ON F-HARMONIC MAPS
AND CONVEX FUNCTIONS

TAE HO KANG

Abstract. We show that any F-harmonic map from a compact manifold $M$ to $N$ is necessarily constant if $N$ possesses a strictly-convex function, and prove 'Liouville type theorems' for F-harmonic maps. Finally, when the target manifold is the real line, we get a result for F-subharmonic functions.

1. F-harmonic maps and F-subharmonic functions

Recently, M. Ara[1] introduced the concept of F-harmonic maps, and unified the theory of harmonic maps, $p$-harmonic maps, exponentially harmonic maps and so on. More precisely, let $F : [0, \infty) \to [0, \infty)$ be a $C^2$ function such that $F' > 0$ on $(0, \infty)$. Let $\phi : (M, g) \to (N, h)$ be a smooth map between riemannian manifolds with metrics $g$ and $h$ respectively. Then $\phi$ is an F-harmonic map if it satisfies the F-tension field equation weakly:

$$\text{Trace} \nabla (F' \left( \frac{||d\phi||^2}{2} \right) d\phi) = 0,$$

i.e., for every compactly supported vector field $X$ along $\phi$

$$\int_M < F' \left( \frac{||d\phi||^2}{2} \right) d\phi, \nabla X > \geq 0,$$

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where $\|d\phi\|$ denotes the Hilbert-Schmidt norm of the differential $d\phi$ of $\phi$, which is the differential 1-form with values in the induced bundle $\phi^{-1}TN$ over $M$. It is harmonic, $p$-harmonic, $\alpha$-harmonic and exponentially harmonic when $F(t) = t$, $(2t)^{p/2} / p (p \geq 4)$, $(1 + 2t)^{\alpha} (\alpha > 1, \dim M = 2)$ and $e^t$, respectively. Finally we define an $F$-subharmonic function. A smooth function $\phi : M \rightarrow R$ is an $F$-subharmonic function if $\phi$ satisfies the inequality

$$\text{Trace}\nabla(F''(\frac{\|d\phi\|^2}{2})d\phi) \geq 0$$

weakly, i.e.,

$$\int_M < F''(\frac{\|d\phi\|^2}{2})d\phi, d\tau > \leq 0$$

for any compactly supported, nonnegative smooth function $\tau$ on $M$. It is subharmonic and $p$-subharmonic when $F(t) = t$ and $(2t)^{p/2} / p (p \geq 4)$, respectively.

2. Main results

In this article we prove the following theorems.

**Theorem 1.** Suppose that a smooth map $\phi : M \rightarrow N$ is $F$-harmonic. If $M$ is compact and there exists a strictly convex function on $N$, then $\phi$ is a constant map.

**Theorem 2.** Let $M$ and $N$ be riemannian manifolds. Suppose that $M$ is complete and noncompact, and $N$ has a strictly convex function $f : N \rightarrow R$ such that the uniform norm $\|df\|$ is bounded. If a smooth map $\phi : M \rightarrow N$ is $F$-harmonic with $\int_M F''(\frac{\|d\phi\|^2}{2})\|d\phi\| < \infty$, then $\phi$ is a constant map.

**Theorem 3.** Let $M$ be a complete noncompact manifold. If any $F$-subharmonic function $\phi : M \rightarrow R$ with

$$\int_M F''(\frac{\|d\phi\|^2}{2})\|d\phi\| < \infty,$$
then $\phi$ is a constant map.

Remark. In the above theorems, for the case of $F(t) = t$, i.e., harmonic maps or $F(t) = (2t)^{p/2}/p$, i.e., $p$-harmonic maps, see [3] and [6], respectively. In these cases, the energy $\int_M F'(\|d\phi\|^2)\|d\phi\|$ reduces to $\int_M \|d\phi\|$ (cf. [7]) and $\int_M \|d\phi\|^{p-1}$ (cf. [4, 5, 6]), respectively.

3. Proofs

First we show the following lemma.

Lemma. Let $\phi : M \rightarrow N$ be a smooth map between Riemannian manifolds and $f : N \rightarrow \mathbb{R}$ be a smooth function. Then the following identity holds for every smooth function $\eta$ on $M$.

$$< F'(\frac{\|d\phi\|^2}{2})d(f \circ \phi), d\eta > = -F'(\frac{\|d\phi\|^2}{2})\text{Trace}(\nabla df)(d\phi, d\phi)\eta + < \nabla(\eta \cdot (\text{grad} f) \circ \phi), F'(\frac{\|d\phi\|^2}{2})d\phi > .$$

Proof. Let $\{e_i\}$ be an orthonormal frame around some point of $M$ which satisfies $\nabla e_i = 0$ at that point. Then

$$< \nabla(\eta \cdot (\text{grad} \circ \phi), F'(\frac{\|d\phi\|^2}{2})d\phi >$$

$$= \sum_i < \nabla e_i(\eta \cdot (\text{grad} \circ \phi), F'(\frac{\|d\phi\|^2}{2})d\phi(e_i) >$$

$$= \sum_i d\eta(e_i) F'(\frac{\|d\phi\|^2}{2}) < \text{grad} f \circ \phi, d\phi(e_i) >$$

$$+ \sum_i \eta F'(\frac{\|d\phi\|^2}{2}) < \nabla d\phi(e_i) (\text{grad} f) \circ \phi, d\phi(e_i) >$$

$$= < F'(\frac{\|d\phi\|^2}{2})d(f \circ \phi), d\eta > + \eta F'(\frac{\|d\phi\|^2}{2})\text{Trace}(\nabla df)(d\phi, d\phi).$$
where the last term was calculated as follows;

\[
\sum_i \left< \nabla_{d\phi(e_i)} (\text{grad} f) \circ \phi, d\phi(e_i) \right>
\]

\[
= \sum_i \nabla_{d\phi(e_i)} < (\text{grad} f) \circ \phi, d\phi(e_i) >
\]

\[
- \sum_i < (\text{grad} f) \circ \phi, \nabla_{d\phi(e_i)} d\phi(e_i) >
\]

\[
= \sum_i \nabla_{d\phi(e_i)} (d\phi(e_i)f) - \sum_i \nabla_{d\phi(e_i)} d\phi(e_i)f
\]

\[
= \sum_i \nabla_{d\phi(e_i)} df(d\phi(e_i)) - \sum_i df(\nabla_{d\phi(e_i)} d\phi(e_i))
\]

\[
= \sum_i (\nabla_{d\phi(e_i)} df)(d\phi(e_i))
\]

\[
= \text{Trace}(\nabla df)(d\phi, d\phi).
\]

\[\Box\]

*Proof of Theorem 1.* Let \( f : N \to R \) be a strictly convex function. Taking \( \eta = 1 \) in Lemma and integrating on \( M \), we obtain

\[
\int_M F'(\frac{||d\phi||^2}{2}) \text{Trace}(\nabla df)(d\phi, d\phi) = 0,
\]

since \( \phi \) is \( F \)-harmonic map. Thus we have \( F'(\frac{||d\phi||^2}{2}) = 0 \), which implies that \( \frac{||d\phi||^2}{2} = 0 \), i.e., \( \phi \) is constant.

\[\Box\]

*Proof of Theorem 2.* Let us fix a point of \( M \) and denote \( B_r \) the geodesic ball with radius \( r \) and centered at this point. Then there exists a smooth function \( \eta \) on \( M \) such that

\[
0 \leq \eta \leq 1, \quad ||d\eta|| \leq \frac{c}{r},
\]
\[ \eta = \begin{cases} 
1 & \text{on } B_r \\
0 & \text{on } M \setminus B_{2r},
\end{cases} \]

where \( c \) is a positive constant which does not depend on \( r \). Then it follows from Lemma that

\[
\int_M F'(\frac{\|d\phi\|^2}{2}) \text{Trace}(\nabla^2 f)(d\phi, d\phi) \\
= -\int_M F'(\frac{\|d\phi\|^2}{2}) < d(f \circ \phi), d\eta > \\
\leq \int_M F'(\frac{\|d\phi\|^2}{2})\|df\|\|d\phi\|d\eta \\
\leq \frac{c}{r} \int_M F'(\frac{\|d\phi\|^2}{2})\|d\phi\| \to 0 \quad (\text{as } r \to \infty). 
\]

Thus we obtain \( F'(\frac{\|d\phi\|^2}{2}) = 0 \), which implies that \( \phi \) is constant. \( \square \)

**Proof of Theorem 3.** Taking a nondecreasing strictly convex function \( f \) with bounded derivative on the real line. Then for any nonnegative smooth function \( \eta \) with compact support, we get

\[
\text{div} \left[ \sum_i F'(\frac{\|d\phi\|^2}{2}) d\phi(e_i) \cdot \eta \cdot (\nabla f) \circ \phi \right] e_i \\
= \sum_i < e_i \{ F'(\frac{\|d\phi\|^2}{2}) d\phi(e_i) \cdot \eta \cdot (\nabla f) \circ \phi \} e_1, e_2 > \\
= \sum_i e_i \{ F'(\frac{\|d\phi\|^2}{2}) d\phi(e_i) \cdot \eta \cdot (\nabla f) \circ \phi \} 
\]
 Integrating this equation over $M$ and using assumptions, we obtain

$$\int_M \nabla \{\eta \cdot (\text{grad} f) \circ \phi\}, F'(\frac{||d\phi||^2}{2})d\phi >$$

$$= -\int_M \text{trace}(\nabla(F'(\frac{||d\phi||^2}{2})d\phi) \cdot \eta \cdot (\text{grad} f \circ \phi) \leq 0.$$ 

From this inequality and Lemma, we have

$$\int_M F'(\frac{||d\phi||^2}{2})\text{Trace}(\nabla f)(d\phi, d\phi)\eta$$

$$= \int_M \nabla \{\eta \cdot (\text{grad} f) \circ \phi\}, F'(\frac{||d\phi||^2}{2})d\phi >$$

$$- \int_M < F'(\frac{||d\phi||^2}{2})d(f \circ \phi), d\eta >$$

$$\leq -\int_M < F'(\frac{||d\phi||^2}{2})d(f \circ \phi), d\eta >.$$

Then we can argue as in Theorem 2. 

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Department of Mathematics
University of Ulsan
Ulsan 680–749, Korea
E-mail: thkang@mail.ulsan.ac.kr