* $y \in U$ *the Cauchy problem (2.1) with* x_0 *replaced by* y *has a unique solution* $x(t, y)$ *for* $t_1 \le t \le t_2$. *The solution is in* $\mathcal{C}^1([t_1, t_2] \times U)$.

Take any function g such that $g(t, x)$ and $\frac{\partial g}{\partial x}(t, x)$ are continuous in the open set $\Omega \subset \mathbb{R}^{1+n}$ of the above theorem.

Corollary 2.2. *Under the conditions of Lemma 2.1 the Cauchy problem*

$$\begin{cases} \frac{dx(t)}{dt} & = f(t, x(t)) + \epsilon g(t, x(t)), \\ x(t_0) & = y \end{cases} \tag{2.2}$$

has a unique solution $x(t, \epsilon, y)$ *for* $t_1 \le t \le t_2$ *where* $t_1 < t_0 < t_2$. *The solution is in* $\mathcal{C}^1([t_1, t_2] \times U \times I)$.

Proof. Apply the Lemma 2.1 to the following Cauchy problem

$$\begin{cases} \frac{dx(t)}{dt} & = f(t, x(t)) + \eta(t) g(t, x(t)), \\ \frac{d\epsilon(t)}{dt} & = 0, \\ x(t_0) & = y, \\ \eta(t_0) & = \epsilon. \end{cases}$$

The existence and regularity of solution $(x(t), \eta(t))$ of this equation is guaranteed by the theorem. Then $x(t)$ is a solution of the Cauchy problem (2.2). $\square$

Lemma 2.3. *Let* $x(t)$ *be a solution in* $[0, T]$ *of the ordinary differential equation*

$$\begin{cases} \frac{dx(t)}{dt} & = a_0(t)x(t)^n + a_1(t)x(t) + a_2(t), \\ x(0) & = x_0 \end{cases} \tag{2.3}$$

with all $a_j, j = 0, 1, 2,$ *are continuous and* $a_0 \ge 0$. *Let*

$$K = \int_0^T |a_2(s)| \exp\Big(-\int_0^s a_1(u)du\Big) ds. \tag{2.4}$$

If $x(0) > K$ *it follows that*

$$K < \frac{1}{n-1} \left(x(0) - K\right)^{1-n}.$$

Proof. Set $x(t) = X(t)\exp\Big(\int_0^t a_1(s)ds\Big)$. Then

$$\begin{aligned}\frac{dx}{dt} &= \frac{dX}{dt}\exp\Big(\int_0^t a_1(s)ds\Big) + a_1(t)X(t)\exp\Big(\int_0^t a_1(s)ds\Big)\\ &= a_0(t)X(t)^n\exp\Big(n\int_0^t a_1(s)ds\Big)\\ &\quad + a_1(t)X(t)\exp\Big(\int_0^t a_1(s)ds\Big) + a_2(t).\end{aligned}$$

Thus we have

$$\begin{aligned}\frac{dX}{dt} &= a_0(t)\exp\Big((n-1)\int_0^t a_1(s)ds\Big)X(t)^n\\ &\quad + a_2(t)\exp\Big(-\int_0^t a_1(s)ds\Big)\\ &= \tilde{a}_0(t)X(t)^n + \tilde{a}_2(t)\end{aligned}$$

where $\tilde{a}_0(t) = a_0(t)\exp\Big((n-1)\int_0^t a_1(s)ds\Big)$ and $\tilde{a}_2(t) = a_2(t)\exp\Big(-\int_0^t a_1(s)ds\Big)$. Let's introduce

$$X_2(t) = \int_0^t |\tilde{a}_2(s)|ds.$$

Then $X_2(0) = 0, X_2(T) = K$. Let X_1 be the solution of the Cauchy problem

$$\begin{cases}\frac{dX_1(t)}{dt} &= \tilde{a}_0(t)\Big(X_1(t)-K\Big)^n,\\ X_1(0) &= x(0).\end{cases}$$

Upon integrating this equation we obtain

$$\Big(X_1(t)-K\Big)^{1-n} - \Big(x(0)-K\Big)^{1-n} = (1-n)\int_0^t \tilde{a}_0(s)ds.$$

Since $\tilde{a}_0 \geq 0$, X_1 is increasing if X_1 exists in $[0,T]$ and

$$\begin{aligned}\int_0^T \tilde{a}_0(s)ds &= \frac{1}{n-1}\Big(x(0)-K\Big)^{1-n} - \frac{1}{n-1}\Big(X_1(T)-K\Big)^{1-n}\\ &< \frac{1}{n-1}\Big(x(0)-K\Big)^{1-n}.\end{aligned}$$

Now

$$\frac{d\Big(X_1(t)-X_2(t)\Big)}{dt} = \tilde{a}_0(t)\Big(X_1(t)-K\Big)^n - |\tilde{a}_2(t)|$$
$$\le \tilde{a}_0(t)\Big(X_1(t)-X_2(t)\Big)^n + \tilde{a}_2(t).$$

Observe that $X_1(t) - X_2(t) = x(t)$ when $t = 0$. Therefore $X_1(t) - X_2(t) \le x(t)$ in $[0, t]$ as long as $X_1(t)$ exists. Thus X_1 cannot become infinite in $[0, T]$, which proves that

$$\int_0^T |a_0(t)|dt \cdot \exp\Big((1-n)\int_0^T |a_1(t)|dt\Big)$$
$$\le \int_0^T |a_0(t)| \cdot \exp\left((1-n)\int_0^t |a_1(u)|du\right)dt$$
$$< \frac{1}{n-1}\Big(x(0)-K\Big)^{1-n}.$$

□

THEOREM 2.4. *Let $a_j, j = 0,1,2,$ be continuous functions in $[0,T]$, set $a_0^+ = \max(a_0, 0)$, and define K by (2.4). If $x_0 \ge 0$ and*

$$\int_0^T a_0^+(t)dt \cdot \exp\left((n-1)\int_0^T |a_1(t)|dt\right) < (x_0+K)^{1-n},$$
(2.5)
$$\int_0^T |a_0(t)|dt \cdot \exp\left((n-1)\int_0^T |a_1(t)|dt\right),$$

then

$$\begin{cases} \frac{dx(t)}{dt} & = a_0(t)x(t)^n + a_1(t)x(t) + a_2(t), \\ x(0) & = x_0 \end{cases}$$

has a solution in $[0,T]$ with $x(0) = x_0$, and

$$x(T)^{1-n} \ge (x_0+K)^{1-n} + (1-n)\int_0^T a_0^+(t)dt \cdot \exp\left((n-1)\int_0^T |a_1(t)|dt\right)$$

if $x(T) \ge 0$.

Proof. We set

$$x(t) = X(t)\exp\Big(\int_0^t a_1(s)ds\Big).$$

Let $X_2(t)$ again be the integral of $|\tilde{a}_2|$ with $X_2(0) = 0$. Thus $X_2(T) = K$. Now let X_1 be the solution of

$$\frac{dX_1(t)}{dt} = \tilde{a}_0^+(t)\Big(X_1(t) + K\Big)^n, \quad X_1(0) = x_0,$$

which is

$$\Big(X_1(t) + K\Big)^{1-n} = (x_0 + K)^{1-n} + (1-n)\int_0^t \tilde{a}_0^+(s)ds.$$

Since

$$\int_0^T a_0^+(t)\cdot\exp\Big((n-1)\int_0^t a_1(s)|ds\Big)dt$$
$$\leq \int_0^T a_0^+(t)dt\cdot\exp\Big((n-1)\int_0^T |a_1(t)|dt\Big)$$

we obtain by (2.5) an increasing function X_1 existing in $[0,T]$. Since

$$\begin{aligned}\frac{d\Big(X_1(t) + X_2(t)\Big)}{dt} &= \tilde{a}_0^+(t)\Big(X_1(t) + K\Big)^n + |\tilde{a}_2(t)| \\ &\geq \tilde{a}_0(t)\Big(X_1(t) + X_2(t)\Big)^n + \tilde{a}_2(t)\end{aligned}$$

and $X_1 + X_2 = X$ at $t = 0$, we have $X \leq X_1 + X_2 \leq X_1 + K$ in $[0,T]$ if X exists. Hence

$$\begin{aligned}X(T)^{1-n} &\geq \Big(X_1(T) + K\Big)^{1-n} \\ &= (x_0 + K)^{1-n} + (1-n)\int_0^T \tilde{a}_0^+(s)ds \\ &\geq (x_0 + K)^{1-n} \\ &\qquad + (1-n)\int_0^T a_0^+(t)dt\cdot\exp\Big((n-1)\int_0^T |a_1(t)|dt\Big).\end{aligned}$$

Since $x(T)^{1-n} = X(T)^{1-n} \exp\left((1-n)\int_0^T a_1(t)dt\right)$ we have

$$x(T)^{1-n} \geq (x_0 + K)^{1-n} + (1-n)\int_0^T a_0^+(t) \exp\left((n-1)\int_0^T |a_1(t)|dt\right). \quad (2.6)$$

□

When $a_0 \equiv 1, a_1 \equiv 0, a_2 \equiv 0$, (2.6) reduces to

$$T \leq \frac{x_0^{1-n}}{n-1}.$$

This agrees with (1.2).

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