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PRODUCT SPACE AND QUOTIENT SPACE IN K_0-PROXIMITY SPACES

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Abstract. We introduce the k_0-proximity space as a generalization of the Efremovič-proximity space. We define a product k_0-proximity and the quotient k_0-proxmity and show some properties of k_0-proximity space.

1. Introduction

In this article, we propose some generalization of the concept of the Efremovič's proximity, which we call "K_0-proximity". We show that K_0-proximity δ induces a topology $\tau(\delta)$ in X and this induced topology is a completely regular topology. Given a K_0-proximity space (X, δ), a subset will be called a proximity δ-neighbourhood of A (symbols $A \ll B$) iff for each $x \in X$, $x \not\delta A$ or $x \not\delta (X - B)$.

A K_0-proximity neighbourhood furnishes an alternative approach to the study of K_0-proximity spaces. We define a product K_0-proximity $\delta = \pi\{\delta_\alpha : \alpha \in I\}$ on X and we shall introduce the concept of quotient k_0-proximity.

2. Preliminaries

The proximity relation δ was introduced in 1950 by Efremovič and he showed that the proximity relation δ induces a topology $\tau(\delta)$ in X and that the induced topology is completely regular in [1].

He also showed that every completely regular space (X, τ) admits a compatible proximity δ on X such that $\tau(\delta) = \tau$. He axiomatically

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characterized the proximity relation, A is near B, which is denoted by $A\delta B$, for subsets A and B of any set X. Efremovič axioms of proximity relation δ are as follows;

E1. $A\delta B$ implies $B\delta A$.

E2. $(A\cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$.

E3. $A\delta B$ implies $A\neq\phi$, $B\neq\phi$.

E4. $A\not\delta B$ implies there exists a subset E such that $A\not\delta E$ and $(X-E)\not\delta B$.

E5. $A\cap B\neq\phi$ implies $A\delta B$.

A binary relation δ satisfying axioms E1-E5 on the power set of X is called a (Efremocič) proximity on X. If δ also satisfies the following;

E6. $x\delta y$ implies $x=y$ then δ is called the separated proximity relation.

Definition 2.1. Let δ be a binary relation between a set X and its power set $P(X)$ such that

$K_0$1. $x\delta\{y\}$ implies $y\delta\{x\}$.

$K_0$2. $x\delta(A\cup B)$ if and only if $x\delta A$ or $x\delta B$.

$K_0$3. $x\not\delta\phi$ for all $x\in X$.

$K_0$4. $x\in A$ implies $x\delta A$.

$K_0$5. For each subset $E\subset X$, if there is a point $x\in X$ such that either $x\delta A$, $x\delta E$ or $x\delta B$, $x\delta(X-E)$, then we have $y\delta A$ and $y\delta B$ for some $y\in X$. The binary relation δ is called the K_0-proximity on X iff δ satisfies the axioms K_01-K_05. The pair (X,δ) is called a K_0-proximity space.

$K_0$6. If $x\delta\{y\}$ implies $x=y$, then δ is called the separated K_0-proximity relation.

Lemma 2.2. *In a K_0-proximity space (X,δ) let δ_1 be a binary relation on $P(X)$ defined as follows;*

If we define $A\delta_1 B$ if and only if there is a point $x\in X$ such that $x\delta A$,$x\delta B$, then δ_1 is an Efremovič proximity.

In what follows, we introduce some properties of the K_0-proximity.

Lemma 2.3. *If $x\delta A$ and $A\subset B$, then $x\delta B$.*

Lemma 2.4. *If there exists a point $x\in X$ such that $x\delta A$, $x\delta\{y\}$ then $y\delta A$.*

Lemma 2.5. *If a subset A of a K_0-proximity space (X,δ) is defined to be closed iff $x\delta A$ implies $x\in A$, then the collection of complements of all closed sets so defined yields a topology $\tau=\tau(\delta)$ on X.*

Lemma 2.6. *Let (X,δ) be a K_0-proximity space and $\tau = \tau(\delta)$. Then the τ-closure $\bar{A}$ of $A \subset X$ is given by $\bar{A} = \{x \mid x\delta A, x \in X\}$.*

Definition 2.7. If on a set X there is a topology τ and a K_0-proximity δ such that $\tau = \tau(\delta)$, then τ and δ are said to be compatible.

Lemma 2.8. *If G is a subset of a K_0-proximity space (X,δ), then $G \in \tau(\delta)$ iff $x\not\delta(X-G)$ for every $x \in G$.*

Lemma 2.9. *If A and B are subsets of a K_0-proximity space (X,δ) then for each $x \in X$, $x\not\delta A$ or $x \not\delta B$ implies (i) $B \subset (X-A)$ and (ii) $B \subset Int(X-A)$, where the closure and interior are taken with respect to $\tau(\delta)$.*

Lemma 2.10. *In a K_0-proximity space (X,δ), if A^δ is defined to be a set $\{x \mid x\delta A, x \in X\}$ for each subset A of X, then δ is a Kuratowski closure operator.*

Definition 2.11. If δ_1 and δ_2 are two K_0-proximities on a set X, we define $\delta_1 < \delta_2$ iff $x\delta_2 A$ implies $x\delta_1 A$.

The above is expressed by saying that δ_1 is finer than δ_2, or δ_1 is coarser than δ_2.

The following Lemma shows that a finer K_0-proximity structure induces a finer topology.

Lemma 2.12. *Let δ_1, δ_2 be two K_0-proximities defined on a set X. Then we have;*

1. *$\delta_1 < \delta_2$ implies $\tau(\delta_1) \subset \tau(\delta_2)$*
2. *Let τ_1 and τ_2 be two completely regular topologies on X, and let δ_1 and δ_2 be the K_0-proximities on X with respect to τ_1 and τ_2 respectively. Then $\tau_1 \subset \tau_2$ implies $\delta_1 < \delta_2$.*

Definition 2.13. A subset B of a K_0-proximity space (X,δ) is a δ-neighbourhood of A (in symbols $A \ll B$) iff for each $x \in X, x\not\delta A$ or $x\not\delta(X-B)$.

Lemma 2.14. *Let (X,δ) be a K_0-proximity space let $\bar{A}$ and Int A denote, respectively, the closure and interior of A in $\tau(\delta)$. Then*

1. *$A \ll B$ implies $\bar{A} \ll B$, and*
2. *$A \ll B$ implies $A \ll$ Int B.*

Therefore $A \subset$ Int B, showing that a δ-neighbourhood is a topological neighbourhood.

LEMMA 2.15. *For each point $x \in X$, $y\not\delta A$ or $x\not\delta B$ implies there exist subsets C and D such that $A \ll C$, $B \ll D$ and for each $y \in X$, $y\not\delta C$ or $y\not\delta D$.*

LEMMA 2.16. *Let δ be a compatible K_0-proximity on a completely regular space (X,τ). If A is compact, B is closed and $A \cap B = \phi$, then for each $x \in X$, $x\not\delta A$ or $x\not\delta B$.*

LEMMA 2.17. *Given a K_0-proximity space (X,δ), the relation $\ll$ satisfies the following properties;*

1. $X \ll X$.
2. $A \ll B$ *implies* $A \subset B$.
3. $A \subset B \ll C \subset D$ *implies* $A \ll D$
4. $A \ll B_i$ *for* $i = 1, 2, \ldots, n$ *iff* $A \ll \cap_{i=1}^{n} B$.
5. $A \ll B$ *implies* $(X - B) \ll (X - A)$.
6. $A \ll B$ *implies there is* C *such that* $A \ll C \ll D$.

LEMMA 2.18. *If $\ll$ is a binary relation on the power set of X satisfying Lemma 2.17 and δ is defined by that for each $x \in X$, $x\not\delta A$ or $x\not\delta B$ iff $A \ll (X - B)$, then δ is an K_0-proximity on X. B is a δ-neighbourhood of A iff $A \ll B$.*

DEFINITION 2.19. Let (X, δ_1) and (Y, δ_2) be two K_0-proximity spaces. A function $f : X \mapsto Y$ is said to be a K_0-proximity mapping iff for some $x \in X$, $x\delta_1 A$, $x\delta_1 B$ implies $f(x)\delta_2 f(A)$, $f(x)\delta_2 f(B)$.

LEMMA 2.20. *Let (X, δ_1) and (Y, δ_2) be two K_0-proximity spaces and let $f : X \mapsto Y$ be a function. The following properties of f are equivalent:*

1. f *is a* K_0*-proximity mapping.*
2. $y\not\delta_2 B$ *implies* $x\not\delta_1 f^{-1}(B)$ *for each* $x \in f^{-1}(y)$.
3. $y \ll_2 B$ *implies* $x \ll_1 f^{-1}(B)$ *for each* $x \in f^{-1}(y)$.

LEMMA 2.21. *A K_0-proximity mapping $f : (X, \delta_1) \mapsto (Y, \delta_2)$ is continuous with respect to $\tau(\delta_1)$ and $\tau(\delta_2)$.*

LEMMA 2.22. *Given a function $f : X \mapsto (Y, \delta_1)$ the binary relation δ defined by $x\delta A$ iff $f(x)\delta_1 f(A)$, is the coarsest K_0-proximity on X such that f is a K_0-proximity mapping.*

DEFINITION 2.23. Two K_0-proximity spaces (X, δ_1) and (Y, δ_2) are said to be K_0-proximity isomorphic iff there exists a one-to-one mapping

f from X onto Y such that both f and f^{-1} are K_0-proximity mappings. Such a mapping f is called a K_0-proximity isomorphism.

It follows from the Lemma 2.21 that two K_0-proximity spaces are K_0-proximity isomorphic iff they are homeomorphic.

Definition 2.24. Let (X, δ) be a K_0-proximity space, and $Y \subset X$. The induced K_0-proximity δ_Y on Y is the coarsest K_0-proximity such that the inclusion mapping $i : Y \mapsto X$ is a K_0-proximity mapping.

The K_0-proximity space (Y, δ_Y) is called the subspace of (X, δ) and δ_Y is called the induced K_0-proximity.

3. Main Results

We next consider the product of a family $\{(X_\alpha, \delta_\alpha) : \alpha \in I\}$ of K_0-proximity spaces. Let $X = \Pi\{X_\alpha : \alpha \in I\}$ denote the Cartesian product of these spaces. We define a product K_0-proximity $\delta = \Pi\{\delta_\alpha : \alpha \in I\}$ on X as follows:

Definition 3.1. Let $x \in X$ and A be a subset of X. Define $x\delta A$ iff for each finite cover $N = \{A_1, A_2, \ldots, A_n\}$ of A there is A_i such that $P_\alpha(x)\delta_\alpha P_\alpha(A_i)$ for each $\alpha \in I$, where P_α denotes the projection of X onto X_α.

Theorem 3.2. *The binary relation δ defined in the Definition 3.1 is a K_0-proximity on the product set X.*

Proof. 1. Since each δ_α is symmetric, so is δ and K_01 is satisfied.

2. Let A and B be subsets of X. If $x\delta A$ and $N = \{E_1, E_2, \ldots E_n\}$ is a finite cover of $A \cup B$ then N is also a cover of A and there is some E_i in N such that $P_\alpha(x)\delta_\alpha P_\alpha(E_i)$ for each $\alpha \in I$. That is, $x\delta(A \cup B)$.

   Suppose that $x\not\delta A$ and $x\not\delta B$. Then there is some finite covers $N = \{A_1, A_2, \ldots, A_m\}$ of A and $L = \{B_1, B_2, \ldots B_n\}$ of B such that for each $A_i \in N$ there is $\alpha_i \in I$ with $P_{\alpha_i}(x)\not\delta_{\alpha_i} P_{\alpha_i}(A_i)$ and for each $B_j \in L$ there is $\alpha_j \in I$ with $P_{\alpha_j}(x)\not\delta_{\alpha_j} P_{\alpha_j}(B_j)$. $N \cup L = \{A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_n\}$ is a cover of $A \cup B$ and there is no member A_i or B_j in $N \cup L$ such that $P_\alpha(x)\delta_\alpha P_\alpha(A_i)$ for each $\alpha \in I$ or $P_\alpha(x)\delta_\alpha P_\alpha(\alpha)(B_j)$ for each $\alpha \in I$. Hence we have $x\not\delta(A \cup B)$.

3. Since $N = \{\phi\}$ is a finite cover of ϕ and $P_\alpha(x) \not\delta_\alpha P_\alpha(\phi)$ for each $\alpha \in I$ we have $x\not\delta\phi$.

4. If $x \in A$ and $A = A_1 \cup A_2 \cup \cdots \cup A_n$ then there is some A_i such that $x \in A_i$. Hence for each $\alpha \in I$ we have $P_\alpha(x)\delta_\alpha P_\alpha(A_i)$, that is, $x\delta A$.
5. $K_0 5$ is clear by K_0-proximity.

□

DEFINITION 3.3. Let $\{(X_\alpha, \delta_\alpha) \mid \alpha \in I\}$ be a family of K_0-proximity spaces $(X_\alpha, \delta_\alpha)$. The pair (X, δ), where $X = \Pi X_\alpha$, $\delta = \Pi\delta_\alpha$, is called the product K_0-proximity space of the family.

THEOREM 3.4. *A mapping f from a K_0-proximity space (Y, δ_1) to a product K_0-proximity space $X = \Pi X_\alpha$ is a K_0-proximity mapping iff the composition $P_\alpha \circ f : Y \mapsto X_\alpha$ is a K_0-proximity mapping for each projection P_α.*

Proof. We need only prove that if each $P_\alpha \circ f$ is a K_0-proximity mapping then so is f. Let $y \in Y$ and $B \subset Y$. And suppose that $y\delta_1 B$ and $f(y)\not\delta f(B)$. Then there is some cover $N = \{A_1, \ldots, A_n\}$ of $f(B)$ such that for each $A_i \in N$, $P_{\alpha_i}(f(y))\not\delta_{\alpha_j} P_{\alpha_i}(A_i)$ for some $\alpha_i \in I$. Since $\{f^{-1}(A_1), \ldots, f^{-1}(A_n)\}$ is a cover of B and $y\delta_1 B$, we have $y\delta_1 f^{-1}(A_j)$ for some A_j. Hence $(P_\alpha \circ f(y))\delta_\alpha(P_\alpha \circ f(f^{-1}(A_j)))$ for each $\alpha \in I$ since $p_\alpha \circ f$ is a K_0-proximity mapping. That is, $P_\alpha(f(y))\delta_\alpha P_\alpha(A_j)$ for each $\alpha \in I$. This contradicts to the fact $P_{\alpha_j}(f(y)) \not\delta_{\alpha_j} P_{\alpha_j}(A_j)$. Therefore $f(y)\delta f(B)$, that is, f is a K_0-proximity mapping. □

COROLLARY 3.5. *The product K_0-proximity $\delta = \Pi\delta_\alpha$ is the coarsest K_0-proximity on $X = \Pi X_\alpha$ for which each projection P_α is a K_0-proximity mapping.*

In the following we shall introduce the concept of a quotient K_0-proximity.

THEOREM 3.6. *Let (X, δ) be a K_0-proximity space and let $f : X \mapsto Y$ be a mapping, where Y is any set. If we define $y\delta_1 B$ iff each f-saturated closed subset of X containing $f^{-1}(B)$ contains $f^{-1}(y)$, then δ_1 is a K_0-proximity on Y and f is a K_0-proximity mapping. (or δ_1 is the finest K_0-proximity on Y such that f is a K_0-proximity mapping.)*

Proof. We first show that δ_1 is a K_0-proximity on Y.

1. $K_0 1$ is clear by definition.

2. Suppose that $y\delta_1(A\cup B)$ and $y\not\delta_1 B$ then each f-saturated closed set F containing $f^{-1}(A\cup B)$ contains $f^{-1}(y)$ and there is f-saturated closed set G containing $f^{-1}(B)$ such that $G\cap f^{-1}(y)=\phi$.

   Consequently each f-saturated closed set H containing $f^{-1}(A)$ contains $f^{-1}(y)$, since if $H\cap f^{-1}(y)=\phi$ then the closed saturated $H\cup G$ containing $f^{-1}(A)\cup f^{-1}(B)$ does not contain $f^{-1}(y)$ and it is a contradiction. Hence $y\delta_1 A$. Suppose that $y\delta_1 A$. Then each f-saturated closed set F containing $f^{-1}(A)$ contains $f^{-1}(y)$. Hence each f-saturated closed set H containing $f^{-1}(A)\cup f^{-1}(B)$ also contains $f^{-1}(y)$. That is, $y\delta_1(A\cup B)$.
3. Since the empty set ϕ is a f-saturated closed set containing $\phi = f^{-1}(\phi)$ such that $f^{-1}(y)\cap\phi=\phi$ for each y in Y, we have $y\not\delta_1\phi$ for each y in Y.
4. If $y\in A$ then $f^{-1}(y)\subset f^{-1}(A)$ and each f-saturated closed set F containing $f^{-1}(A)$ also contains $f^{-1}(y)$. Therefore we have $y\delta_1 A$.
5. $K_0 5$ is clear.

□

THEOREM 3.7. *In the Theorem 3.6, δ_1 is the finest K_0-proximity on Y such that f is a K_0-proximity mapping.*

Proof. Let δ_0 be any K_0-proximity on Y such that f is a K_0-proximity mapping. And let $y\not\delta_0 B$. Then $y\not\delta_0\bar{B}$ and we have $x\not\delta_0 f^{-1}(\bar{B})$ for each x in $f^{-1}(y)$, that is, $f^{-1}(y)\cap f^{-1}(\bar{B})=\phi$. Since $f^{-1}(\bar{B})$ is a f-saturated closed set containing $f^{-1}(B)$, $y\not\delta_1 B$. □

DEFINITION 3.8. Let (X,δ) be a K_0-proximity space and let $f: X\mapsto Y$ be a mapping. The finest K_0-proximity δ_1 on Y such that f is a K_0-proximity mapping is called the quotient K_0-proximity for Y relative to f and the K_0-proximity δ on X.

THEOREM 3.9. *Let f be a K_0-proximity mapping of a space X onto a space Y and let Y have the quotient K_0-proximity. Then a mapping g on Y to a K_0-proximity space Z is a K_0-proximity mapping iff the composition $g\circ f$ is a K_0-proximity mapping.*

Proof. Let $g\circ f$ be a K_0-proximity mapping and let $g(y)\not\delta_Z g(B)$ then $g(y)\not\delta_Z\overline{g(B)}$.

Since $g\circ f$ is a K_0-proximity mapping, for each x in $f^{-1}g^{-1}(g(y))$, $x\not\delta_X f^{-1}g^{-1}(\overline{g(B)})$ or for each x in $f^{-1}(y)$, $x\not\delta_X f^{-1}g^{-1}(\overline{g(B)})\supset f^{-1}(B)$

or $x \not\delta_X f^{-1}(B)$ and $f^{-1}(g^{-1}(\overline{g(B)}))$ is a f-saturated closed set containing $f^{-1}(B)$ in X. Hence $y \not\delta_X B$.

The converse is clear. □

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