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NOTE ON THE FUZZY PROXIMITY SPACES

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Abstract. This paper is devoted to the study of the role of fuzzy proximity spaces. We define a fuzzy K-proximity space, a fuzzy R-proximity space and prove some of its properties. Furthermore, we discuss the topological structure based on these fuzzy K-proximity and fuzzy R-proximity.

1. Introduction

The concept of fuzzy set was introduced by Zadeh [11] in 1965. This idea was used by Chang [2],who in 1968 defined fuzzy topological spaces, and by Lowen [6],who in 1974 defined fuzzy uniform spaces. More recently, Katsaras [3], who in 1979, defined fuzzy proximities, on the base of the axioms suggested by Efremovič [8].

In this paper we propose some generalization of the concept of the fuzzy proximity, which we call a "fuzzy K-proximity" and a "fuzzy R-proximity". We also try to examine some of its properties and characterize the topological structure based on these fuzzy K-proximity and fuzzy R-proximity.

2. Preliminaries

As a preparation, we briefly review some basic definitions concerning a fuzzy proximity space. Throughout this paper, X is reserved to denote a nonempty set and let I^X be the collection of all mappings from X to the unit closed interval $I = [0, 1]$ of the real line. A member

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μ of I^X is called a *fuzzy set* of X. For any $\mu, \rho \in I^X$, the *join* $\mu \vee \rho$, and the *meet* $\mu \wedge \rho$ of μ and ρ defined as followings: For any $x \in X$,

$$(\mu \vee \rho)(x) = sup\{\mu(x), \rho(x)\} \text{ and } (\mu \wedge \rho)(x) = inf\{\mu(x), \rho(x)\},$$

respectively. And $\mu \leq \rho$ if for each $x \in X$, $\mu(x) \leq \rho(x)$. The *complement* μ' of a fuzzy set μ in X is $1 - \mu$ defined by $\mu'(x) = (1 - \mu)(x) = 1 - \mu(x)$ for each $x \in X$. 0 and 1 denote constant functions mapping all of X to 0 and 1, respectively. Now we give the definitions of a fuzzy topology and a closure operator.

Definition 2.1. A *fuzzy topology* on X is a subset α of I^X which satisfies the following conditions:

(FT1) $0, 1 \in \alpha$.
(FT2) If $\mu, \rho \in \alpha$, then $\mu \wedge \rho \in \alpha$.
(FT3) If $\mu_i \in \alpha$ for each $i \in A$, then $sup_{i \in A} \mu_i \in \alpha$.

The pair (X, α) is called a *fuzzy topological space*, or *fts* for short.

Definition 2.2. A map $\mu \mapsto cl(\mu)$, from I^X into I^X, is said to be a *closure operator* if it satisfies the following conditions:

(C1) $\mu \leq cl(\mu)$.
(C2) $cl(cl(\mu)) = cl(\mu)$.
(C3) $cl(\mu \vee \rho) = cl(\mu) \vee cl(\rho)$.
(C4) $cl(0) = 0$.

Given a closure operator on I^X, the collection

$$\{\mu \in I^X \mid cl(1 - \mu) = 1 - \mu\}$$

is a fuzzy topology on X.

In the following we first define a fuzzy proximity space and a fuzzy point. Let δ be a binary relation on I^X, i.e., $\delta \subset I^X \times I^X$. The facts that $(\mu, \rho) \in \delta$ and $(\mu, \rho) \notin \delta$ are denoted by $\mu\delta\rho$ and $\mu\bar{\delta}\rho$, respectively.

Definition 2.3. A binary relation δ on I^X is called a *fuzzy proximity* if δ satisfies the following conditions:

(FP1) $\mu\delta\rho$ implies $\rho\delta\mu$.
(FP2) $(\mu \vee \rho)\delta\sigma$ if and only if $\mu\delta\sigma$ or $\rho\delta\sigma$.

(FP3) $\mu\delta\rho$ implies $\mu \neq 0$ and $\rho \neq 0$.
(FP4) $\mu\overline{\delta}\rho$ implies that there exists a $\sigma \in I^X$ such that $\mu\overline{\delta}\sigma$ and $(1-\sigma)\overline{\delta}\rho$.
(FP5) $\mu \wedge \rho \neq 0$ implies $\mu\delta\rho$.

The pair $(X, \delta)$is called a *fuzzy proximity space*.

Definition 2.4. A fuzzy set in X is called a *fuzzy point* if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is $\gamma(0 < \gamma < 1)$, we denote this fuzzy point by x_γ, where the point x is called its *support*.

Definition 2.5. The fuzzy point x_γ is said to be *contained in a fuzzy set* μ, or to *belong to* μ, denoted by $x_\gamma \in \mu$, if $\gamma < \mu(x)$. Evidently, every fuzzy set μ can be expressed as the union of all the fuzzy points which belong to μ.

3. Fuzzy K-Proximity

We define a fuzzy K-proximity space and we investigate some properties of this structure.

Definition 3.1. A binary relation δ on I^X is called a *fuzzy K-proximity* if δ satisfies the following conditions:

(FK1) $x_\gamma\delta(\mu \vee \rho)$ if and only if $x_\gamma\delta\mu$ or $x_\gamma\delta\rho$.
(FK2) $x_\gamma\overline{\delta}0$ for all x_γ.
(FK3) $x_\gamma \in \mu$ implies $x_\gamma\delta\mu$.
(FK4) $x_\gamma\overline{\delta}\mu$ implies that there exists a $\rho \in I^X$ such that $x_\gamma\overline{\delta}\rho$ and $y_\gamma\overline{\delta}\mu$ for all $y_\gamma \in (1-\rho)$.

The pair (X, δ) is called a *fuzzy* K-*proximity space*.

One can easily show that the fuzzy proximity on I^X implies the fuzzy K-proximity on I^X.

Theorem 3.2. *Every fuzzy proximity on I^X implies the fuzzy K-proximity on I^X.*

Proof. (FP1) and (FP2) implies (FK1), (FP3) implies (FK2), and (FP5) implies (FK3). If $\mu = \{x_\gamma\}$ and $\mu\overline{\delta}\rho$, then (FP4) there exists a

$\sigma \in I^X$ with $x_\gamma \overline{\delta} \sigma$, and $(1-\sigma)\overline{\delta}\rho$. Hence for each $y_\gamma \in (1-\sigma)$, we have $y_\gamma \overline{\delta} \rho$. This means that (FP1) and (FP4) implies (FK4). □

Now we shall introduce the fuzzy proximity δ_1 from the fuzzy K-proximity δ replacing the axiom (FK4) in the fuzzy K-proximity by the stronger one.

Definition 3.3. A binary relation δ on I^X is called the *fuzzy proximity* if δ satisfies the axioms (FP1), (FP2), (FP3) in the Definition 2.3, and (FP4′) For each $\sigma \in I^X$ there is a fuzzy point x_γ such that either $x_\gamma \delta \mu$, $x_\gamma \delta \sigma$ or $x_\gamma \delta \rho$, $x_\gamma \delta (1-\sigma)$, then we have $x_\gamma \delta \mu$ and $x_\gamma \delta \rho$.

Definition 3.4. In a fuzzy K-proximity space (X, δ), let δ_1 be a binary relation on I^X defined as follows : For each $\mu, \rho \in I^X$,

$\mu \delta_1 \rho$ if and only if there is a fuzzy point x_γ such that $x_\gamma \delta \mu$ and $x_\gamma \delta \rho$.

Theorem 3.5. *The binary relation δ_1 on I^X defined in Definition 3.4 is the fuzzy proximity*

Proof. We will show that δ_1 satisfies (FP1) $\sim$ (FP5).

(FP1) It is clear that $\mu \delta_1 \rho$ implies $\rho \delta_1 \mu$.

(FP2)

$$\begin{aligned}
(\mu \vee \rho)\delta_1 \sigma &\iff \exists \text{ a fuzzy point } x_\gamma \text{ such that } x_\gamma \delta (\mu \vee \rho) \text{ and } x_\gamma \delta \sigma \\
&\iff (x_\gamma \delta \mu \text{ or } x_\gamma \delta \rho) \text{ and } x_\gamma \delta \rho \\
&\iff (x_\gamma \delta \mu, x_\gamma \delta \sigma) \text{ or } (x_\gamma \delta \rho, x_\gamma \delta \sigma) \\
&\iff \mu \delta_1 \sigma \text{ or } \rho \delta_1 \sigma.
\end{aligned}$$

(FP3)

$$\begin{aligned}
\mu \delta_1 \rho &\implies \exists \text{ a fuzzy piont } x_\gamma \text{ such that } x_\gamma \delta \mu \text{ and } x_\gamma \delta \rho \\
&\implies \mu \neq 0 \text{ and } \rho \neq 0.
\end{aligned}$$

(FP4) Suppose that for each $\sigma \in I^X$, $\mu \delta_1 \sigma$ or $\rho \delta_1 (1-\sigma)$. Hence for some fuzzy point x_γ we have either $x_\gamma \delta \mu$, $x_\gamma \delta \sigma$ or $x_\gamma \delta \rho$, $x_\gamma \delta (1-\sigma)$, therefore by (FP4′) $x_\gamma \delta \mu$ and $x_\gamma \delta \rho$, that is, $\mu \delta_1 \rho$.

(FP5)

$$\begin{aligned}\mu\wedge\rho\neq 0 &\implies \exists \text{ a fuzzy point } x_\gamma \text{ such that } x_\gamma\in\mu \text{ and } x_\gamma\in\rho\\ &\implies x_\gamma\delta\mu \text{ and } x_\gamma\delta\rho\\ &\implies \mu\delta_1\rho.\end{aligned}$$

□

In what follows we introduce some properties of the fuzzy K-prox imity.

LEMMA 3.6. *If $x_\gamma\delta\mu$ and $\mu\leq\rho$, then $x_\gamma\delta\rho$.*

Proof. By (FK1) $x_\gamma\delta\mu \implies x_\gamma\delta(\mu\vee\rho) \implies x_\gamma\delta\rho$. □

THEOREM 3.7. *In the fuzzy K-proximity space (X,δ) if μ^δ is defined to be a set $\bigvee\{x_\gamma \mid x_\gamma\delta\mu$ and x_γ is a fuzzy point in $X\}$ for each fuzzy set μ in X, then δ is a closure operator. Hence we can introduce the fuzzy topology $\mathcal{T}(\delta)$ on X by δ.*

Proof. Since the other axioms are easily verified, it suffices to show that δ satisfies (C2). So, we assume that $x_\gamma\overline{\delta}\mu$. Then by (FK4) there exists a $\rho\in I^X$ such that $x_\gamma\overline{\delta}\rho$ and $y_\gamma\overline{\delta}\mu$ for all $y_\gamma\in(1-\rho)$. If $z_\gamma\in\mu^\delta$, then $z_\gamma\delta\mu$. Hence $z_\gamma\in\rho$, that is $\mu^\delta\leq\rho$. Since $x_\gamma\overline{\delta}\rho$ we have $x_\gamma\overline{\delta}\mu^\delta$. This means that $x_\gamma\in\mu^{\delta\delta}$ implies $x_\gamma\in\mu^\delta$ or $\mu^{\delta\delta}\subset\mu^\delta$. Therefore $\mu^{\delta\delta}=\mu^\delta$. □

THEOREM 3.8. *Let (X,α) be a fuzzy topological space. If a binary relation δ is defined by $x_\gamma\delta\mu$ if and only if $x_\gamma\in cl(\mu)$, then δ is a fuzzy K-proximity on I^X and the fuzzy topology $\mathcal{T}(\delta)$ induced by δ is the given topology α.*

Proof. Now we will show that δ satisfies (FK1) $\sim$ (FK4).
(FK1)

$$\begin{aligned}x_\gamma\delta(\mu\vee\rho) &\iff x_\gamma\in cl(\mu\vee\rho)\\ &\iff x_\gamma\in cl(\mu)\vee x_\gamma\in cl(\rho)\\ &\iff x_\gamma\delta\mu \text{ or } x_\gamma\delta\rho.\end{aligned}$$

(FK2)

$$cl(0) = 0 \implies x_\gamma \overline{\delta} 0 \text{ for all } x_\gamma.$$

(FK3)

$$\begin{aligned} x_\gamma \in \mu &\implies x_\gamma \in cl(\mu) \\ &\implies x_\gamma \delta \mu. \end{aligned}$$

(FK4)

$$\begin{aligned} x_\gamma \overline{\delta} \mu &\iff x_\gamma \notin cl(\mu) \\ &\iff x_\gamma \notin cl(cl(\mu)) \\ &\iff x_\gamma \overline{\delta} cl(\mu) \\ &\iff \text{ if } cl(\mu) = \rho, \text{ then } x_\gamma \overline{\delta} \rho \text{ and } y_\gamma \overline{\delta} \mu \text{ for all } y_\gamma \in (1 - cl(\mu)). \end{aligned}$$

Since $x_\gamma \in cl(\mu) \iff x_\gamma \delta \mu \iff x_\gamma \in \mu^\delta$, we have $cl(\mu) = \mu^\delta$, that is, $\mathcal{T}(\delta) = \alpha$. □

THEOREM 3.9. *The fuzzy topological space X is T_1 if and only if there is a fuzzy K-proximity δ on I^X satisfying the following condition:*
(FK5) $x_\gamma \delta \{y_\gamma\} \implies x_\gamma = y_\gamma$.

Proof. Assume X is T_1. Then there is a binary relation δ on I^X satisfying conditions (FK1) $\sim$ (FK4). So, $x_\gamma \in \mu^\delta \iff x_\gamma \delta \mu$. Hence $x_\gamma \delta \{y_\gamma\} \implies x_\gamma \in \{y_\gamma\}^\delta = \{y_\gamma\}$, since X is T_1. That is, $x_\gamma = y_\gamma$.

Conversely, if $x_\gamma \delta \{y_\gamma\}$ implies that $x_\gamma = y_\gamma$ then $\{y_\gamma\}^\delta = \{y_\gamma\}$, that is, X is T_1. □

LEMMA 3.10. $x_\gamma \delta \{y_\gamma\}$ *and* $y_\gamma \delta \mu \implies x_\gamma \delta \mu$.

Proof.

$$\begin{aligned} x_\gamma \overline{\delta} \mu &\implies \exists \rho \text{ such that } x_\gamma \overline{\delta} \rho \text{ and } z_\gamma \overline{\delta} \mu \text{ for all } z_\gamma \in (1 - \rho) \\ &\implies y_\gamma \notin \rho (\text{ if } y_\gamma \in \rho \text{ then } x_\gamma \delta \{y_\gamma\},\ y_\gamma \in \rho \text{ so we have } x_\gamma \delta \rho) \\ &\implies y_\gamma \in (1 - \rho),\ \text{that is},\ y_\gamma \overline{\delta} \mu. \end{aligned}$$

It is a contradiction. □

4. Fuzzy R-Proximity

We introduce a fuzzy R-proximity and we prove that some of properties of this notion.

DEFINITION 4.1. A binary relation δ on I^X is called a *fuzzy R-proximity* if δ satisfies the following conditions:

(FR1) $\mu\delta\rho$ implies $\rho\delta\mu$.
(FR2) $(\mu \vee \rho)\delta\sigma$ if and only if $\mu\delta\sigma$ or $\rho\delta\sigma$.
(FR3) $\mu\delta\rho$ implies $\mu \neq 0$ and $\rho \neq 0$.
(FR4) $x_\gamma\overline{\delta}\mu$ implies that there exists a $\rho \in I^X$ such that $x_\gamma\overline{\delta}\rho$ and $(1-\rho)\overline{\delta}\mu$.
(FR5) $\mu \wedge \rho \neq 0$ implies $\mu\delta\rho$.

The pair (X, δ) is called a *fuzzy* R-*proximity space*.

THEOREM 4.2. *In a fuzzy R-proximity space (X, δ) if μ^δ is defined to be a set $\bigvee\{x_\gamma \mid x_\gamma\delta\mu$ and x_γ is a fuzzy point in $X\}$ for each fuzzy set μ in X, then δ is a closure operator. Hence we can introduce the fuzzy topology $\mathcal{T}(\delta)$ on X by δ.*

Proof. Now we will show that δ is a closure operator.

(C1) Suppose that $\mu \neq 0$. There exists $y \in X$ such that $\mu(y) \neq 0$. Consider the fuzzy point $y_\gamma \in I^X$. Here $y_\gamma \wedge \mu \neq 0$ and therefore $y_\gamma\delta\mu$. Also, $\mu = \bigvee_{\mu(y)\neq 0} y_\gamma$. Hence, $\mu^\delta = \bigvee\{x_\gamma \mid x_\gamma\delta\mu\} \geq \bigvee_{\mu(y)\neq 0} y_\gamma = \mu$. Consequently $\mu^\delta \geq \mu$.

(C2) For this, it suffices to show that $x_\gamma\delta\mu^\delta$ if and only if $x_\gamma\delta\mu$. Suppose that $x_\gamma\delta\mu$. Then $x_\gamma\delta\mu^\delta$ because of $\mu \leq \mu^\delta$. Conversely, suppose that $x_\gamma\delta\mu^\delta$. Now $y_\gamma \leq \mu^\delta$ implies $y_\gamma \leq \bigvee\{x_\gamma \mid x_\gamma\delta\mu\}$, which gives $y_\gamma \leq x_p$ for some x_p such that $x_p\delta\mu$. We have $y_\gamma\delta\mu$. Thus, we get $x_\gamma\delta\mu^\delta$ and $y_\gamma\delta\mu$ for each $y_\gamma \leq \mu^\delta$. Hence $x_\gamma\delta\mu$.

(C3)

$$\begin{aligned}
(\mu \vee \rho)^\delta &= \bigvee\{x_\gamma \mid x_\gamma\delta(\mu \vee \rho)\} \\
&= \bigvee\{x_\gamma \mid x_\gamma\delta\mu \text{ or } x_\gamma\delta\rho)\} \\
&= (\bigvee\{x_\gamma \mid x_\gamma\delta\mu\}) \vee (\bigvee\{x_\gamma \mid x_\gamma\delta\rho\}) \\
&= \mu^\delta \vee \rho^\delta
\end{aligned}$$

(C4) It is also easy to see that $0^\delta = 0$. □

THEOREM 4.3. *If (X,δ) is a fuzzy R-proximity space, then $\mathcal{T}(\delta)$ is fuzzy R_0 regular.*

Proof. Let μ be a fuzzy closed set and x_γ a fuzzy point such that $x_\gamma\overline{\delta}\mu$. Then there is a ρ such that $x_\gamma\overline{\delta}\rho$ and $(1-\rho)\overline{\delta}\mu$. Hence $x_\gamma \wedge \rho^\delta = 0$ or $x_\gamma \leq 1-\rho^\delta = \sigma$. On the other hand $\mu \wedge (1-\rho)^\delta = 0$ or $\mu \leq 1-(1-\rho)^\delta = \lambda$, that is, $1-\lambda \leq 1-\mu$. Since $\sigma \wedge \lambda = 0$, there exist fuzzy open sets σ, λ such that $x_\gamma \leq \sigma \leq 1-\lambda \leq 1-\mu$.

To prove that the induced fuzzy topology $\mathcal{T}(\delta)$ also satisfies the R_0 axiom, i.e., $x_\gamma \in y_\gamma^\delta$ implies $y_\gamma \in x_\gamma^\delta$, let $x_\gamma \in y_\gamma^\delta$. Then $x_\gamma\delta y_\gamma$ if and only if $y_\gamma\delta x_\gamma$ if and only if $y_\gamma \in x_\gamma^\delta$. □

THEOREM 4.4. *In a fuzzy R_0 regular space $(X,\mathcal{T})$, let δ be a binary relation on I^X define as follows:*

$$\mu\delta\rho \text{ if and only if } \mu^\delta \wedge \rho^\delta \neq 0,$$

then δ is the fuzzy R-proximity, which is compatible with $\mathcal{T}$.

Proof. We will show that δ satisfies (FR1)∼(FR5).

(FR1) $\mu\delta\rho \implies \mu^\delta \wedge \rho^\delta \neq 0 \implies \rho^\delta \wedge \mu^\delta \neq 0 \implies \rho\delta\mu$.

(FR2)

$$\begin{aligned}(\mu\vee\rho)\delta\sigma &\iff (\mu\vee\rho)^\delta \wedge \sigma^\delta \neq 0\\ &\iff (\mu^\delta \vee \rho^\delta) \vee \sigma^\delta \neq 0\\ &\iff (\mu^\delta \wedge \sigma^\delta) \vee (\rho^\delta \wedge \sigma^\delta) \neq 0\\ &\iff \mu^\delta \wedge \sigma^\delta \neq 0 \text{ or } \rho^\delta \wedge \sigma^\delta \neq 0\\ &\iff \mu\delta\sigma \text{ or } \rho\delta\sigma.\end{aligned}$$

(FR3)

$$\begin{aligned}\mu\delta\rho &\implies \mu^\delta \wedge \rho^\delta \neq 0\\ &\implies \mu^\delta \neq 0 \text{ and } \rho^\delta \neq 0\\ &\implies \mu \neq 0 \text{ and } \rho \neq 0.\end{aligned}$$

(FR4) Suppose that $x_\gamma \overline{\delta} \mu$. Applying the definition of δ to $x_\gamma \overline{\delta} \mu$ we obtain $x_\gamma^\delta \wedge \mu^\delta = 0$ and hence either $x_\gamma^\delta = 0$ or $\mu^\delta = 0$. Since X is regular, there exist fuzzy open sets ρ, σ such that $x_\gamma \leq \rho \leq 1 - \sigma \leq 1 - \mu^\delta$. The following two cases arise:

Cases(1). $x_\gamma^\delta = 0$. Take $\sigma = 1$. Then $x_\gamma^\delta \wedge \sigma^\delta = 0$ implies $x_\gamma \overline{\delta} \sigma$, and $(1-\sigma)^\delta \wedge \mu^\delta = 0$ implies $(1-\sigma)\overline{\delta}\mu$.

Cases(2). $\mu^\delta = 0$. Take $\sigma = 0$. Then $x_\gamma^\delta \wedge \sigma^\delta = 0$ implies $x_\gamma \overline{\delta} \sigma$, and $(1-\sigma)^\delta \wedge \mu^\delta = 0$ implies $(1-\sigma)\overline{\delta}\mu$.

(FR5) $\mu \wedge \rho \neq 0 \implies \mu^\delta \wedge \rho^\delta \neq 0 \implies \mu \delta \rho$. □

THEOREM 4.5. *A fuzzy K-proximity space is also R-proximity.*

Proof. Let (X, δ) be a fuzzy K-proximity space. Then, $\mathcal{T}(\delta)$ is a fuzzy completely regular [4, 8]. Since a completely regular space is a regular, $\mathcal{T}(\delta)$ is a fuzzy R_0 regular. Hence, (X, δ) is a fuzzy R-proximity space. □

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