UNIFORM DECAY OF SOLUTIONS
FOR VISCOELASTIC PROBLEMS

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Abstract. In this paper we prove the existence of solution and uniform decay rates of the energy to viscoelastic problems with nonlinear boundary damping term. To obtain the existence of solutions, we use Faedo-Galerkin’s approximation, and also to show the uniform stabilization we use the perturbed energy method.

1. Introduction

In this paper, we consider the uniform decay of solutions for viscoelastic problems with nonlinear boundary damping of the following form:

\[ Ku'' - (1 + ||\nabla u||^2)\Delta u + \int_0^t h(t - \tau)\Delta u(\tau)d\tau = 0 \]
\[ \quad \text{on } Q = \Omega \times (0, \infty), \]
\[ u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x) \quad \text{on } x \in \Omega, \]
\[ u = 0 \quad \text{on } \Sigma_1 = \Gamma_1 \times (0, \infty), \]
\[ (1 + ||\nabla u||^2)\frac{\partial u}{\partial \nu} - \int_0^t h(t - \tau)\frac{\partial u}{\partial \nu} - g(u') = f(u) \]
\[ \quad \text{on } \Sigma_0 = \Gamma_0 \times (0, \infty), \]

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where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with \( C^2 \) boundary \( \Gamma := \partial \Omega \) such that \( \Gamma = \Gamma_0 \cup \Gamma_1, \overline{\Gamma_0} \cap \Gamma_1 = \emptyset \) and \( \Gamma_0, \Gamma_1 \) have positive measures, and \( \nu \) denotes the unit outer normal vector pointing towards \( \Gamma \). When \( \Gamma_0 = \emptyset \), problem (1.1) with \( h = 0 \) results from the mathematical description of small amplitude vibrations of an elastic string (see [7]). In fact, a mathematical model for the deflection of an elastic string of length \( L > 0 \) is given by the mixed problem for the nonlinear wave equation

\[
\rho h \frac{\partial^2 u}{\partial t^2} = \{ p_0 + \frac{Eh}{2L} \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 dx \} \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad 0 < x < L, \quad t \geq 0,
\]

where \( u \) is the lateral deflection, \( x \) the space coordinate, \( t \) the time, \( E \) the Young modulus, \( \rho \) the mass density, \( h \) the cross section area and \( p_0 \) the initial axial tension.

There exists many literature about viscoelastic problems with the memory term acting in the domain. Among the numerous works in this direction, we can cite Jiang and Rivera[4]. When \( K = I \) Georgiev and Todorova[3] investigated blow-up properties of the solutions of wave equation with nonlinear damping and source term acting in the domain. For the existence results for Kirchhoff type wave equation with \( \partial \Omega = \Gamma_1 \) and \( K = I \), see Brito[1], Matsuyama[5], Ikehata[6] and Yamada[9]. When \( h = 0 \) and \( K = I \), Bae[8] has studied the uniform decay of solution for the Kirchhoff type wave equations with nonlinear boundary damping \( g(t)|u'|^\alpha u' \) and boundary source term \( \int_0^t g(t-r)|u(r)|\gamma u(r)dr \).

On the other hand, Cavalcanti et al [2] have studied the global existence and uniform decay of strong solutions of linear wave equation;

\[
K_1 u'' + K_2 u' - \Delta u = 0 \quad \text{on} \quad Q = \Omega \times (0, \infty),
\]

\[
u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{on} \quad \Omega,
\]

\[
u \in C^2 \quad \text{on} \quad \Sigma_1 = \Gamma_1 \times (0, \infty),
\]

\[
\frac{\partial u}{\partial \nu} + u' + \alpha(t)(|u'|^\alpha u' - |u|^\gamma u) = 0 \quad \text{on} \quad \Sigma_0 = \Gamma_0 \times (0, \infty).
\]
In this paper, we will study the existence of solutions of the Kirchoff type viscoelastic problem (1.1) with nonlinear boundary damping, nonlinear boundary source term. Moreover, we will consider the uniform decay of the energy of the problem (1.1).

2. Assumptions and main result

Throughout this paper we define

\[ V = \{ u \in H^1(\Omega); \ u = 0 \ \text{on} \ \Gamma_1 \}, \ (u, v) = \int_{\Omega} u(x)v(x)dx, \]

\[ (u, v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)d\Gamma, \ ||u||_{L^p, \Gamma_0} = \int_{\Gamma_0} |u(x)|^pdx \]

and \[ ||u||_\infty = ||u||_{L^\infty(\Omega)}. \]

For simplicity we denote \[ || \cdot ||_{L^2(\Omega)} \] and \[ || \cdot ||_{2, \Gamma_0} \] by \[ || \cdot || \] and \[ || \cdot ||_{\Gamma_0} \], respectively. For our result, we need the following assumptions.

(A1) Let us consider \( u_0, u_1 \in V \cap H^2(\Omega) \) verifying the compatibility conditions

\[ u_0 = \Delta u_0 = u_1 = 0 \ \text{on} \ \Gamma_1, \]

\[ (1 + ||\nabla u_0||^2)^{\frac{1}{2}} \frac{\partial u_0}{\partial \nu} + g(u_1) = f(u_0) \ \text{on} \ \Gamma_0. \]

(A2) Let \( K \) be a function in \( W^{1,\infty}(0, \infty) \cap L^\infty(0, \infty), K \geq 0 \) such that

\[ -K'(t) \geq \delta > 0, \ \forall \ t \geq 0. \]

(A3) \( f : R \rightarrow R \) is a \( C^1 \) function such that for some positive constant \( C_0 \),

\[ |f(s)| \leq C_0 |s|^{\gamma+1}, \quad |f'(s)| \leq C_0 |s|^\gamma, \]

where \( 0 < \gamma \leq \frac{1}{n-2} \) if \( n \geq 3 \) or \( \gamma > 0 \) if \( n = 1, 2 \).
(A4) \( g : \mathbb{R} \to \mathbb{R} \) is a nondecreasing \( C^1 \)-function such that for some positive constants \( C_1, C_2 \),
\[
C_1|s|^\rho + 2 \leq g(s)s \leq C_2|s|^\rho + 2, \quad s \in \mathbb{R},
\]
where \( 0 < \rho \leq \frac{1}{n-2} \) if \( n \geq 3 \) or \( \rho > 0 \) if \( n = 1, 2 \).

(A5) Let the function \( h : \mathbb{R}^+ \to \mathbb{R}^+ \) be a nonnegative and bounded \( C^2 \)-function such that \( t = 1 - \int_0^\infty h(r)dr > 0 \) and for some \( \xi_1, \xi_2, \xi_3 \),
\[
-\xi_1 h(t) \leq h'(t) \leq -\xi_2 h(t) \quad \text{and} \quad 0 \leq h''(t) \leq \xi_3 h(t), \quad \forall \ t \geq 0.
\]

Now we state our main result.

**Theorem 2.1.** Under assumptions \((A_1)-(A_5)\) and \( \rho \geq \gamma \), problem \((1.1)\) has a unique strong solution \( u : \Omega \to \mathbb{R} \) such that \( u \in L^\infty(0, \infty; V) \), \( u' \in L^\infty(0, \infty; V) \), \( \sqrt{K} u'' \in L^\infty(0, \infty; L^2(\Omega)) \), \( u'' \in L^2(0, \infty; L^2(\Omega)) \). Moreover, if \( \rho = \gamma \), then there exist positive constants \( C_1 \) and \( C_2 \) such that
\[
E(t) \leq C_1 E(0) \exp(-C_2 \varepsilon t), \quad t \geq 0.
\]

### 3. Proof of Theorem 2.1

In this section we are going to show the existence of solution for problem \((1.1)\). Now, we represent by \( \{w_j\}_{j \in \mathbb{N}} \) a basis in \( V \cap H^2(\Omega) \) which is orthonormal in \( L^2(\Omega) \), by \( V_m \) the finite dimensional subspace of \( V \) generated by the first \( m \) vectors. Next we define for each \( \varepsilon > 0 \), \( K_\varepsilon(t) = K(t) + \varepsilon \) and \( u_\varepsilon(t) = \sum_{j=1}^m \gamma_{\varepsilon m}(t) w_j \), where \( u_\varepsilon(t) \) is the solution of the following problem

\[
(K_\varepsilon(t)u''_\varepsilon(t), w) + (1 + \|\nabla u_\varepsilon(t)\|^2)(\nabla u_\varepsilon(t), \nabla w) + (g(u'_\varepsilon(t)), w)_{\Gamma_0} = (f(u_\varepsilon(t)), w)_{\Gamma_0} + \int_0^t h(t - r)(\nabla u_\varepsilon(r), \nabla w)dr,
\]

\[
u_\varepsilon(0) = u'_\varepsilon(0) = 0 \quad \text{for all} \quad w \in V_m.
\]
with the initial conditions,

\[ u_{\text{em}}(0) = u_{0m} = \sum_{j=1}^{m} (u_0, w_j)w_j \rightarrow u_0 \]

strongly in \( H_0^1(\Omega) \cap H^2(\Omega) \),

\[ u'_{\text{em}}(0) = u'_{1m} = \sum_{j=1}^{m} (u_1, w_j)w_j \rightarrow u_1 \]

strongly in \( H_0^1(\Omega) \).

Note that we can solve the system (3.1)-(3.2) by Picard's iteration method. In fact, the problem (3.1)-(3.2) has a unique solution on some interval \([0, T_m]\). The extension of these solutions to the whole interval \([0, T]\) is a consequence of the estimates which we are going to prove below.

A Priori Estimate I.

Multiplying (3.1) by \( \gamma'(t) \), summing over \( j \), we obtain

\[
\frac{d}{dt} \left[ \frac{1}{2} \| \sqrt{K_e(t)}u'_{\text{em}}(t) \|^2 + \frac{1}{2} \| \nabla u_{\text{em}}(t) \|^2 + \frac{1}{4} \| \nabla u_{\text{em}}(t) \|^4 \right] + \frac{1}{\gamma + 2} \| u_{\text{em}}(t) \|_{\gamma+2, \Gamma_0}^{\gamma+2} + (g(u'_{\text{em}}(t)), u'_{\text{em}}(t))_{\Gamma_0} \\
= \frac{1}{2} (K'(t), |u'_{\text{em}}(t)|^2) + (f(u_{\text{em}}(t)), u'_{\text{em}}(t))_{\Gamma_0} \\
+ \frac{d}{dt} \int_0^t h(t - r)(\nabla u_{\text{em}}(r), \nabla u_{\text{em}}(t))dr \\
- h(0)\| \nabla u_{\text{em}}(t) \|^2 - \int_0^t h'(t - r)(\nabla u_{\text{em}}(r), \nabla u_{\text{em}}(t))dr \\
+ (|u_{\text{em}}(t)|^\gamma u_{\text{em}}(t), u'_{\text{em}}(t))_{\Gamma_0}.
\]

Note that the assumption (A3), Holder's inequality, Young's inequal-
ity and imbedding \( L^{p+2}(\Gamma_0) \hookrightarrow L^{\gamma+2}(\Gamma_0) \) give us

\[
(f(u_{em}(t)), u'_{em}(t))_{\Gamma_0} + (|u_{em}(t)|^\gamma u_{em}(t), u'_{em}(t))_{\Gamma_0} \\
\leq (C_0 + 1) \int_{\Gamma_0} |u_{em}(t)|^{\gamma+1} |u'_{em}(t)| d\Gamma \\
\leq C_1(\eta) \|u_{em}(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + \eta \|u'_{em}(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} \\
\leq C_1(\eta) \|u_{em}(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + C_2(\eta) + \eta \|u'_{em}(t)\|_{p+2,\Gamma_0}^{p+2}.
\]

Considering Schwarz's inequality and taking the assumption \((A_5)\) into account, we deduce

\[
\int_0^t h'(t-r)(\nabla u_{em}(r), \nabla u_{em}(t)) dr \leq \frac{\xi_1^2}{2} \|\nabla u_{em}(t)\|^2 \\
+ \frac{1}{2} \|h\|_{L^1(0,\infty)} \int_0^t h(t-r) \|\nabla u_{em}(r)\|^2 dr.
\]

Combining the above inequalities, and integrating it over \((0, t)\), assumptions \((A_2)\) and \((A_4)\) imply

\[
E_{em}(t) + \frac{1}{\gamma+2} \|u_{em}(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + \frac{\delta}{2} \int_0^t \|u'_{em}(s)\|^2 ds \\
+ \int_0^t (C_1 - \eta) \|u'_{em}(s)\|_{p+2,\Gamma_0}^{p+2} ds \\
\leq E_{em}(0) + \frac{1}{\gamma+2} \|u_{0em}\|_{\gamma+2,\Gamma_0}^{\gamma+2} + \int_0^t C_2(\eta) \\
+ C_1(\eta) \|u_{em}(s)\|_{p+2,\Gamma_0}^{p+2} ds \\
+ (\frac{\xi_1^2}{2} - h(0)) \int_0^t \|\nabla u_{em}(s)\|^2 ds \\
+ \int_0^t h(t-r)(\nabla u_{em}(r), \nabla u_{em}(t)) dr \\
+ \frac{1}{2} \|h\|_{L^1(0,\infty)} \int_0^t \int_0^s h(s-r) \|\nabla u_{em}(r)\|^2 dr ds.
\]
\begin{equation}
\begin{aligned}
&\leq E_{em}(0) + \frac{1}{\gamma + 2} \|u_{0em}\|_{\gamma + 2, \Gamma_0}^2 \\
&+ \int_0^t C_2(\eta) + C_1(\eta) \|u_{em}(s)\|_{\gamma + 2, \Gamma_0}^2 ds \\
&+ \left( \frac{\xi_2}{2} - h(0) \right) + \frac{1}{2} \|h\|_{L^2(0, \infty)}^2 \\
&+ \|h\|_{L^1(0, \infty)} \|h\|_{L^\infty(0, \infty)} \int_0^t \|\nabla u_{em}(s)\|^2 ds, 
\end{aligned}
\end{equation}

where \( E_{em}(t) = \frac{1}{2} \|\sqrt{K_\gamma(t)}u'_{em}(t)\|^2 + \frac{1}{2} \|\nabla u_{em}(t)\|^2 + \frac{1}{4} \|\nabla u_{em}(t)\|^4. \)

Employing Gronwall’s lemma we obtain the first estimate:

\begin{equation}
\begin{aligned}
\|\sqrt{K_\gamma(t)}u'_{em}(t)\|^2 + \|\nabla u_{em}(t)\|^2 + \int_0^t \|u'_{em}(s)\|^2 ds \\
+ \int_0^t \|u_{em}(s)\|_{\rho + 2, \Gamma_0}^{\rho + 2} \leq L_1,
\end{aligned}
\end{equation}

where \( L_1 > 0 \) is a positive constant independent of \( m \) and \( t > 0. \)

From the assumptions \((A_4)\) on \( g \) and \((3.7), \) we get

\begin{equation}
\int_0^t \|g(u'_{em}(s))\|_{\Gamma_0}^2 ds \leq L_2,
\end{equation}

where \( L_2 > 0 \) is a positive constant independent of \( m \) and \( t > 0. \)

**A Priori Estimate II.**

Now differentiating \((3.1), \) multiplying the result by \( \gamma(t) \) and sum-
Making use of the Schwarz inequality, Young Inequality and taking assumption (\(A_3\)) into account, we have

\[
\frac{1}{2} \frac{d}{dt} \left[ \| \sqrt{K_e(t)} u''_{cm}(t) \|^2 + (1 + \| \nabla u_{em}(t) \|^2) \| \nabla u'_{cm}(t) \|^2 \right]
\]
\[
+ 2(\nabla u_{em}(t), \nabla u'_{em}(t))^2 + \frac{1}{2} (K_e'(t), |u''_{em}(t)|^2)
\]
\[
+ \int_{\Gamma_0} g'(u'_{em}(t))(u''_{em}(t))^2 d\Gamma
\]
\[
= (f'(u_{em}(t))u'_{em}(t), u''_{em}(t))_{\Gamma_0}
\]
\[
+ 3(\nabla u_{em}(t), \nabla u'_{em}(t)) \| \nabla u'_{em}(t) \|^2 - h(0) \| \nabla u'_{em}(t) \|^2
\]
\[
+ h(0) \frac{d}{dt} (\nabla u_{em}(t), \nabla u'_{em}(t))
\]
\[
+ \frac{d}{dt} \int_0^t h'(t - r)(\nabla u_{em}(r), \nabla u'_{em}(t)) dr
\]
\[
- h'(0)(\nabla u_{em}(t), \nabla u'_{em}(t))
\]
\[
- \int_0^t h''(t - r)(\nabla u_{em}(r), \nabla u'_{em}(t)) dr.
\]

\[
(f'(u_{cm}(t))u'_m(t), u''_{cm}(t))_{\Gamma_0}
\]
\[
\leq C_0 \| u_{em}(t) \|_{2(\gamma + 1), \Gamma_0} \| u'_{em}(t) \|^2_{2(\gamma + 1), \Gamma_0} \| u''_{em}(t) \|_{\Gamma_0}
\]
\[
\leq C \| \nabla u_{em}(t) \|^2 \| \nabla u'_{em}(t) \| \| u''_{em}(t) \|_{\Gamma_0}
\]
\[
\leq C(\eta)L_1^{2\gamma} \| \nabla u'_m(t) \|^2 + \eta \| u''_{em}(t) \|^2_{\Gamma_0}.
\]
Integrating (3.9) over \((0, t)\), we have

\[
E_{\epsilon m}(t) + \frac{\delta}{2} \int_0^t \|u_{\epsilon m}'(s)\|^2 ds
\]

\[
+ \int_0^t \int_\Gamma (g'(u_{\epsilon m}(s)) - \eta) (u_{\epsilon m}'(s))^2 d\Gamma ds
\]

\[
\leq E_{\epsilon m}(0) + 3 \int_0^t \|\nabla u_{\epsilon m}(s)\| \|\nabla u_{\epsilon m}'(s)\|^3 ds
\]

\[
+ h(0) \|\nabla u_{\epsilon m}(t)\| \|\nabla u_{\epsilon m}'(t)\| + \|\nabla u_{\epsilon m}\| \|\nabla u_{\epsilon m}\|
\]

\[
+ \frac{1}{2} \|h\|_{L^1(0, \infty)}^2 + \|h\|_{L^1(0, \infty)} \|h\|_{L^\infty(0, \infty)}
\]

\[
+ |h'(0)| \int_0^t \|\nabla u_{\epsilon m}(r)\|^2 dr
\]

\[
+ \frac{1}{2} (\xi_1^2 + \xi_3^2 - 2h(0) + |h'(0)|)
\]

\[
+ C(\eta) L_1 \int_0^t \|\nabla u_{\epsilon m}'(s)\|^2 ds,
\]

where \(E_{\epsilon m}(t) = \frac{1}{2} [\|K_{\epsilon}(t) u_{\epsilon m}(t)\|^2 + (1 + \|\nabla u_{\epsilon m}(t)\|^2) \|\nabla u_{\epsilon m}'(t)\|^2 + 2(\nabla u_{\epsilon m}(t), \nabla u_{\epsilon m}'(t))^2] \). Considering the first estimate and employing Gronwall’s inequality, for sufficiently small \(\eta\),

\[
\|\sqrt{K_{\epsilon}(t)u_{\epsilon m}''(t)}\|^2 + \|\nabla u_{\epsilon m}'(t)\|^2 + \int_0^t \|u_{\epsilon m}'(s)\|^2 ds \leq L_4,
\]

where \(L_4\) is a positive constant independent of \(m \in N\) and \(t \in [0, T]\).

By the estimates we can extract subsequence \((u_{\epsilon m})\) of \((u_{\epsilon m})\) such
that
\begin{equation}
(3.13)
\quad u_{e\mu} \rightarrow u_e \quad \text{weak star} \quad L^\infty(0,T;V),
\end{equation}
\begin{equation}
(3.14)
\quad u_{e\mu}' \rightarrow u_e' \quad \text{weak star} \quad L^\infty(0,T;V),
\end{equation}
\begin{equation}
\sqrt{K_e}u_{e\mu}'' \rightarrow \sqrt{K_e}u_e'' \quad \text{weak star} \quad L^\infty(0,T;L^\infty(\Omega)),
\end{equation}
\begin{equation}
(3.15)
\quad L^\infty(0,T;L^{\gamma+2}(\Gamma_0)),
\end{equation}
\begin{equation}
(3.16)
\quad u_{e\mu} \rightarrow u_e \quad \text{weak star} \quad L^\infty(0,T;L^{\rho+2}(\Gamma_0)),
\end{equation}
\begin{equation}
(3.17)
\quad u_{e\mu}' \rightarrow u_e' \quad \text{weak star} \quad L^\infty(0,T;L^\rho(\Gamma_0)),
\end{equation}
\begin{equation}
(3.18)
\quad u_{e\mu}' \rightarrow u_e' \quad \text{weak star} \quad L^2(0,T;L^2(\Gamma_0)),
\end{equation}
\begin{equation}
(3.19)
\quad u_{e\mu}'' \rightarrow u_e'' \quad \text{weak star} \quad L^2(0,T;L^2(\Gamma_0)).
\end{equation}

The convergence (3.13)-(3.15) and (3.19) are sufficient to pass to the limit in the linear terms of problem (3.1). Next we are going to consider the nonlinear ones.

**Analysis of the nonlinear terms.**

Taking (3.7) into account, we note that
\[
\int_0^T \int_{\Gamma_0} |f(u_{e\mu})|^{\frac{\gamma+2}{\gamma+1}} d\Gamma dt \leq C_1 \int_0^T \int_{\Gamma_0} |u_{e\mu}|^{\gamma+2} d\Gamma dt \leq L,
\]
\[
\int_0^T \int_{\Gamma_0} |g(u_{e\mu}')|^{\frac{\rho+2}{\rho+1}} d\Gamma dt \leq C_1 \int_0^T \int_{\Gamma_0} |u_{e\mu}'|^{\rho+2} d\Gamma dt \leq L,
\]
so we have \( \phi, \xi \in L^2(0,T;L^2(\Gamma_0)) \) such that
\[
f(u_{e\mu}) \rightarrow \phi \quad \text{weakly in} \quad L^{\frac{\gamma+2}{\gamma+1}}(\Gamma_0 \times (0,T)),
\]
\[
g(u_{e\mu}') \rightarrow \xi \quad \text{weakly in} \quad L^{\frac{\rho+2}{\rho+1}}(\Gamma_0 \times (0,T)).
From the first and second estimates we deduce
\begin{align*}
(\mathbf{u}_{\epsilon\mu}) & \text{ is bounded in } L^2(0, T; H^{\frac{1}{2}}(\Gamma_0)), \\
(\mathbf{u}'_{\epsilon\mu}) & \text{ is bounded in } L^2(0, T; H^{\frac{1}{2}}(\Gamma_0)), \\
(\mathbf{u}''_{\epsilon\mu}) & \text{ is bounded in } L^2(0, T; L^2(\Gamma_0)).
\end{align*}
Taking into consideration that the imbedding $H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^2(\Gamma)$ is continuous and compact and using Aubin compactness theorem, we can extract a subsequence, still represented by $(\mathbf{u}_{\epsilon\mu})$, such that
(3.20) \begin{align*}
\mathbf{u}_{\epsilon\mu} & \rightharpoonup \mathbf{u}_\epsilon \quad \text{strongly in } L^2(0, T; L^2(\Gamma_0)), \\
\mathbf{u}'_{\epsilon\mu} & \rightharpoonup \mathbf{u}'_\epsilon \quad \text{strongly in } L^2(0, T; L^2(\Gamma_0)),
\end{align*}
which imply that
(3.21) \begin{align*}
\mathbf{u}_{\epsilon\mu} & \to \mathbf{u}_\epsilon \quad \text{a.e. on } \Sigma_0 \quad \text{and} \quad \mathbf{u}'_{\epsilon\mu} \to \mathbf{u}'_\epsilon \quad \text{a.e. on } \Sigma_0
\end{align*}
and therefore
\begin{align*}
f(\mathbf{u}_{\epsilon\mu}) & \to f(\mathbf{u}_\epsilon) \quad \text{weakly in } L^{\frac{3+\frac{2}{p}}{2}}(\Gamma_0 \times (0, T)), \\
g(\mathbf{u}'_{\epsilon\mu}) & \to g(\mathbf{u}'_\epsilon) \quad \text{weakly in } L^{\frac{2+\frac{2}{p}}{2}}(\Gamma_0 \times (0, T)).
\end{align*}
Thus
\begin{align*}
f(\mathbf{u}_{\epsilon\mu}) & \to f(\mathbf{u}_\epsilon) \quad \text{weakly in } L^2(0, T; L^2(\Gamma_0)), \\
g(\mathbf{u}'_{\epsilon\mu}) & \to g(\mathbf{u}'_\epsilon) \quad \text{weakly in } L^2(0, T; L^2(\Gamma_0)).
\end{align*}
Using standard arguments, we can show from the above estimates that
(3.22) \begin{align*}
K(t)u''(t) - \Delta [u(t) + ||\nabla u(t)||^2 u(t)] - \int_0^t h(t-r)u(r)dr &= 0
\end{align*}
in $L^2_{\text{loc}}(0, T; L^2(\Omega))$
Making use of the generalized Green's formula we deduce that
(3.23) \begin{align*}
\frac{\partial}{\partial \nu} [1 + ||\nabla u(t)||^2]u(t) - \int_0^t h(t-r)u(r)dr + g(u') &= f(u)
\end{align*}
in $L^2_{\text{loc}}(0, T; L^2(\Gamma_0))$
This completes the proof of the existence of solutions of (1.1). The uniqueness is obtained in a usual way, so we omit the proof here. □
4. Uniform decay

In this section we derive the decay estimates for the energy of (1.1). We define the energy \( E(t) \) of the problem (1.1) by

\[
E(t) = \frac{1}{2} \| \sqrt{K(t)} u'(t) \|^2 + \frac{1}{2} \| \nabla u(t) \|^2 + \frac{1}{4} \| \nabla u(t) \|^4.
\]  

Then the derivative of energy is given by

\[
E'(t) = \frac{1}{2} (K'(t), |u'(t)|^2) + \int_0^t h(t-r)(\nabla u(r), \nabla u'(r)) dr - (g(u'(t)), u'(t))_{\Gamma_0} + (f(u(t)), u'(t))_{\Gamma_0}.
\]  

Define \((h \square u)\) by

\[
(h \square u)(t) := \int_0^t h(t-r)||u(t)-u(r)||^2dr.
\]  

Next consider the modified energy \( e(t) \):

\[
e(t) = \frac{1}{2} \| \sqrt{K(t)} u'(t) \|^2 + \frac{1}{2} \| \nabla u(t) \|^2 + \frac{1}{2} (h \square \nabla u)(t)
\]  

\[
+ \left( 1 - \int_0^t h(r) dr \right) ||\nabla u(t)||^2 + \frac{1}{\gamma + 2} \alpha(t)||u(t)||_{\gamma+2,\Gamma_0} + \int_{\Gamma_0} \int_0^\infty f(\eta) d\eta d\Gamma,
\]  

where \( \alpha \in W^{1,\infty}(0,\infty) \cap L^1(0,\infty) \) with \(-m_0 \alpha(t) \leq \alpha'(t) \leq -m_1 \alpha(t), \alpha(t) \geq m_2 \) for all \( t \geq 0 \), \( m_i \geq 0 \), \( i = 0,1,2 \), \( m_1 > 2(\gamma + 2) \) and \( ||\alpha||_\infty \leq C_1 \), \( C_1 \) is a constant in assumption \((A_4)\). Then (4.2) and (4.4) imply

\[
e'(t) = \frac{1}{2} (K'(t), |u'(t)|^2) - (g(u'(t)), u'(t))_{\Gamma_0}
\]  

\[
- \frac{1}{2} h(t)||\nabla u(t)||^2 + \frac{1}{2} (h' \square \nabla u)(t)
\]  

\[
+ \alpha(t)||u(t)||_{\gamma+2,\Gamma_0} + \frac{1}{\gamma + 2} \alpha'(t)||u(t)||_{\gamma+2,\Gamma_0}.
\]
Considering Young's inequality, we get

\[
\alpha(t)(|u(t)|^\gamma u(t), u'(t))_{\Gamma_0} \\
\leq \eta \alpha(t) |u'(t)|^{\gamma+2}_{\gamma+2, \Gamma_0} + \eta^{-\frac{1}{\gamma+1}} \alpha(t) |u(t)|^{\gamma+2}_{\gamma+2, \Gamma_0}.
\]

Choosing \( \eta = 2^{-(\gamma+1)} \) then \( C_1 - \eta \|\alpha\|_\infty > 0 \). Thus for \( \gamma = \rho \) and sufficiently small \( \eta > 0 \), the assumptions \((A_2)\) and \((A_5)\) imply

\[
e'(t) \leq -\frac{\delta}{2} |u'(t)|^2 - \beta |u'(t)|^{\gamma+2}_{\gamma+2, \Gamma_0} - \frac{1}{2} h(t) \|\nabla u(t)\|^2 \\
- \frac{\xi_2}{2} (h \Box \nabla u)(t) - \beta_1 |u(t)|^{\gamma+2}_{\gamma+2, \Gamma_0},
\]

where \( \beta = C_1 - \eta \|\alpha\|_\infty > 0 \) and \( \beta_1 = m_2(\frac{m}{\gamma+2} - \eta^{-\frac{1}{\gamma+1}}) > 0 \). On the other hand we note that from assumption \((A_5)\)

\[
E(t) \leq \frac{1}{2} |u'(t)|^2 + \frac{1}{2l} (1 - \int_0^t h(r) dr) \|\nabla u(t)\|^2 \\
+ \frac{1}{4} \|\nabla u(t)\|^4 \leq t^{-1} e(t)
\]

and therefore it is enough to obtain the desired exponential decay for the modified energy \( e(t) \) which will be done below.

For this purpose let \( \lambda \) be the positive number such that \( \|v\|^2 \leq \lambda \|\nabla v\|^2 \), \( \forall v \in V \) and for every \( \epsilon > 0 \) let us define the perturbed modified energy by

\[
e_\epsilon(t) = e(t) + \epsilon \psi(t), \quad \text{where} \quad \psi(t) = (K(t)u'(t), u(t)).
\]

Applying Cauchy Schwarz's inequality, we easily obtain the following proposition.

**Proposition 4.1.** We have the inequality for any \( \epsilon > 0 \)

\[
|e_\epsilon(t) - e(t)| \leq \epsilon \lambda^\frac{1}{4} \|K\|\|e(t)\|, \quad \forall t \geq 0.
\]

The following proposition is the useful instrument for the energy decay of problem (1.1).
Proposition 4.2. There exist $C > 0$ and $\epsilon_1$ such that for $\epsilon \in (0, \epsilon_1]$

$$c'_c(t) \leq -\epsilon C c(t).$$

Proof. Using the equation (1.1), we have

$$\psi'(t) = \|\sqrt{K(t)}u'(t)\|^2 + (K'(t)u'(t), u(t))$$

$$- (1 + \|\nabla u(t)\|^2)\|\nabla u(t)\|^2 - (g(u'(t)), u(t))_{\Gamma_0}$$

$$+ (f(u(t)), u(t))_{\Gamma_0} + \int_0^t h(t - r)(\nabla u(r), \nabla u(t))dr$$

$$= -c(t) + \frac{3}{2}\|\sqrt{K(t)}u'(t)\|^2 + (K'(t)u'(t), u(t))$$

$$- \frac{1}{2}\|\nabla u(t)\|^2 - \frac{3}{4}\|\nabla u(t)\|^4 + \frac{1}{2}(h \nabla \nabla u(t))$$

$$- \frac{1}{2}\int_0^t h(r)dr\|\nabla u(t)\|^2 - (g(u'(t)), u(t))_{\Gamma_0}$$

$$+ (f(u(t)), u(t))_{\Gamma_0} + \int_0^t h(t - r)(\nabla u(r), \nabla u(t))dr$$

$$+ \frac{1}{\gamma + 2}\alpha(t)\|u(t)\|^{\gamma + 2}_{\gamma + 2, \Gamma_0}.$$  \hfill (4.9)

Now applying Schwarz's inequality and (4.3), we get

$$\int_0^t h(t - r)(\nabla u(r), \nabla u(t))dr$$

$$\leq \frac{1}{2}(h \nabla \nabla u(t)) + \frac{3}{2}\|\nabla u(t)\|^2 \int_0^t h(r)dr. \hfill (4.10)$$

From Schwarz's inequality and Young's inequality we get

$$|(g(u'(t)), u(t))_{\Gamma_0}| \leq C_2\|u'(t)\|_{\rho + 2, \Gamma_0}^{\rho + 1}\|u(t)\|_{\rho + 2, \Gamma_0}$$

$$\leq C_4(\eta)\|u'(t)\|_{\rho + 2, \Gamma_0}^{\rho + 2} + \eta\|u(t)\|_{\rho + 2, \Gamma_0}^{\rho + 2}. \hfill (4.11)$$
and

\[
(K'(t)u'(t), u(t)) \leq \lambda \frac{1}{2} ||K'|| \frac{1}{2} ||u'(t)|| ||\nabla u(t)|| \\
\leq \lambda C_5(\eta) ||K'||_\infty ||u'(t)||^2 + \eta ||\nabla u(t)||^2.
\]

Thus we have

\[
\psi'(t) \leq -\epsilon(t) + \left( \frac{3}{2} ||K||_\infty + \lambda C_5(\eta) ||K'||_\infty \right) ||u'(t)||^2 \\
- \frac{1}{2} h(t) ||\nabla u(t)||^2 - \frac{3}{4} \eta ||\nabla u(t)||^4 \\
+ (h \Box \nabla u)(t) + \int_0^t h(r) dr ||\nabla u(t)||^2 \\
+ C_4(\eta) ||u'(t)||_{\rho + 2, \Gamma_0}^2 + \eta ||u(t)||_{\rho + 2, \Gamma_0}^2 \\
+ (C_0 + \frac{1}{\gamma + 2} ||\alpha||_\infty) ||u(t)||_{\gamma + 2, \Gamma_0}^2
\]

(4.12)

From (4.7), (4.12) and the assumption (A2) and considering \( \rho = \gamma \), we get

\[
\epsilon'(t) = \epsilon'(t) + \epsilon \psi'(t) \\
\leq -\epsilon \epsilon(t) - \left( \frac{\delta}{2} - \epsilon \left[ \frac{3}{2} ||K||_\infty + \lambda C_5(\eta) ||K'||_\infty \right] \right) ||u'(t)||^2 \\
- \frac{1}{2} h(t) ||\nabla u(t)||^2 - \frac{3}{2} \epsilon ||\nabla u(t)||^4 - (\delta - \frac{\xi_2}{2} - \epsilon)(h \Box \nabla u)(t) \\
- (\beta - C_4(\eta) \epsilon) ||u'(t)||_{\gamma + 2, \Gamma_0}^{\gamma + 2} \\
- \left[ \beta_1 - (C_0 + \frac{1}{\gamma + 2} ||\alpha||_\infty) \right] ||u(t)||_{\gamma + 2, \Gamma_0}^{\gamma + 2}
\]

(4.13)

Defining

\[
\epsilon_1 = \min\left\{ \frac{\xi_2}{2}, \frac{\delta}{2}, \frac{3}{2} ||K||_\infty + \lambda C_5(\eta) ||K'||_\infty \right\}^{-1}, \frac{\beta}{C_4(\eta)}, \frac{\beta_1(\gamma + 2)}{C_0(\gamma + 2) + ||\alpha||_\infty}\}
\]
Then for each $\epsilon \in (0, \epsilon_1]$, we have

$$
e'(t) \leq -\epsilon C e(t) \tag{4.14}$$

if $\|h\|_{L^1(0, \infty)}$ is sufficiently small. \hfill \Box

**Continuing the proof of Theorem 2.1**

Let $\epsilon_0 = \min\{\frac{1}{2\lambda^{\frac{1}{2}}\|K\|_{L^2}}, \epsilon_1\}$ and let us consider $\epsilon \in (0, \epsilon_0]$. As we have $\epsilon < \frac{1}{2\lambda^{\frac{1}{2}}\|K\|_{L^2}}$, we conclude from Proposition 4.1

$$(1 - \epsilon\lambda^{\frac{1}{2}}\|K\|_{L^2})e(t) < e_e(t) < (1 + \epsilon\lambda^{\frac{1}{2}}\|K\|_{L^2})e(t)$$

and so

$$
\frac{1}{2} e(t) < e_e(t) < \frac{3}{2} e(t). \tag{4.15}
$$

Thus we have

$$e'(t) \leq -\frac{2}{3} C_1 \epsilon e(t)$$

and

$$
\frac{d}{dt} [e_e(t) \exp\left(\frac{2}{3} C_1 \epsilon t\right)] \leq 0. \tag{4.16}
$$

Integrating (4.16), inequality (4.15) implies

$$
e(t) \leq 3e(0) \exp\left(-\frac{2}{3} C_1 \epsilon t\right). \tag{4.17}
$$

Hence from (4.9) and (4.17) we get

$$E(t) \leq t^{-1} e(t) \leq 3e(0) t^{-1} \exp\left(-\frac{2}{3} C_1 \epsilon t\right), \quad t \geq 0.$$

This completes the proof of Theorem 2.1. \hfill \Box
REFERENCES


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