

## ON THE SHAPE DERIVATIVE IN THE DOMAIN INCLUSION

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ABSTRACT. The shape derivative for the domain functional will be discussed in the situation of domain inclusion. Hadamard's shape structure is sought by using the material derivative in conjunction with the domain imbedding technique.

### 1. Introduction

Shape variations are closely connected with the optimal shape control problem associated with the applicable studies in fluid mechanics, flexible structures, free boundary problems. The first result concerning the differentiability with respect to perturbations of a domain was obtained in 1907 for the first eigenvalue of a membrane by Hadamard. He also computed the derivative of the Green's function for the Laplace operator with respect to the normal variations of the domain (c.f. [8]).

Zolesio ([11], [12]) introduced a new technique that is called the material derivative method to show the existence of a solution of a domain identification problem given a velocity field of domain perturbations. This method utilized the idea of spatial and material descriptions in the motion of a body in continuum mechanics. Recently, in conjunction with shape control problem in the Navier-Stokes flows, shape perturbation in the situation of domain inclusion has been studied expanding the material derivative method ([6], [7]). This methodology have been applied specifically to figure out the optimal forbody shape ([9]).

In this paper, we are concerned with the general discussion for the shape derivative in the situation of domain inclusion. Hadamard's shape structure in regard to the shape perturbation will be sought

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by employing the material derivative in conjunction with the domain imbedding technique.

Before continuing our discussion, we need to recall the following extension theorem about domains that will be of use in the sequel (Calderon's extension theorem); see [1].

LEMMA 1. *For every uniform Lipschitz domain  $\Omega \subset \mathbb{R}^n$  and positive integer  $m$ , there exists a linear continuous operator*

$$E : H^m(\Omega) \longrightarrow H^m(\mathbb{R}^n)$$

such that for every  $u \in H^m(\Omega)$  we have  $\|Eu\|_m \leq c\|u\|_m$ , where the positive constant  $c$  depends only on the cone imbedding in  $\Omega$ .

Here,  $H^m(\Omega)$  denotes the canonical Sobolev space. Note that  $H^m(\Omega) = \gamma_\Omega(H^m(\mathbb{R}^n))$ , where  $\gamma_\Omega$  denotes the restriction of  $H^m(\mathbb{R}^n)$  to  $H^m(\Omega)$ . Hence, Lemma 1 states that for a uniform Lipschitz domain  $\Omega$ , it holds  $\gamma_\Omega \circ E =$  the identity map over  $H^m(\Omega)$ . We shall use  $\gamma_\Gamma$  to denote the trace of  $H^m(\Omega)$  onto its boundary.

## 2. The Material Derivative in the Domain Inclusion

To study domain perturbation in the domain inclusion, let us introduce a one parameter family of domains  $\{\Omega_t\}_{t \geq 0}$  in the following manner;

Given a domain  $\Omega \subset \mathbb{R}^n$  and a (smooth) vector field  $\vec{V}(t, \cdot)$  defined in a neighborhood of  $\Omega$ , each point  $\vec{p}$  of  $\Omega$  is continuously transported in a one-to-one fashion onto a point  $\vec{x}(t)$  at  $t > 0$  through the following system of differential equations

$$(2.1) \quad \begin{cases} \frac{d}{dt} \vec{x}(t) = \vec{V}(t, \vec{x}(t)), \\ \vec{x}(0) = \vec{p}. \end{cases}$$

The solution of this system introduces a one-to-one transformation  $\mathcal{H}_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $\sigma > 0$  such that

$$(2.2) \quad \mathcal{H}_t(\vec{p}) = \vec{x}(t) = \vec{p} + \int_0^t \vec{V}(s, \vec{x}(s)) ds,$$

for  $0 \leq t < \sigma$  and  $\vec{p} \in \Omega \subset \mathbb{R}^n$ . The domain perturbed in the  $\vec{V}$ -direction is defined by  $\Omega_t \equiv \mathcal{H}_t(\Omega)$  with  $\Omega = \Omega_0 = \mathcal{H}_0(\Omega)$ . Naturally, we assume the boundary is preserved under the transformation  $\mathcal{H}_t$ , so that  $\Gamma_t \equiv \partial\Omega_t = \mathcal{H}_t(\Gamma)$  and  $\Gamma = \Gamma_0 = \mathcal{H}_0(\Gamma)$ .

In technical terms,  $\mathcal{H}_t$  is called a local configuration,  $\Omega$  a reference domain and  $\Omega_t$  a domain of spatial fields. The mapping  $t \mapsto \mathcal{H}_t(\vec{p})$  is called the shape trajectory of  $\vec{p}$  and  $\vec{V}$  the shape velocity. In case that the dependency of  $\mathcal{H}_t$  on  $\vec{V}$  should be emphasized, we will write  $\mathcal{H}_t = \mathcal{H}_t(\vec{V})$ .

By the local existence and uniqueness theorem for the system of differential equations and the integral representation (2.2) for the trajectory, it is not difficult to show that

$$(2.3) \quad \mathcal{H}_{t_1+t_2}(\vec{V})(\vec{q}) = \mathcal{H}_{t_2}(\vec{V}_{t_1})(\mathcal{H}_{t_1}(\vec{V})(\vec{q}))$$

for all  $t_1, t_2 \geq 0$  such that  $0 \leq t_1, t_2, t_1 + t_2 < \sigma$  and all points  $\vec{q}$  in a neighborhood  $\mathcal{O}_{\vec{p}}$  of  $\vec{p}$ , where  $\vec{V}_{t_1}(s, \cdot) = \vec{V}(t_1 + s, \cdot)$ .

For practical applications, we consider the case when all the perturbations of a domain are constrained in a fixed domain  $\widehat{\Omega}$ . To be more precise, let us assume that  $\widehat{\Omega}$  is a bounded open set containing  $\Omega$  and that  $\vec{V} : [0, \tilde{t}] \times \widehat{\Omega} \rightarrow \mathbb{R}^n$  denotes a continuous vector field. Suppose that  $t \mapsto \vec{V}(t, \vec{x})$  is continuous for each  $\vec{x} \in \Omega_t$  and  $\vec{V}(t, \cdot)$  is Lipschitz continuous, so that there exists a positive constant  $c$  such that

$$(2.4) \quad |\vec{V}(t, \vec{x}_1) - \vec{V}(t, \vec{x}_2)| \leq c |\vec{x}_1 - \vec{x}_2|,$$

for  $t \in [0, \tilde{t}]$  and every  $\vec{x}_1$  and  $\vec{x}_2$  in  $\widehat{\Omega}$ . Then for any  $\vec{p} \in \Omega$ , there exists an  $\tilde{\sigma} \in (0, \tilde{t})$ , an open neighborhood  $\mathcal{O}_{\vec{p}}$  of  $\vec{p}$  in  $\widehat{\Omega}$  and a one-to-one transformation

$$\mathcal{H}_t : \mathcal{O}_{\vec{p}} \rightarrow \mathcal{H}_t(\mathcal{O}_{\vec{p}}) \subset \mathbb{R}^n \quad \text{for } 0 \leq t < \tilde{\sigma},$$

such that  $t \mapsto \mathcal{H}_t(\vec{p})$  is a unique solution of (2.1) for  $0 \leq t < \tilde{\sigma}$ . Let us assume that

$$(2.5) \quad \bigcup_{\substack{\mathcal{O}_{\vec{p}} \subset \widehat{\Omega} \\ 0 \leq t < \tilde{\sigma}}} \mathcal{H}_t(\mathcal{O}_{\vec{p}}) \subset \widehat{\Omega} \quad \text{for every } \vec{p} \in \Omega.$$

This is the condition for the existence of the inverse  $\mathcal{H}_t^{-1}$  of  $\mathcal{H}_t$  for  $0 \leq t < \tilde{\sigma}$ , when  $\vec{V}(t, \cdot)$  is defined on  $\widehat{\Omega}$ .

Since  $\overline{\Omega}$  is compact, there exists a finite open covering  $\{\mathcal{O}_i\}_{i=1}^m$  of  $\overline{\Omega}$  in  $\widehat{\Omega}$  with corresponding positive numbers  $\sigma_1, \dots, \sigma_m$  and transformations  $\{\mathcal{H}_t^{(i)}\}$  such that  $\mathcal{H}_t^{(i)} : \mathcal{O}_i \rightarrow \mathcal{H}_t^{(i)}(\mathcal{O}_i) \subset \widehat{\Omega}$  is one-to-one for  $0 \leq t < \sigma_i$ . Let us take  $\mathcal{O} = \bigcup_{i=1}^m \mathcal{O}_i$  and  $\sigma = \min\{\sigma_1, \dots, \sigma_m\}$ . Notice that by the uniqueness of the solution of the differential equations,  $\mathcal{H}_t^{(i)}(\vec{q}) = \mathcal{H}_t^{(j)}(\vec{q})$  for  $\vec{q} \in \mathcal{O}_i \cap \mathcal{O}_j$ . So if we patch them together by defining

$$\mathcal{H}_t(\vec{q}) = \mathcal{H}_t^{(i)}(\vec{q}) \quad \text{if } \vec{q} \in \mathcal{O}_i,$$

then obviously  $\mathcal{H}_t : \Omega \rightarrow \Omega_t \subset \widehat{\Omega}$  is a one-to-one transformation for  $0 \leq t < \sigma$ . The continuity of  $\mathcal{H}_t(\cdot)$  for all  $0 \leq t < \sigma$  easily follows from the expression (2.2) of  $\mathcal{H}_t$  and the Lipschitz continuity (2.4) of  $\vec{V}(t, \cdot)$ . Furthermore, if  $\vec{V}(t, \cdot)$  is of class  $\mathcal{C}^k$  over  $\widehat{\Omega}$ , from the classical regularity result, it follows that  $\mathcal{H}_t(\cdot)$  is also of class  $\mathcal{C}^k$  over  $\widehat{\Omega}$ . Clearly,  $(0, \sigma) \ni t \mapsto \mathcal{H}_t(\vec{x})$  is continuously differentiable for each  $\vec{x} \in \Omega_t$ .

Next, we consider the inverse  $\mathcal{H}_t^{-1}$  of  $\mathcal{H}_t$ . Note that if  $t \mapsto \vec{V}(t, \cdot)$  is defined in a neighborhood  $(-\sigma, \sigma)$  of 0 and  $\frac{\partial \vec{V}}{\partial t} = \vec{0}$ , then  $\mathcal{H}_{t_1+t_2}(\vec{q}) = \mathcal{H}_{t_1}(\mathcal{H}_{t_2}(\vec{q}))$  for all  $t_1$  and  $t_2$  such that  $-\sigma < t_1, t_2, t_1 + t_2 < \sigma$  and all points  $\vec{q}$  in a neighborhood  $O_{\vec{p}}$  of  $\vec{p}$ . In this case,  $\{\mathcal{H}_t\}_{-\sigma < t < \sigma}$  is a local one parameter group of transformation whose inverse is given by  $\mathcal{H}_t^{-1} = \mathcal{H}_{-t}$  for  $-\sigma < t < \sigma$ .

Since this is not the case, to discuss its inverse  $\mathcal{H}_t^{-1}$ , we consider the following system of differential equations:

$$(2.6) \quad \begin{aligned} \frac{d}{ds} \vec{p}(s) &= -\vec{V}(t-s, \vec{p}(s)) \quad 0 \leq s \leq t, \\ \vec{p}(0) &= \vec{x} = \mathcal{H}_t(\vec{p}(t)) \quad \text{for } \vec{x} \in \Omega_t \subset \widehat{\Omega}. \end{aligned}$$

This introduces a unique Lipschitzian solution  $\mathcal{J}_t(\vec{x}) = \vec{p}(t)$ .

LEMMA 2. *Under the assumptions of (2.4) and (2.5), the transformation  $\mathcal{J}_t$  induced from (2.6) is an inverse of  $\mathcal{H}_t$ . Moreover, if  $\vec{V}(t, \cdot)$  belongs to the class  $\mathcal{C}^k(\widehat{\Omega})$ , so does  $\mathcal{H}_t^{-1} = \mathcal{J}_t$ .*

*Proof.* Consider the map  $s \mapsto \mathcal{H}_{t-s}(\vec{p})$  for  $0 \leq s \leq t$ . Since  $\mathcal{H}_{t-s}(\vec{p}) = \vec{x}(t-s) = \vec{V}(t-s, \vec{x}(t-s))$ ,  $s \mapsto \mathcal{H}_{t-s}(\vec{p})$  is a solution (2.6), i.e.,  $\mathcal{H}_{t-s} = \mathcal{J}_s$ . Hence

$$\mathcal{J}_t(\mathcal{H}_t(\vec{p})) = \vec{p}(t) = \mathcal{H}_{t-t}(\vec{p}) = \vec{p},$$

whence  $\mathcal{J}_t$  is a left inverse of  $\mathcal{H}_t$ . To show that  $\mathcal{J}_t$  is also a right inverse of  $\mathcal{H}_t$ , we consider a function  $\vec{y}(s) = \vec{p}(t-s)$ . Since

$$\frac{d}{ds}\vec{y}(s) = \frac{d}{ds}\vec{p}(t-s) = \vec{V}(t-(t-s), \vec{p}(t-s)) = \vec{V}(s, \vec{y}(s))$$

from (2.6),  $\vec{y}(s)$  is a solution of

$$\frac{d}{ds}\vec{y}(s) = \vec{V}(s, \vec{y}(s)), \quad \vec{y}(0) = \vec{p}(t).$$

So, it follows that

$$\vec{x} = \vec{p}(0) = \vec{y}(t) = \mathcal{H}_t(\vec{p}(t)) = \mathcal{H}_t(\mathcal{J}_t(\vec{x})),$$

and hence that  $\mathcal{J}_t = \mathcal{H}_t^{-1}$ . It can be simply written as  $\mathcal{H}_t(\vec{V})^{-1} = \mathcal{H}_t(-\vec{V}_t)$ , where  $\vec{V}_t(s, \cdot) = \vec{V}(t-s, \cdot)$ . The second assertion can be proved in the same way as the regularity of  $\mathcal{H}_t$  is allowed, using the regularity of  $\vec{V}_t(s, \cdot)$  and the equations (2.6).  $\square$

Note that from (2.3),  $\mathcal{H}_{t+h}(\vec{V}) - \mathcal{H}_t(\vec{V}) = (\mathcal{H}_h(\vec{V}_t) - \mathcal{I}) \circ \mathcal{H}_t(\vec{V})$  for  $t, h > 0$ . So,  $\frac{\partial}{\partial t}\mathcal{H}_t(\vec{V})(\vec{p}) = \vec{V}(t, \mathcal{H}_t(\vec{V})(\vec{p}))$ . Hence, it follows that

$$(2.7) \quad \vec{V}(t, \vec{x}) = \frac{\partial \mathcal{H}_t}{\partial t}(\mathcal{H}_t^{-1}(\vec{x})) \quad \text{for every } \vec{x} \in \Omega_t, 0 \leq t < \sigma.$$

The argument hitherto can be simply stated as follows: if  $\vec{V}(t, \cdot)$  is of class  $\mathcal{C}^k$  with  $k \geq 0$  over  $\widehat{\Omega}$ , there exists a  $\mathcal{C}^k$ -diffeomorphism  $\mathcal{H}_t$  from  $\Omega$  onto  $\Omega_t$  and vice versa, if  $\{\mathcal{H}_t\}_{0 \leq t < \sigma}$  is a family of  $\mathcal{C}^k$ -diffeomorphisms,  $\vec{V}$  can be recovered from (2.7) and  $\vec{V}(t, \cdot)$  is also of class  $\mathcal{C}^k$ . Note that if  $\vec{V}(t, \cdot)$  is Lipschitz continuous, so is  $\mathcal{H}_t$ , and vice versa.

REMARK. Zolesio [11] showed that for any domain  $D$  in  $\mathbb{R}^n$  and a (smooth) velocity  $\vec{V} : [0, \tau] \times \bar{D} \rightarrow \mathbb{R}^n$  satisfying

$$\begin{aligned} \vec{V}(t, \vec{x}) \cdot \vec{n}(\vec{x}) &= 0 \quad \text{if the outward normal } \vec{n}(\vec{x}) \text{ is defined a.e. } \vec{x} \in \partial D \\ \vec{V}(t, \vec{x}) &= \vec{0} \quad \text{otherwise,} \end{aligned}$$

the solution  $\mathcal{H}_t$  of (2.1) maps  $\bar{D}$  into  $\bar{D}$  for all  $0 \leq t \leq \tau$  (c.f. Sokolowski et al. [10]).

Delfour and Zolesio ([4], [5]) extended the class of  $\vec{V}$  such that  $\mathcal{H}_t(\bar{D}) \subset \bar{D}$ ,  $\forall t \in [0, \tau]$ , using the “viability theory” introduced by Aubin-Cellina [2]: The general motivation for the viability theory is to study the viable phenomena such that a trajectory  $t \rightarrow x(t)$  belongs to a fixed closed subset  $M$  of a Hilbert space  $H$ . Let  $T_M(x)$  be a Bouligand contingent cone to  $M$  at  $x$  which is characterized by

$$v \in T_M(x) \iff \liminf_{h \rightarrow 0^+} \frac{d_M(x + hv)}{h} = 0,$$

where  $d_M(x) = \inf\{\|x - y\|_H \mid y \in M\}$ . Obviously,  $T_M(x)$  is a generalization of the tangent space to  $M$  at  $x$ , when  $M$  is a smooth manifold in  $\mathbb{R}^n$ . The fundamental theorem for the study of the viability follows from the following version of Nagumo’s theorem which can be found in [2]:

Nagumo’s theorem: Under the above setting, let  $f : M \rightarrow H$  be a continuous mapping satisfying the tangential condition

$$\forall x \in M, \quad f(x) \in T_M(x).$$

Then for all  $x_0 \in M$ , there exists  $\tau > 0$  such that the differential equation  $\dot{x}(t) = f(x(t))$ ,  $x(0) = x_0$  has a viable trajectory on  $[0, \tau]$ .  $\square$

Take  $D \subset \mathbb{R}^n$ . Let us consider vector fields  $\vec{V}$  which satisfy the following conditions:

- (i)  $\vec{V}(t, \cdot)$  is Lipschitz continuous,
- (ii)  $\vec{V}(t, \vec{x})$  and  $-\vec{V}(t, \vec{x})$  belong to the Bouligand contingent cone  $T_{\bar{D}}(\vec{x})$  to  $\bar{D}$  at  $\vec{x} \in \partial D$ , for all  $(t, \vec{x}) \in [0, \tau] \times \partial D$ .

The condition (ii) is a double viability condition by Nagumo's theorem, which guarantees the existence of a homeomorphism  $\mathcal{H}_t : \bar{D} \rightarrow \bar{D}$  (for details, refer to Delfour et al. [3]).

We are now ready to discuss the variation of a function due to the domain perturbation. Throughout this paper, we assume

$$(2.8) \quad \bigcup_{t \in [0, \tilde{t}]} \{t\} \times \Omega_t \subset [0, \tilde{t}] \times \widehat{\Omega}.$$

Let  $y_t$  be a regular function defined on  $\Omega_t = \mathcal{H}_t(\Omega)$ . Then the composite  $y_t \circ \mathcal{H}_t$  is defined on a fixed reference domain  $\Omega$ . The (pointwise) material derivative of  $y_t$  at  $\vec{p} \in \Omega$  in the  $\vec{V}$ -direction is defined by the following semi-derivative :

$$(2.9) \quad \dot{y} = \dot{y}(\vec{p}; \vec{V}) = \left. \frac{d}{dt} y_t(\mathcal{H}_t(\vec{p})) \right|_{t=0^+}.$$

If  $\{\Omega_t\}_{0 \leq t \leq \tilde{t}}$  is a class of domains with the uniform extension property, we can consider  $y_t$  as a restriction of  $y$  to  $\{t\} \times \Omega_t$ , where  $y$  is defined globally in  $[0, \tilde{t}] \times \widehat{\Omega}$ , i.e.,

$$(2.10) \quad y(t, \vec{x}) = E_{\widehat{\Omega}}(y_t(\mathcal{H}_t))(\mathcal{H}_t^{-1}(\vec{x})) \quad \text{and} \quad y_t(\vec{x}) = y(t, \vec{x}),$$

where  $E_{\widehat{\Omega}}$  is the Calderon extension operator (defined on  $\Omega$ ) to  $\widehat{\Omega}$ . Then, using the chain rule, the material derivative (2.9) can be written as

$$(2.11) \quad \begin{aligned} \dot{y}(\vec{p}; \vec{V}) &= \lim_{t \rightarrow 0^+} \frac{y(t, \mathcal{H}_t(\vec{p})) - y(0, \vec{p})}{t} \\ &= \frac{\partial y}{\partial t}(0, \vec{p}) + (\nabla y \cdot \vec{V})(0, \vec{p}), \end{aligned}$$

where  $\nabla y = (\frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_n})$ . Similarly, if  $\vec{u}_t$  is a vector-valued function defined in  $\Omega_t$  and  $\vec{u}(t, \cdot)$  is its extension to  $\widehat{\Omega}$ , the material derivative of  $\vec{u}_t$  can be written as

$$(2.12) \quad \dot{\vec{u}}(0, \vec{p}) = \frac{\partial \vec{u}}{\partial t}(0, \vec{p}) + (\vec{V}(0, \vec{p}) \cdot \nabla) \vec{u}(0, \vec{p}).$$

This concept can be naturally generalized into Sobolev spaces. For example, let  $y_t \in H^m(\Omega_t)$  and  $\vec{V}(t, \cdot) \in \mathcal{C}^k(\widehat{\Omega}; \mathbb{R}^n)$  for  $0 \leq t \leq \tilde{t}$ , where  $0 \leq k \leq m$ . Since  $\mathcal{H}_t$  is  $\mathcal{C}^k$ -diffeomorphism,  $y_t \circ \mathcal{H}_t \in H^k(\Omega)$ , which can be verified by Leibniz's rule. In fact,

$$H^k(\Omega) = \{y_t \circ \mathcal{H}_t \mid \vec{V}(t, \cdot) \in \mathcal{C}^k(\widehat{\Omega}; \mathbb{R}^n) \text{ and } y_t \in H^k(\Omega_t)\}.$$

Then,  $\dot{y} = \dot{y}(\Omega; \vec{V}) \in H^k(\Omega)$  is called the material derivative of  $y$  at  $\Omega \subset \widehat{\Omega}$  in the  $\vec{V}$ -direction in the Sobolev space  $H^k(\Omega)$  if

$$\lim_{t \rightarrow 0^+} \left\| \frac{y(t, \mathcal{H}_t(\vec{p})) - y(0, \vec{p})}{t} - \dot{y}(\Omega; \vec{V}) \right\|_{k, \Omega} = 0.$$

Notice that unless  $k > \frac{n}{2}$ , pointwise expressions such as (2.9) are meaningless. It makes sense only almost everywhere. To avoid notational confusion, we write it by

$$\dot{y} = \dot{y}(\Omega; \vec{V}) = \lim_{t \rightarrow 0^+} \frac{y_t \circ \mathcal{H}_t - y \circ \mathcal{I}}{t},$$

where the limit is taken in  $H^k(\Omega)$ . The material derivative in the weak space can be defined in a similar manner via duality.

If  $y \in H^k(\widehat{\Omega})$  is a uniform extension of  $y_t \in H^k(\Omega_t)$ , the shape derivative  $y'(\Omega; \vec{V})$  at  $\Omega \subset \widehat{\Omega}$  of the uniform extension  $y \in H^k(\widehat{\Omega})$  in the  $\vec{V}$ -direction is defined by

$$(2.13) \quad y'(\Omega; \vec{V}) = \dot{y}(\Omega; \vec{V}) - \nabla y(\Omega) \cdot \vec{V}(0).$$

Note that  $y'(\Omega; \vec{V}) \in H^{k-1}(\Omega)$  from (2.13). If  $k-1 > \frac{n}{2}$ , since  $H^k(\widehat{\Omega}) \subset \mathcal{C}^1(\widehat{\Omega})$ , the shape derivative can be defined pointwise

$$\begin{aligned} y'(\Omega; \vec{V}) &= \frac{\partial y}{\partial t}(0, \vec{p}) \\ &= \dot{y}(0, \vec{p}; \vec{V}) - (\nabla y \cdot \vec{V})(0, \vec{p}). \end{aligned}$$



### 3. Hadamard's Structure in the Shape Derivative

Now, we are concerned with the domain functional. Let  $J(\Omega)$  be any domain functional whose value is dependent on the domain  $\Omega \subset \widehat{\Omega}$ . Then the rate of variation of  $J(\Omega)$  at the reference domain  $\Omega$  with respect to the domain perturbation may be measured as a directional semi-derivative

$$dJ(\Omega; \vec{V}) = \lim_{t \rightarrow 0^+} \frac{J(\Omega_t) - J(\Omega)}{t} = \frac{d}{dt} J(\mathcal{H}_t(\vec{V})(\Omega)) \Big|_{t=0^+} .$$

The domain functional  $J(\Omega)$  is said to be shape differentiable if

- (a)  $dJ(\Omega; \vec{V})$  exists for all directions  $\vec{V}$
- (b)  $\vec{V} \mapsto dJ(\Omega; \vec{V})$  is linear and continuous over appropriate admissible vector fields.

If  $J(\Omega)$  is shape differentiable, under the sophisticated structure, we can interpret  $\vec{V} \mapsto dJ(\Omega; \vec{V})$  in the distribution sense:

$$(3.1) \quad dJ(\Omega; \vec{V}) = \langle \mathcal{G}(\Omega), \vec{V} \rangle .$$

Then  $\mathcal{G}(\Omega)$  appears to be a vector valued distribution of a finite order acting on the appropriate test function space which is determined by the regularity of the admissible domains. In this case,  $\mathcal{G}(\Omega)$  is called the shape gradient of the domain functional  $J(\Omega)$  and is usually written as

$$(3.2) \quad \mathcal{G}(\Omega) = \text{grad } J(\Omega) .$$

The problems associated with the shape optimizations are often rendered into the problem of finding  $\mathcal{G}(\Omega)$  so that  $J(\Omega_{t^*}) < J(\Omega_t)$  for  $t > 0$ .

REMARK. In the above definition, condition (b) requires an appropriate topology for the admissible vector fields and the continuity of  $\vec{V} \mapsto dJ(\Omega; \vec{V})$ . For a vector field  $\vec{V} \in \mathcal{C}^0([0, \tilde{t}]; \mathcal{C}_0^k(\widehat{\Omega}; \mathbb{R}^n))$ , i.e.,  $\vec{V}(t)(\cdot) = \vec{V}(t, \cdot)$  a vector valued  $\mathcal{C}^k$ -function with compact support in  $\widehat{\Omega}$ , Delfour et al. [4] introduced the inductive limit topology to utilize the sheaf structure of the distribution:

Let  $\mathfrak{L}^{m,k} = \{\vec{V} \in \mathcal{C}^m([0, \tilde{t}]; \mathcal{C}_0^k(\widehat{\Omega}; \mathbb{R}^n)) \mid \vec{V}(t, \vec{x}) \text{ and } -\vec{V}(t, \vec{x}) \text{ belong to the Bouligand contingent cone to } \overline{\widehat{\Omega}} \text{ at } \vec{x} \in \partial\widehat{\Omega}\}$ . Let  $\mathfrak{L}_K^{m,k}$  denote the closed subspace of  $\mathfrak{L}^{m,k}$  with  $\vec{V}(t, \cdot) \in \mathcal{C}_0^k(K; \mathbb{R}^n)$ , where  $K$  is a relatively compact set in  $\widehat{\Omega}$ . The inductive limit can be introduced by

$$\overrightarrow{\mathfrak{L}}_{\widehat{\Omega}}^{m,k} = \varinjlim_K \{\mathfrak{L}_K^{m,k} \mid \forall K \Subset \widehat{\Omega}\},$$

where  $\varinjlim$  denotes the inductive limit with respect to relative compact subsets of  $\widehat{\Omega}$  endowed with the natural inductive limit topology. Under this structure,  $\overrightarrow{\mathfrak{L}}_{\widehat{\Omega}}^{m,k} \ni \vec{V} \mapsto dJ(\Omega; \vec{V})$  is continuous (for details, refer to [4] and [5]).

We are now ready to demonstrate the structure for the shape derivative of a domain functional in the situation of the domain inclusion.

**MAIN THEOREM.** *We assume  $\vec{V}$  belongs to a class of vector fields satisfying (2.4) and (2.5). Suppose  $J(\Omega)$  is shape differentiable at  $\Omega \subset \widehat{\Omega}$ . Then*

- (i)  $dJ(\Omega; \vec{V}) = dJ(\Omega; \vec{V}(0)), \quad \forall \vec{V} \in \mathcal{C}^0([0, \tilde{t}]; \mathcal{C}_0^k(\widehat{\Omega}; \mathbb{R}^n)).$
- (ii)  $\text{supp } \mathcal{G}(\Omega) \subset \Gamma = \partial\Omega.$
- (iii) *There exists a scalar distribution  $g(\Gamma)$  of a finite order such that*

$$(3.3) \quad dJ(\Omega; \vec{V}) = \langle \mathcal{G}(\Omega), \vec{V}(0) \rangle_{\Omega}$$

$$(3.4) \quad = \langle g(\Gamma), \vec{V}(0) \cdot \vec{n} \rangle_{\Gamma},$$

where  $\vec{V}(0) \cdot \vec{n}$  is the normal component of  $\vec{V}(0)$  on  $\Gamma$ .

*Proof.* (i); Let  $\sigma$  be a positive number found in (2.2). For a given  $\vec{V}$ , take a positive integer  $m_0$  so large that  $\frac{\tilde{t}}{m_0} < \sigma$ . Define  $\vec{V}_m(t) = \vec{V}(\frac{t}{m})$  for  $m \geq m_0$ . Since  $\text{supp } \vec{V}(t) \subset \widehat{\Omega}$ , there exists a compact subset  $K$  of  $\widehat{\Omega}$  such that

$$\bigcup_{t \in [0, \tilde{t}]} \text{supp } \vec{V}_m(t) \subset K, \quad \forall m \geq m_0.$$

For every  $|\alpha| \leq k$ , since

$$\begin{aligned} & \sup_{\substack{0 \leq t \leq \tilde{t} \\ \vec{x} \in \bar{K}}} \left| \left( \frac{\partial}{\partial \vec{x}} \right)^\alpha (\vec{V}_m(t, \vec{x}) - \vec{V}(0, \vec{x})) \right|_{\mathbb{R}^n} \\ &= \sup_{\substack{0 \leq t \leq \tilde{t} \\ \vec{x} \in \bar{K}}} \left| \left( \frac{\partial}{\partial \vec{x}} \right)^\alpha \left( \vec{V}\left(\frac{t}{m}, \vec{x}\right) - \vec{V}(0, \vec{x}) \right) \right|_{\mathbb{R}^n} \longrightarrow 0 \quad \text{as } m \rightarrow \infty, \end{aligned}$$

it follows that  $dJ(\Omega; \vec{V}_m) \rightarrow dJ(\Omega; \vec{V}(0))$ .

(ii); In the sense of distribution, we note that

$$(3.5) \quad \text{supp } \mathcal{G}(\Omega) \cap \text{supp } \vec{V}(t, \cdot) = \emptyset \implies \langle \mathcal{G}(\Omega), \vec{V} \rangle = 0.$$

Let  $\vec{V}(t) \in \mathcal{C}_0^k(\widehat{\Omega}; \mathbb{R}^n)$  be a vector field such that  $\text{supp } \vec{V}(t) \cap \bar{\Omega} = \emptyset$ . Since  $\vec{V} = \vec{0}$  in  $\Omega$ , (2.1) yields  $\mathcal{H}_t(\vec{V}) = \mathcal{I}$ . Hence we have  $\mathcal{H}_t(\vec{V})(\Omega) \equiv \Omega_t = \Omega$  and  $dJ(\Omega; \vec{V}) = 0$ . So, it follows from (3.5) that  $\text{supp } \vec{V}(t) \subset \Omega$  and

$$(3.6) \quad \text{supp } \mathcal{G}(\Omega) \subset \bar{\Omega}.$$

Next, we suppose that  $\text{supp } \vec{V}(t) \subset \Omega$ . Then, there exists an open set  $\mathcal{O}$  and a one-to-one transformation  $\tilde{\mathcal{H}}_t$  such that  $\text{supp } \vec{V}(t) \subset \mathcal{O} \subset \Omega$  and  $\tilde{\mathcal{H}}_t(\vec{V})(\mathcal{O}) = \mathcal{O}$ , which can be obtained by using the similar technique as the previous section. We consider a one-to-one transformation  $\mathcal{H}_t(\vec{V})$  defined on  $\widehat{\Omega}$  by

$$\mathcal{H}_t(\vec{p}) = \begin{cases} \tilde{\mathcal{H}}_t(\vec{V})(\vec{p}), & \text{if } \vec{p} \in \bar{\mathcal{O}}, \\ \vec{p} & \text{if } \vec{p} \in \widehat{\Omega} - \bar{\mathcal{O}}. \end{cases}$$

Then, clearly  $\mathcal{H}_t(\vec{V})$  satisfies (2.1) and we have  $\mathcal{H}_t(\Omega) = \Omega$ . This implies that  $\gamma_\Omega(\mathcal{G}(\Omega)) = \vec{0}$  and hence that  $\text{supp } \mathcal{G}(\Omega) \subset \widehat{\Omega} - \bar{\Omega}$ . So, combined with (3.6), the result follows.

(iii); (3.3) immediately follows from (i). Since the distribution with a compact support has a finite order, from (ii),  $g(\Gamma)$  has a finite

order. Let  $\Omega \subset \widehat{\Omega}$  be a domain of class  $\mathcal{C}^k$ , ( $k \geq 1$ ). Then the normal vector field  $\vec{n}$  exists and belongs to  $\mathcal{C}^{k-1}(\Gamma; \mathbb{R}^n)$ . Consider the continuous linear mapping

$$dJ(\Omega; \cdot) : \mathcal{C}^k(\widehat{\Omega}; \mathbb{R}^n) \ni \vec{V} \longmapsto dJ(\Omega; \vec{V}) = \langle \mathcal{G}(\Omega), \vec{V}(0) \rangle .$$

Let  $\mathcal{K}(\Omega) = \{ \vec{V} \in \mathcal{C}^k(\widehat{\Omega}; \mathbb{R}^n) \mid \vec{V}(0) \cdot \vec{n} = 0 \text{ on } \Gamma \}$ . We first show that

$$(3.7) \quad \mathcal{K}(\Omega) \subset \ker dJ(\Omega; \cdot);$$

Let  $\vec{V}(0) \cdot \vec{n} = 0$  on  $\Gamma$ . Then  $\vec{V}(0)$  is a tangent vector field to  $\Gamma$ . Hence transversality along  $\Gamma$  does not occur, i.e.,  $\mathcal{H}_t(\vec{V})(\Omega) = \Omega$ . Hence  $dJ(\Omega; \vec{V}) = 0$ , so that  $\vec{V} \in \ker dJ(\Omega; \cdot)$ . Therefore, there exists a continuous linear functional  $\Lambda$  on  $\mathcal{C}^k(\widehat{\Omega}; \mathbb{R}^n)/\mathcal{K}(\Omega)$  such that

$$dJ(\Omega; \vec{V}) = \Lambda \circ \pi ,$$

where  $\pi : \mathcal{C}^k(\widehat{\Omega}; \mathbb{R}^n) \ni \vec{V}(0) \mapsto [\vec{V}(0)] \in \mathcal{C}^k(\widehat{\Omega}; \mathbb{R}^n)/\mathcal{K}(\Omega)$  denotes the canonical projection. It is not difficult to show that  $\mathcal{C}^k(\widehat{\Omega}; \mathbb{R}^n)/\mathcal{K}(\Omega)$  is isomorphic to  $\mathcal{C}^{k-1}(\Gamma)$  (see [5]). So, we can regard  $\Lambda$  as a continuous linear functional  $dJ(\Gamma; \cdot)$  on  $\mathcal{C}^{k-1}(\Gamma)$  and  $\pi$  as a projection map onto  $\mathcal{C}^{k-1}(\Gamma)$  via  $\pi(\vec{V}(0)) = [\vec{V}(0) \cdot \vec{n}]$ . Hence there exists a distribution  $g(\Gamma)$  of  $(k-1)$ -order in this case such that

$$dJ(\Omega; \vec{V}) = dJ(\Gamma; \vec{V}(0) \cdot \vec{n}) = \langle g(\Gamma), \vec{V}(0) \cdot \vec{n} \rangle_{\Gamma} .$$

□

REMARK. From (3.7),  $dJ(\Omega; \vec{V})$  can be written as

$$dJ(\Omega; \vec{V}) = \langle \mathcal{G}(\Omega), (\vec{V}(0) \cdot \vec{N})\vec{N} \rangle ,$$

where  $\vec{N}$  is a unitary extension of  $\vec{n}$  to  $\widehat{\Omega}$ . Such an extension always exists if  $\Gamma$  is of class  $\mathcal{C}^k$ , ( $k \geq 1$ ), which can be verified by using the local atlas along the boundary of the domain and patching them using cutoff functions (see [10] and [12], for details). Hence  $dJ(\Omega; \vec{V})$  can be written in the integral form as

$$dJ(\Omega; \vec{V}) = \int_{\widehat{\Omega}} (\mathcal{G}(\Omega) \cdot \vec{N})(\vec{V}(0) \cdot \vec{N}) d\Omega .$$

Consequently, in the representation of (3.3) and (3.4),  $g(\Gamma)$  can be related to  $\mathcal{G}(\Omega)$  via  $g(\Gamma) = \gamma_{\Gamma}(\mathcal{G}(\Omega) \cdot \vec{N})$ .

REMARK. Hadamard took the variation of the domain functional only in the normal direction to the boundary of the smooth domain to obtain necessary optimality conditions for specific problems(c.f. [5]). This method is usually called the normal variation method. On contrast, our approach yields a generalization of the normal variation method.

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