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INVESTIGATION OF THE ERROR DUE TO THE PRESENCE OF THE MAPPED ELEMENT

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Abstract. We analyze the error in the p version of the of the finite element method when the effect of the quadrature error is taken into account. We investigate source of quadrature error due to the presence of mapped elements. We present theoretical and computational examples regarding the sharpness of our results.

1. Introduction

Let Ω be a polygonal domain in R^2, or a line segment in R^1, and consider the following model problem on Ω,

$$Lu = -\text{div}\,(a\,\nabla u) \;=\; f \quad \text{in } \Omega \subset R^2, \tag{1.1}$$

$$Lu = -\frac{d}{dx}\left(a\,\frac{du}{dx}\right) \;=\; f \quad \text{in } \Omega \subset R^1, \tag{1.2}$$

with one of the following boundary conditions

$$u \;=\; 0 \quad \text{on } \Gamma, \tag{1.3}$$

$$\frac{\partial u}{\partial n} \;=\; g \quad \text{on } \Gamma. \tag{1.4}$$

Here, ∇ is the gradient operator and n denotes the unit outward normal to Γ defined almost everywhere on Γ. For the case of (1.4), we assume f, g satisfy a compatibility condition to ensure existence of a solution.

Now we define Sobolev spaces

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^i u \in L^p(\Omega), 0 \leq |i| \leq m\},$$

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equipped with norm

$$\|u\|^p_{m,p,\Omega} = \sum_{0\le|i|\le m} \|D^i u\|^p_{0,p,\Omega},$$

where m is a non-negative integer and $1 \le p < \infty$.

Let us define $H = H^1_0(\Omega)$, $H^1(\Omega)$ or $H^1_{per}(\Omega)$ corresponding to the boundary condition (1.3), (1.4) respectively. Then the variational form of (1.1)–(1.4) is to find $u \in H$ satisfying

$$B(u,v) = F(v) \quad \text{for all } v \in H, \tag{1.5}$$

where

$$B(u,v) = \int_\Omega a\,\nabla u \cdot \nabla v\,dx, \tag{1.6}$$

and

$$F(v) = (f,v) + < g, v > = \int_\Omega f v\,dx + \int_\Gamma g v\,ds, \tag{1.7}$$

with $g = 0$ unless (1.4) is in force.

2. Error Estimates under Finite Element Method

We now consider the approximation of the solution of problem (1.1)–(1.4) by the p version of the finite element method. Let τ be a fixed triangulation of $\bar{\Omega}$ by elements K_i which are line segments in R^1 and closed triangles and quadrilaterals in R^2. We will also consider the corresponding curvilinear elements. $K_i \cap K_j$ will be assumed to be either empty, a common vertex or an entire side of K_i and K_j. We assume that every $v \in V$ is a vertex of some K_i. The mesh on $\Omega \subset R^2$ also subdivides the boundary Γ into segments, which we denote by $\tilde{K}_i$.

We will often refer to the reference interval $I = [-1,1]$, the reference triangle $T = \{(x,y) : x \ge 0,\, y \ge 0,\, x + y \le 1\}$ and the reference square $Q = I^2$. For each $K \in \tau$, we assume that there exists an invertible map F_K such that

$$K = F_K(\hat{K}),$$

where $\hat{K} = I$, T or Q is the corresponding reference element. Hence we obtain the correspondences

$$\hat{x} \in \hat{K} \leftrightarrow x = F_K(\hat{x}) \in K, \tag{2.8}$$

$$(\hat{v} : \hat{K} \to R) \leftrightarrow (v = \hat{v} \circ F_K^{-1} : K \to R), \tag{2.9}$$

between the points $\hat{x} \in \hat{K}$ and $x \in K$, and between functions defined on $\hat{K}$ and K.

We will also assume that for each $\tilde{K} \in \Gamma$, there exists an invertible map $\tilde{F}_{\tilde{K}}$ such that

$$\tilde{K} = \tilde{F}_{\tilde{K}}(I),$$

which gives the correspondence

$$(\hat{v} : I \to R) \leftrightarrow (v = \hat{v} \circ \tilde{F}_{\tilde{K}}^{-1} : \tilde{K} \to R). \tag{2.10}$$

Our basis functions may be complex valued when we consider periodic boundary conditions.

Unless otherwise stated, the mappings F_K, F_K^{-1}(respectively $\tilde{F}_{\tilde{K}}$, $\tilde{F}_{\tilde{K}}^{-1}$) will be assumed to be sufficiently smooth with the Jacobian J_K (respectively $\tilde{J}_{\tilde{K}}$) positive, bounded below away from zero.

We now define the following polynomial spaces $\{U_p(\hat{K})\}$ on the reference elements. For $\hat{K} = I$, $U_p(\hat{K}) = \mathcal{P}_p(I)$, the set of polynomials of degree $\le p$ on I.

We then define

$$U_p(K) = \{v : \hat{v} \in U_p(\hat{K})\},$$

$$S_p = \{v \in C^0(\Omega) : \, v|_K \in U_p(K), \; \forall K \in \tau\}.$$

We also define

$$S_{p,0} = S_p \cap H.$$

Then the p version of the FEM to approximate the solution of (1.5) consists of finding, for $p = 1, 2, 3, \ldots$, a $u_p \in S_{p,0}$ satisfying

$$B(u_p, v) = F(v) \qquad \text{for all } v \in S_{p,0}. \tag{2.11}$$

Let us now give some results which will be used in the next section.

THEOREM 2.1. *Let $\{U_p(\hat{K})\}$ be a sequence of polynomial spaces described previously and let $S_{p,0}$ be the corresponding spaces on Ω. Then the sequence of projections*

$$P_p^1 : H_0^1(\Omega) \to S_{p,0}, \quad p = 1, 2, 3, \ldots,$$

defined by

$$\int_\Omega \nabla(P_p^1 u) \cdot \nabla v \, dx = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad \textit{for all } v \in S_{p,0}, \tag{2.12}$$

satisfy

$$\|v - P_p^1 v\|_{s,\Omega} \le Cp^{-(r-s)}\|v\|_{r,\Omega} \quad 0 \le s \le 1 < r, \tag{2.13}$$

with C a constant independent of p and v but dependent upon r.

Proof. The two-dimensional case has been proved in [3], the argument from which can be generalized to the three-dimensional case. A different proof of the n-dimensional case may be found in [10], the result being optimal up to arbitrary $\epsilon > 0$. □

3. Error Estimates under Numerical Quadrature

The p version introduced in the previous chapter and related results for it assume that all integrations have been performed exactly. In practice, even if the functions a, f and g have simple analytical expressions, the integrals which appear in (1.6) and (1.7) are seldom computed exactly. Instead, they are approximated through the process of numerical quadrature, which we now describe.

Let us assume that we are given a family of quadrature rules $\{R_p\}$ defined on the reference element $\hat{K}$ by

$$\int_{\hat{K}} \hat{\psi}(\hat{x})\, d\hat{x} \sim \sum_{l=1}^{L_p} \hat{\omega}_l^p\, \hat{\psi}(\hat{b}_l^p). \tag{3.14}$$

Then on each $K \in \tau$, we get a quadrature rule defined by

$$\int_K \psi(x)\, dx \sim \sum_{l=1}^{L_p} \omega_{l,K}^p\, \psi(b_{l,K}^p), \tag{3.15}$$

where $\omega_{l,K}^p = J_K(\hat{b}_l^p)\, \hat{\omega}_l^p$ and $b_{l,K}^p = F_K(\hat{b}_l^p)$. In the case of Neumann conditions in two-dimensional problems, we also assume that $\{\tilde{R}_p\}$ is a family of rules on I, defined analogously to (3.14), such that for each $\tilde{K} \subset \Gamma$, we have

$$\int_{\tilde{K}} \psi(x)\, dx \sim \sum_{l=1}^{\tilde{L}_p} \tilde{\omega}_{l,\tilde{K}}^p\, \psi(\tilde{b}_{l,\tilde{K}}^p).$$

Using these rules, the actual problem solved by numerical integration becomes: Find $\tilde{u}_p \in S_{p,0}$ satisfying

$$B_p(\tilde{u}_p, v) = F_p(v) = (f, v)_p + <g, v>_p, \quad \text{for all } v \in S_{p,0}, \tag{3.16}$$

where

$$(3.17)\quad B_p(u,v) = \sum_{K\in\tau} B_{p,K}(u,v) = \sum_{K\in\tau}\sum_{l=1}^{L} \omega_{l,K}\,[a\,\nabla u\cdot\nabla v](b_{l,K}),$$

$$(3.18)\quad (f,v)_p = \sum_{K\in\tau}(f,v)_{p,K} = \sum_{K\in\tau}\sum_{l=1}^{L}\omega_{l,K}\,(fv)(b_{l,K}),$$

$$(3.19)\quad < g,v >_p = \sum_{\tilde K\subset\Gamma} < g,v >_{p,\tilde K} = \sum_{\tilde K\subset\Gamma}\sum_{l=1}^{\tilde L}\tilde\omega_{l,\tilde K}\,(gv)(\tilde b_{l,\tilde K}),$$

where the dependence of L, $\omega_{l,K}$, etc on p is understood.

As in [5] and [6], we will only consider a family of quadrature rules $\{R_p\}$ that satisfy the following assumptions,

(A) $\hat\omega_l^p > 0$ and $\hat b_l^p \in \hat K$.

(B1) There exists a constant C_1, independent of p and $\hat v$ such that

$$\sum_{l=1}^{L_p}\hat\omega_l^p\,\hat v^2(\hat b_l^p) \le C_1\|\hat v\|_{0,\hat K}^2, \qquad \text{for all } \hat v\in U_p(\hat K).$$

(B2) There exists a constant C_2, independent of p and $\hat v$ such that

$$\sum_{l=1}^{L_p}\hat\omega_l^p\,\hat v^2(\hat b_l^p) \ge C_2\|\hat v\|_{0,\hat K}^2, \qquad \text{for all } \hat v\in \tilde U_p(\hat K),$$

where

$$\tilde U_p(\hat K) = \{\frac{\partial\hat v}{\partial\hat x_i} : \hat v\in U_p(\hat K), 1\le i\le n\}\subset U_p(\hat K).$$

(B3) R_p is exact for all $\hat v\in U_m(\hat K)$ with $m = m(p)\ge m_0(p)$.

The minimum value of $m_0(p)$ in (B3) depends upon the space $U_p(\hat K)$ (see [6]).

Often, (B1)–(B3) are satisfied because $\{R_p\}$ satisfies the following stronger condition,

(B) R_p is exact for all $\hat v\in U_m(\hat K)$ with $m\ge 2p$.

We will be particularly interested in Gauss-Legendre (G-L) rules. In the one-dimensional case, the p-point G-L rule on I satisfies (B1)–(B3) with $m = 2p-1$. In the two-dimensional case, the cross product of $(p+1)$-point G-L rules along the x and y axes on Q will satisfy (B) with $m = 2p+1$.

The conditions (A) and (B2) above guarantee solvability of our approximate problem, as seen from the following lemma, which was originally proved in [6].

LEMMA 3.1. *If the mappings F_K are smooth, then there exists a constant $C > 0$ such that*

$$C|v|^2_{1,\Omega} \leq B_p(v,v), \quad \text{for all } v \in S_{p,0},$$

with C independent of p.

Next, the error will depend upon the smoothness of the coefficient $a(x,y)$ and mappings F_K. To this end, let us note that

$$Dv(x) = D\hat{v}(\hat{x})DF_K^{-1}(x) \quad \text{for any } v \longleftrightarrow \hat{v},$$

so that

$$\begin{aligned} E &= E_{\hat{K}}\left(aJ_K(D\hat{u}DF_K^{-1})(D\hat{v}DF_K^{-1})^\top\right) \\ &= E_{\hat{K}}\left((D\hat{u})B^K(D\hat{v})^\top\right) \\ &= \sum_{i,j=1}^{n} E_{\hat{K}}\left(b_{ij}^K \frac{\partial \hat{u}}{\partial \hat{x}_i}, \frac{\partial \hat{v}}{\partial \hat{x}_j}\right). \end{aligned} \tag{3.20}$$

Here, the coefficients b_{ij}^K are entries of the matrix

$$B^K = aJ_K(DF_K^{-1})(DF_K^{-1})^T, \tag{3.21}$$

and for $P : R^n \to R^n$, DP denotes the Jacobian matrix of P. Suppose the b_{ij}^K are approximated by the polynomials $\bar{b}_{ij}^K \in U_q(\hat{K})$. Let us define

$$\rho_{t,s}(B) = \max_{i,j,K} \left\|b_{ij}^K\right\|_{t,s,\hat{K}},$$

with the subscript s omitted when $s = 2$. Then we get the following lemma which gives a bound for the approximation of b_{ij}^K.

We can prove the following Lemmas using Lemma 3.1.

LEMMA 3.2. *For $\Omega \subset R^n$, let $\rho_t(B) < \infty$, where $t > n/2$. Then there exists $\bar{b}_{ij}^K \in U_q(\hat{K})$ for which*

$$K_B^q = \max_{i,j,K} \left\|\bar{b}_{i,j}^K\right\|_{0,\infty,\hat{K}} \leq C(\rho_{0,\infty}(B) + q^{-(t-n/2)}\rho_t(B)), \tag{3.22}$$

$$e_B^q = \max_{i,j,K} \left\|b_{i,j}^K - \bar{b}_{i,j}^K\right\|_{0,\infty,\hat{K}} \leq C\, q^{-(t-n/2)}\rho_t(B). \tag{3.23}$$

LEMMA 3.3. *Let u_p and $\tilde{u}_p$ be the finite element solutions of (2.11) and (3.16) respectively and $\bar{b}_{ij}^K \in U_q(\hat{K})$. Then*

$$\|\tilde{u}_p - u_p\|_{1,\Omega} \le C\{e_B^q \|u\|_{1,\Omega} + K_B^q(\|u-u_p\|_{1,\Omega} + \|u - P_r^1 u\|_{1,\Omega}) + \mathcal{E}_F^p\}, \tag{3.24}$$

where $r = m - p - q$, $P_r^1 u$ is defined by (2.12) and K_B^q, e_B^q are as in (3.22), (3.23) respectively.

The following result gives an asymptotic estimate for the convergence rate in the $H^1(\Omega)$ norm using numerical quadrature.

THEOREM 3.1. *Let $\Omega \subset R^n$. Let $f \in H^s(\Omega)$ with $s > \frac{n}{2}$ and for $n = 2$, $g \in H^{\tilde{s}}(\Gamma)$ with $\tilde{s} > 1/2$ when Neumann conditions are used. Let $b_{ij}^K \in H^t(\hat{K})$ for each i, j, K, with $t > \frac{n}{2}$.*

Let $\tilde{u}_p$ denote the solution of (3.16), with the quadrature rule satisfying (A) and (B1)-(B3), or (B), with m large enough. Let q be a positive integer such that $r = m-p-q > 0$. Let for $n = 2$, $g \neq 0$, the quadrature rule on I satisfy (A) and (B), with $\tilde{m} - 2p > 0$. Then with K_B^q and e_B^q as in (3.22) and (3.23) respectively,

$$\begin{aligned} \|u - \tilde{u}_p\|_{1,\Omega} &\le C\{e_B^q \|u\|_{1,\Omega} + K_B^q\} \\ (3.25) \quad &\le C\left\{q^{-(t-n/2)}\rho_t(B)\|u\|_{1,\Omega} + \max\{\tilde{p}^{-(k-1)}, |\log \tilde{p}|^\gamma \tilde{p}^{-2\alpha}\} K_u K_B^q\right\} \end{aligned}$$

where $\tilde{p} = \min(p, r)$ and the constant C is independent of u, m, $\tilde{m}$, p and q.

Proof. By the triangle inequality, we have

$$\|u - \tilde{u}_p\|_{1,\Omega} \le \|u - u_p\|_{1,\Omega} + \|u_p - \tilde{u}_p\|_{1,\Omega}. \tag{3.26}$$

Using Lemma 3.3, we get

$$\|u_p - \tilde{u}_p\|_{1,\Omega} \le C\{e_B^q \|u\|_{1,\Omega} + E + \mathcal{E}_F^p\}, \tag{3.27}$$

where E is defined and estimated in (3.30). Also, using Lemma 3.2,

$$e_B^q \|u\|_{1,\Omega} \le C q^{-(l-\frac{n}{2})} \rho_t(B) \|u\|_{1,\Omega}. \tag{3.28}$$

Finally, with $r = m - p - q$, using

$$\|u - u_p\|_{1,\Omega} \le C E_u^p := C \inf_{v \in S_{p,0}} \|u - v\|_{1,\Omega}. \tag{3.29}$$

to bound $\|u - u_p\|_{1,\Omega}$ and using

$$\|u - P_r^1 u\|_{1,\Omega} \le C\, E_u^r,$$

we have

$$E = K_B^q(\|u-u_p\|_{1,\Omega} + \|u - P_r^1 u\|_{1,\Omega}) \le C K_B^q(E_u^p + E_u^r) \tag{3.30}$$

Using

$$\|u-u_p\|_{1,\Omega} \le C\, E_u^p = C\, K_u \max\{|\log p|^\gamma\, p^{-2\alpha}, p^{-(k-1)}\}, \tag{3.31}$$

where

$$\alpha = \min_v\{\alpha_1^v\} = \alpha_1^{v_0}, \quad \gamma = N^{v_0}(1), \quad K_u = \sum_{j,l,v} |d_{jl}^v| + \|u^1\|_{k,\Omega}.$$

we can now bound the right hand side of (3.30). □

The above theorem shows that essentially the rate of convergence is

$$\begin{aligned} &O\left\{q^{-(t-n/2)} + \max\{\tilde{p}^{-(k-1)}, |\log\tilde{p}|^\gamma \tilde{p}^{-2\alpha}\}\right. \\ &\qquad \left. + \max\{(m-p)^{-(s-n/2)}, (\tilde{m}-p)^{-(\tilde{s}-1/2)}\}\right\}. \end{aligned} \tag{3.32}$$

Also we see from (3.25) that if e_B^q, $\mathcal{E}_F^p$ are small enough (which happens when the coefficient a, mappings F_K and input data f, g are smooth enough), then the convergence rate is $E_u^{\tilde{p}}$. More precisely, suppose $m \approx 2p$ (as happens when the usual p point G-L rules are used). Let $q = \beta p$, $\beta \ll 1$, so that $r = m - p - q \approx (1-\beta)p$. Suppose t, s, $\tilde{s}$ in (3.32) are large enough so that the middle term dominates. Then the error will once again satisfy

$$\|u - \tilde{u}_p\|_{1,\Omega} \le C(\beta)\max\{p^{-(k-1)}, |\log p|^\gamma p^{-2\alpha}\}, \tag{3.33}$$

so that the asymptotic rate is the same as that using exact integration.

If, however, a, F_K, f or g are not smooth enough, then one of the errors e_B^q and $\mathcal{E}_F^p$ may dominate. Using overintegration with a sufficient number of points would then reduce these errors in many cases until the term $E_u^{\tilde{p}}$ dominates again and the same estimate as in (3.33) was observed. As overintegration is introduced, the value of m, $\tilde{m}$ is increased. For fixed p and r, this increases the values of q (which decreases e_B^q) and $m-p$, $\tilde{m}-p$ (which decreases $\mathcal{E}_F^p$). For more details on the $H^1(\Omega)$ error, including computational results, we refer to [6].

4. Numerical Experiments

In this section, we will consider the effect of using mapped elements in the one-dimensional case. Our numerical results indicate that the effect of the mapping is apparent only in certain cases.

In the one-dimensional case, consider problem (1.2) and (1.3) on $\Omega = [0,1]$ with $a(x) = 1$. Let the function f be chosen so that the exact solution u is given by

$$u(x) = x \sin \pi x. \tag{4.34}$$

Now consider the mapping

$$G(\xi) = \frac{(1+\xi+\epsilon)^\alpha - \epsilon^\alpha}{(2+\epsilon)^\alpha - \epsilon^\alpha}, \tag{4.35}$$

then $x = G(\xi)$ maps the reference element $[-1,1]$ to $[0,1]$. We have an affine mapping for $\alpha = 1$. If $\alpha \neq 1$, we have a nonlinear mapping whose smoothness depends on the parameter ϵ. For ϵ close to 0, the inverse of the Jacobian J_K^{-1} will be very large at $\xi = -1$, giving an unsmooth mapping.

The above choice of u and G has been numerically analyzed in [6]. It was shown there that if the mapping is smooth ($\alpha = 1$ or $\alpha = 2$, $\epsilon = 1.0$, for example), no overintegration is necessary for the stiffness matrix, when the H^1 norm error is of interest. We performed experiments that showed the same to be true for the L_2 norm. When G is not smooth ($\alpha = 2$, $\epsilon = 0.01$, for example), then overintegration helps slightly. However, as seen from Figure 1 of [6], the bulk of the error is due to the deterioration in the approximability properties of the underlying subspaces, which is independent of the quadrature used. The same behavior was observed by us for the L_2 norm.

As further shown in [6], there are situations where the accuracy of the quadrature, rather than the approximability, plays the dominant role. The example considered in [6] was where f was chosen so that

$$u(x) = \sin\{\pi((cx + \epsilon^\alpha)^{1/\alpha} - (1+\epsilon))\}, \tag{4.36}$$

where $c = (2+\epsilon)^\alpha - \epsilon^\alpha$ with the mapping G once again given by (4.35). Using the function in (4.36) and the mapping in (4.35), we see that

$$u(G(\xi)) = \sin \pi\xi, \tag{4.37}$$

which is a very smooth function on $[-1,1]$. In Figure 1, we have plotted the H^1 error on a log-log scale with $\alpha = 1.8$ and $\epsilon = 0.1$ (originally done in [6]). The solid line represents the error when exact integration is used, while the broken line represents the errors with G-L quadrature rule using p, $p+1$, $p+2$, $p+3$ and $2p$ points on the stiffness matrix with the load vector being calculated exactly (ie, with a sufficiently high quadrature rule). As we see, if the quadrature rule for the stiffness

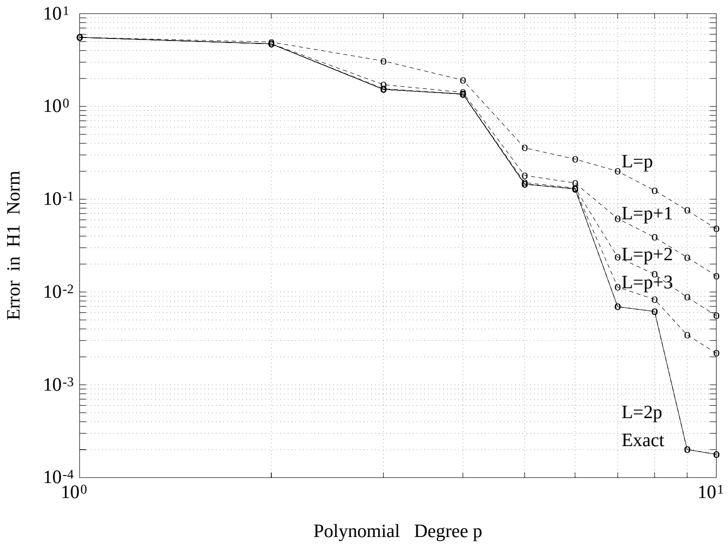

FIGURE 1. The H^1 error for $\alpha = 1.8$, $\epsilon = 0.1$

matrix is not sufficiently accurate, there is a significant loss in the rate of convergence. As we take more G-L points, the rate of convergence is closer to the exact case. In fact, using $2p$ points, the rate of convergence is essentially the same as that obtained using exact integration.

If we take a more unsmooth mapping by choosing a smaller value of ϵ, we expect to use more G-L points to get the correct rate of convergence. In Figure 2, we have plotted the H^1 error with $\alpha = 1.8$ and $\epsilon = 0.01$. Once again, the solid line represents the error when exact integration is used. We see that practically no convergence is observed using p points (compared to Figure 1). As expected, we need more G-L points, in fact p^2 points, to recover the correct rate of convergence. This number will increase further as ϵ is made smaller.

Let us remark that the above kind of example, where overintegration may play a significant role, does not arise except in special situations. One such situation (in two-dimensions) is the method of auxiliary mappings [2] where the (unsmooth) mappings are used to smooth out r^α type singularities.

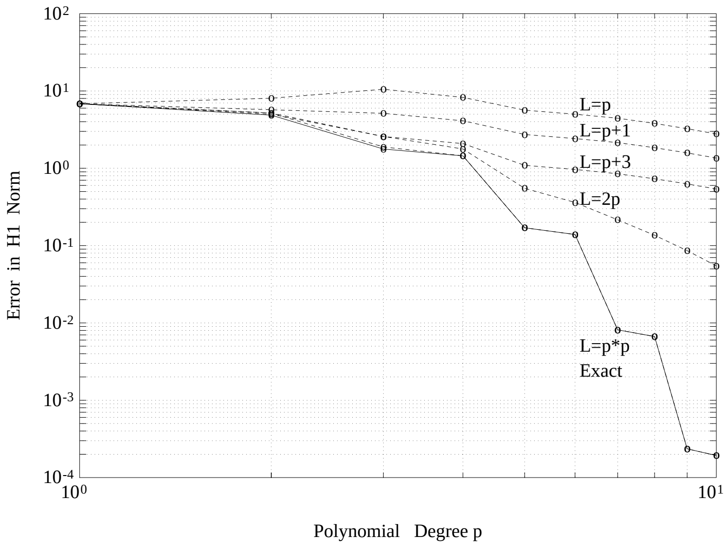

FIGURE 2. The H^1 error for $\alpha = 1.8, \epsilon = 0.01$

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