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ON LEFT(RIGHT) SEMI-REGULAR AND g-REGUAR po-SEMIGROUPS

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Abstract. In this paper, we give some properties of left(right) semi-regular and g-regular po-semigroups.

1. Introduction

Lee introduced the concepts of the left(right) semi-regularity([8]) and the g-regularity ([9]) in a po-semigroup S which are the generalized regularities. The author investigates some characterizations of the left(right) semi-regularity in terms of some types of ideals([8], [10], [11]).

In this paper, we give some properties of left(right) semi-regular and g-regular po-semigroups.

A po-semigroup(: ordered semigroup) is an ordered set $(S, \leq)$ at the same time a semigroup such that $a \leq b \Longrightarrow ca \leq cb$ and $ac \leq bc$ for all $c \in S$.

An element a of a po-semigroup S is *regular* if $a \leq axa$ for some $x \in S$ and S is *regular* if every element of S is regular. An element a of a po-semigroup S is *left*(resp. *right*) *regular* if $a \leq xa^2$(resp. $a \leq a^2x$) for some $x \in S$ and S is *left*(resp. *regular*) if every element of S is left(resp. right) regular([3], [4]). An element a of a po-semigroup S is *left*(resp. *right*) *semi-regular* if $a \leq xaya$(resp. $a \leq ax'ay'$) for some $x, y, x', y' \in S$ and S is *left*(resp. *right*) *semi-regular* if every element of S is left(resp. right) semi-regular([8]). An element a of a po-semigroup S is *g-regular* if $a \leq xay$ for some $x, y \in S$ and S is *g-regular* if every element of S is g-regular([9]).

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Remark. (1) A regular(left regular) *po*-semigroup S is left semi-regular.

(2) A regular(right regular) *po*-semigroup S is right semi-regular.

(3) A left(right) semi-regular *po*-semigroup S is g-regular.

The converses of (1), (2) and (3) are not true, in general(cf. Example 1, 2).

We denote $(H] := \{t \in S | t \leq h \ \text{for some} \ h \in H\}$ for a subset H of a *po*-semigroup S.

A non-empty subset A of a *po*-semigroup S is called a *left*(resp. *right*) *ideal* of S if (1) $SA \subseteq A$ (resp. $AS \subseteq A$), (2) $a \in A,\ b \leq a$ for $b \in S$ imply $b \in A$. A non-empty subset A of a *po*-semigroup S is an *ideal* of S if it is both a left and a right ideal of S([5]).

We note that the left, right ideal and the ideal of *po*-semigroup S generated by $a \in S$ are respectively:

$$L(a) = (a \cup Sa],\ R(a) = (a \cup aS],\ I(a) = (a \cup Sa \cup aS \cup SaS].$$

For a *po*-semigroup S, the Green's relations $\mathcal{L}, \mathcal{R}$ are defined as follows:

$$\mathcal{R} := \{(x, y) \mid R(x) = R(y)\}, \qquad \mathcal{L} := \{(x, y) \mid L(x) = L(y)\}.$$

Then $\mathcal{L}$ and $\mathcal{R}$ are equivalence relations of a *po*-semigroup S.

2. Main Theorems

Theorem 1. *If an element a of a po-semigroup S is left*(resp. *right*) *semi-regular and $b\mathcal{L}a$*(resp. *$b\mathcal{R}a$,*)*$(b \in S)$, then b is left*(resp. *right*) *semi-regular.*

Proof. If $b\mathcal{L}a$, then $(b \cup Sb] = (a \cup Sa]$. Thus ($b \leq a$ or $b \leq ua$) and ($a \leq b$ or $a \leq vb$) for some $u, v \in S$. Thus we have four cases: (1) $b = a$, (2) $b \leq a,\ a \leq vb$, (3) $b \leq ua,\ a \leq b$ or (4) $b \leq ua,\ a \leq vb$ for some $u, v \in S$.

Case (1) If $b = a$, then $b = a \leq xaya = xbyb$ for some $x, y \in S$.

Case (2) If $b \leq a$, and $a \leq vb$, then $b \leq a \leq xaya \leq x(vb)y(vb) = (xv)b(yv)b$ for some $x, y, v \in S$.

Case (3) If $b \leq ua$, and $a \leq b$, then $b \leq ua \leq uxaya \leq (ux)byb$ for some $x, y, u \in S$.

Case (4) If $b \leq ua$, and $a \leq vb$, then $b \leq ua \leq uxaya \leq ux(vb)y(vb)$ $= (uxv)by(vb)$ for some $x, y, u, v \in S$.

For any cases, $b \in (SbSb]$. Therefore b is left semi-regular.

If $b\mathcal{R}a$, then we can show that b is right semi-regular by the similar method. □

By the Example 2 in the next section, we have the following theorem.

THEOREM 2. *If an element a of a po-semigroup S is left(*resp. *right) semi-regular, then $L(a)$(*resp. *$R(a)$) need not be left(*resp. right*) semi-regular.*

THEOREM 3. *If an element a of a po-semigroup S is g-regular and $b\mathcal{L}a(b\mathcal{R}a)(b \in S)$, then b is g-regular.*

Proof. If $b\mathcal{L}a$, then $(b \cup Sb] = (a \cup Sa]$. Thus ($b \leq a$ or $b \leq ua$) and ($a \leq b$ or $a \leq vb$) for some $u, v \in S$. Thus we have four cases: (1) $b = a$, (2) $b \leq a,\ a \leq vb$, (3) $b \leq ua,\ a \leq b$ or (4) $b \leq ua,\ a \leq vb$ for some $u, v \in S$.

Case (1) If $b = a$, then $b = a \leq xay = xby$ for some $x, y \in S$.

Case (2) If $b \leq a,\ a \leq vb$: $b \leq a \leq xay \leq x(vb)y = (xv)by$ for some $x, y, v \in S$.

Case (3) If $b \leq ua,\ a \leq b$: $b \leq ua \leq uxay \leq (ux)by$ for some $x, y, u \in S$.

Case (4) If $b \leq ua,\ a \leq vb$: $b \leq ua \leq uxay \leq ux(vb)y = (uxv)by$ for some $x, y, u, v \in S$.

For any cases, $b \in (SbS]$. Therefore b is g-regular.

If $b\mathcal{R}a$, then we can show that b is g-regular by the similar method. □

By the Example 3 in the next section, we have the following theorem.

Theorem 4. *If an element x of a po-semigroup S is regular and $y\mathcal{L}x(y\mathcal{R}x)(y \in S)$, then y need not to be a regular element.*

3. Examples

EXAMPLE 1([1]). Let $S := \{a, b, c, d, f, g\}$ be a po-semigroup with Cayley table (Table 1) and Hasse diagram (Figure 1) as follows:

$\cdot$	a	b	c	d	f	g
a	b	b	a	d	a	a
b	b	b	b	d	b	b
c	a	b	c	d	c	c
d	d	d	d	d	d	d
f	a	b	c	d	c	c
g	a	b	c	d	f	g

Table 1

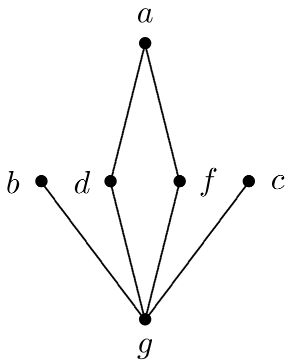

Figure 1

S is g-regular. Indeed: $cac = ac = a \geq a$ and $afc = ac = a \geq f$. And for other elements, it is trivial.

S is not left(right) semi-regular. Indeed: $(aSaS] = (SaSa] = \{b, d, g\}$ for $a \in S$. Thus there does not exist $x, y \in S$ such that $a \leq axay$ and $a \leq xaya$ for $a \in S$.

Hence the g-regularity is a generalised concept than the left(right) semi-regularity in po-semigroups.

Example 2([2]). Let $S := \{a, b, c, d, e\}$ be a po-semigroup with Cayley table (Table 2) and Hasse diagram (Figure 2) as follows:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	a	c
d	d	d	d	d	d
e	d	d	e	d	e

Table 2

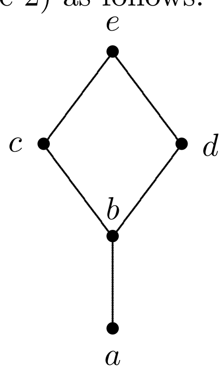

Figure 2

S is left semi-regular but not right semi-regular. Indeed: If $x \in S$ is idempotent, then $x = x^2 = x^4 \in SxSx \subseteq (SxSx]$ and $x = x^2 = x^4 \in SxSx \subseteq\in (xSxS]$. Thus x is left(right) semi-regular. Since all elements of S except b are idempotent, it is sufficient to show that b is left semi-regular. Since $(SbSb] = (\{a,d\}Sb] = (\{a,d\}b] = (\{a,d\}] = \{a,b,d\}$, $b \in (SbSb]$. Thus $b \leq xbyb$ for some $x,y \in S$, and so S is left semi-regular.

But, since $(bSbS] = (\{a\}bS] = (aS] = (a] = \{a\}$, $b \notin (bSbS]$. Thus b is not right semi-regular, and so S is not right semi-regular. Also S is not regular.
Left ideals generated by an element of S are $L(a) = L(b) = L(d) = \{a,b,d\}$ and $L(c) = L(e) = S$. Right ideals generated by an element of S are $R(a) = \{a\}, R(b) = \{a,b\}, R(c) = \{a,b,c\}, R(d) = \{a,b,d\}$ and $R(e) = S$.

Moreover d is right semi-regular. But $R(d)$ is not right semiregular because $R(d)$ contains b which is not right semi-regular.

Example 3([6]). Let $S := \{a,b,c,d,e\}$ be a po-semigroup with Cayley table (Table 3) and Hasse diagram (Figure 3) as follows:

$\cdot$	a	b	c	d	e
a	b	b	b	b	b
b	b	b	b	b	b
c	c	c	c	c	c
d	c	c	c	d	d
e	c	c	c	d	e

Table 3

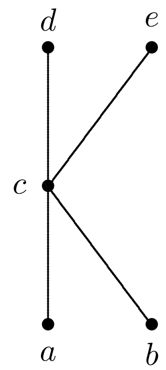

Figure 3

Left ideals generated by an element of S are $L(a) = L(b) = L(c) = \{a,b,c\}, L(d) = \{a,b,c,d\}$ and $L(e) = S$. Right ideals generated by an element of S are $R(a) = \{a,b\}$, $R(b) = \{b\}$, $R(c) = \{a,b,c\}$, $R(d) = \{a,b,c,d\}$ and $R(e) = S$.

Since $(aSa] = (b] = \{b\}$, $a \notin (aSa]$. Thus a is not a regular element. But b,c,d,e are regular elements. Since $L(a) = \{a,b,c\} = L(b)$, $a\mathcal{L}b$. b is a regular element of S, but a is not a regular element.

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